Higher order gravity theories

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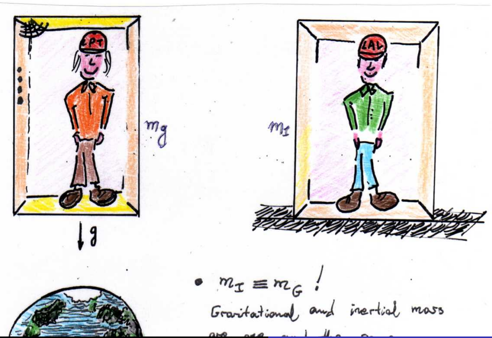
General Relativity

GR is based on two important principles:

- Mach's principle The presence of matter curves the geometry of spacetime
- Equivalence principle Locally a free-falling observer and an inertial observer are indistinguishable



Question: Why we should not modify GR

- Theoretical consistency: In 4 dimensions, consider $\mathcal{L} = \mathcal{L}(\mathcal{M}, g, \nabla g, \nabla\nabla g)$. Then Lovelock's theorem in $D = 4$ states that GR with cosmological constant is the unique metric theory emerging from,

$$S_{(4)} = \int_{\mathcal{M}} d^4x \sqrt{-g^{(4)}} \left[R - 2\Lambda\right]$$

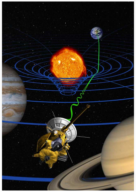
giving,

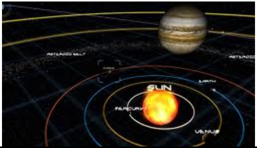
  - Equations of motion of 2^{nd}-order
  - given by a symmetric two-tensor, $G_{\mu\nu} + \Lambda g_{\mu\nu}$
  - and admitting Bianchi identities.

Under these hypotheses GR is the unique massless-tensorial 4 dimensional theory of gravity!

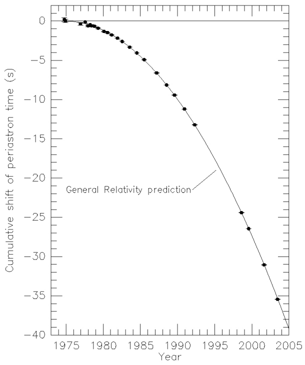
Experimental and observational data in weak gravity

- Experimental consistency:
  -Excellent agreement with solar system tests
  -Strong gravity tests on binary pulsars
  -Laboratory tests of Newton's law (tests on extra dimensions)

Time delay of light  Planetary tajectories



For strong gravity



Orbital decay of the Hulse-Taylor binary pulsar [Taylor&Weisberg04].

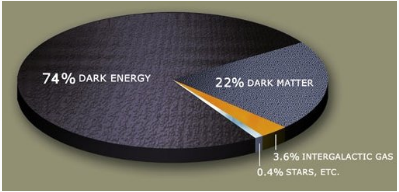
General Relativity is **very well tested** for strong gravity-Hulse-Taylor pulsar

Q: What is the matter content of the Universe today?

Assuming homogeinity-isotropy and GR

$$G_{\mu\nu} = 8\pi G T_{\mu\nu}$$

cosmological and astrophysical observations dictate the matter content of the Universe today:



A: -Only a 4% of matter has been discovered in the laboratory. We hope to see more at LHC. But even then...

If we assume only ordinary sources of matter (DM included) there is disagreement between local, astrophysical and cosmological data.

Universe is accelerating → Enter the cosmological constant

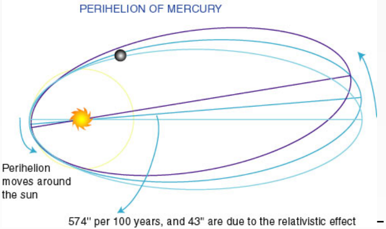
Easiest way out: Assume a tiny cosmological constant $\rho_\Lambda = \frac{\Lambda_{obs}}{8\pi G} = (10^{-3} eV)^4$, ie modify Einstein's equation by,

$$G_{\mu\nu} + \Lambda_{obs} g_{\mu\nu} = 8\pi G T_{\mu\nu}$$

- Cosmological constant introduces $\sqrt{\Lambda}$ and generates a cosmological horizon
- $\sqrt{\Lambda}$ is as tiny as the inverse size of the Universe today, $r_0 = H_0^{-1}$
- In fact had we used planetary trajectories ,SchdS rather than Sch, we would get, $\Lambda_{local} < 10^{11} \Lambda_{obs}$
- Note that $\frac{\textbf{Solar system scales}}{\textbf{Cosmological Scales}} \sim \frac{\textbf{10 A.U.}}{H_0^{-1}} = 10^{-14}$
- Typical mass scale for neutrinos... In fact Λ for GR is a bit like m_I and m_G in Newtonnian theory...

Maybe Λ_{obs} is not a cosmological constant.

What if the need for exotic matter or cosmological constant is the sign for novel gravitational physics at very low energy scales or large distances.



-Same situation at the advent of GR.

-Even then a next order correction with one additional parameter was enough to save Newton's laws (at the experimental precision of the time..)

-Just like successes of GR are not only the advance of Mercury's perihelion, modification of gravity cannot only be "an explanation" of the cosmological constant.

-Furthermore, modifying GR is similar to opening Pandora s box...

Modified gravity theories

- Extra dimensions : Extension of GR to Lovelock theory with modified yet second order field equations. Braneworlds, Kaluza-Klein compactfication
- 4-dimensional modification of GR: Scalar-tensor theories, Einstein-Aether, Hoȓava gravity, Galileon/Hornedski.
- Massive gravity, decoupling limit of DGP
- Theories modifying geometry: torsion, choice of geometric connection

A metric gravity modification: Lovelock theory

- In 4 dimensions, consider $\mathcal{L} = \mathcal{L}(\mathcal{M}, g, \nabla, \nabla^2)$. GR with cosmological constant Λ is the unique metric theory emerging from,

$$S_{(4)} = \int_{\mathcal{M}} d^4x \sqrt{-g^{(4)}}\, [R - 2\Lambda]$$

  depending on a,
  - symmetric two-tensor, $G_{\mu\nu} + \Lambda g_{\mu\nu} + \alpha H_{\mu\nu}$
  - *giving* Equations of motion as 2^{nd}-order metric PDE' s
  - *and* Bianchi identities.

with $H_{AB} = \frac{g_{AB}}{2}\hat{G} - 2RR_{AB} + 4R_{AC}R^{C}_{\ B} + 4R_{CD}R^{C\ D}_{\ A\ B} - 2R_{ACDE}R_{B}^{\ CDE}$

- In $d > 4$ consider $\mathcal{L} = \mathcal{L}(\mathcal{M}, g, \nabla, \nabla^2)$. The above action is no longer unique. The unique, d-dimensional theory with the same properties is due to [Lovelock71'] and its $d = 5, 6$-dimensional version includes the 2nd order term,

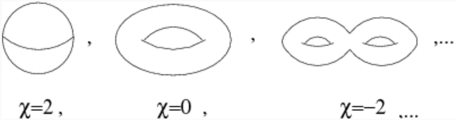
$$S_{(6)} = \int d^6x \sqrt{-g^{(6)}}\, [R - 2\Lambda + \alpha\hat{G}]$$

Lovelock theory basics

As a result Lovelock theory is the classical extension of GR in higher dimensions. It is one of the rare well defined modifications of gravity

- Term $\hat{\mathcal{G}} = R_{ABCD}R^{ABCD} - 4R_{AB}R^{AB} + R^2$ has dimensionful coupling α with dimension l^2
- Some 4 dimensional black hole "theorems" still valid.
- Relation to 4 dimensional higher order scalar-tensor Hordenski theory via Kaluza-Klein compactification
- Well defined perturbation operator and stable vacua
- Generalised junction conditions ([CC and Zegers]) and applications to braneworlds (codimension 2 braneworld cosmology ([CC, Kofinas, Papazoglou])).
- EOM deeply complex for lesser spacetime symmetry. No extension of Kerr, black string, any Weyl metric etc.
- Perturbative relation to string theory leading α' correction to certain string theory effective actions; taking into account, the finite length of

- Gauss working on surface theory noted that scalar curvature only depended on the first fundamental form. Theorema Egregium
-
- In $d = 2$, manifolds are topologically classified by their Euler number χ: $\chi[\mathcal{M}] = 2 - 2h$
- This number is related to curvature via the Gauss-Bonnet theorem
- For each even dimensional manifold, $d = 2n$ an analogue theorem relating this topological number to curvature is due to Chern.



$d = 2$	$d = 4$
$\chi[\mathcal{M}_2] = \frac{1}{4\pi} \int_{\mathcal{M}} R$	$\chi[\mathcal{M}_4] = \frac{1}{32\pi^2} \int_{\mathcal{M}} \hat{\mathcal{G}}$

- These are topological invariants in $d = 2n$.
- Dimensionally extending these scalar densities we obtain just the right densities whose variation leads to second order field equations. All higher derivatives conveniently end up as total divergent terms

Staticity theorems in GR

Einstein's theory for d=4

The spherically symmetric solutions of Einstein's field equations in the vacuum are locally static. In the presence of a cosmological constant the horizon metric is a constant curvature space.
$\longrightarrow$ a generalised version of Birkhoff's theorem with cosmological constant.

Black hole solution:

$$\mathrm{d}s^2 = -V(r)\,\mathrm{d}t^2 + \frac{\mathrm{d}r^2}{V(r)} + r^2 k^{(4)}_{\mu\nu}(x)\,\mathrm{d}x^\mu \mathrm{d}x^\nu$$

with horizon sections $k^{(4)}_{\mu\nu} = \frac{\delta_{\mu\nu}}{1+\frac{\kappa}{4}\delta_{\mu\nu}x^\mu x^\nu}$ homogeneous space
and lapse function, $V = \kappa - \frac{\Lambda}{3}r^2 - \frac{2m}{r}$.
In adS space-time we can have black holes with planar or hyperbolic topology.

Einstein's theory for $d > 4$

In $d > 4$ any Ricci flat (Einstein) space horizon is admissible [Gibbons& Hartnoll].

Horizon degeneracy in higher dimensions?

- What of this horizon degeneracy in Lovelock's theory? Are these degenerate solutions valid due to the fact that we are generalizing GR only by name?
- To answer this question we assume an arbitrary 4 dimensional internal space $(\mathcal{H}, h_{\mu\nu})$ for $d = 6$ dimensional spacetime.
- Are arbitrary Einstein spaces allowed as horizon sections of Lovelock black holes [Gleiser and Dotti], [Bogdanos et.al.]?

An integrabilility condition

- Assume an arbitrary 4 dimensional internal space $(\mathcal{H}, h_{\mu\nu})$ for $d = 6$ dimensional spacetime.
- Consider a time and space dependent warped Anzatz
- Go to light cone coordinates

$$\mathrm{d}s^2 = -2e^{2\nu(u,v)}B(u,v)^{-3/4}\,\mathrm{d}u\mathrm{d}v + B(u,v)^{1/2}\,h^{(4)}_{\mu\nu}(x)\mathrm{d}x^\mu\mathrm{d}x^\nu$$

Insert into Lovelock eqs in 6 dims (GR limit for $\alpha = 0$),

$$\mathcal{E}_{AB} = G_{AB} + \Lambda g_{AB} + \alpha H_{AB} = 0\,,$$
$$H_{AB} = \frac{g_{AB}}{2}\hat{G} - 2RR_{AB} + 4R_{AC}R^C_{\ B} + 4R_{CD}R^{C\ D}_{\ A\ B} - 2R_{ACDE}R_B^{\ CDE}\,.$$

The *uu*- and *vv*-equations

$$\mathcal{E}_{uu} = \left[1 + \alpha\left(B^{-1/2}R^{(4)} + \frac{3}{2}e^{-2\nu}B^{-5/4}B_{,u}B_{,v}\right)\right](2\nu_{,u}B_{,u} - B_{,uu}) = 0\,,$$
$$\mathcal{E}_{vv} = \left[1 + \alpha\left(B^{-1/2}R^{(4)} + \frac{3}{2}e^{-2\nu}B^{-5/4}B_{,u}B_{,v}\right)\right](2\nu_{,v}B_{,v} - B_{,vv}) = 0.$$

They allow to classify the solutions:

1. Class-I Pure GB solutions
2. Class-II Black hole GR like solutions ($\supset$[Boulware&Deser85,CC&Dufaux02,Dotti&Gleiser05]);
3. Class-III ($B(u,v)=const$, $\supset$[Dadhich&Maeda06]), Kaluza-Klein metrics

Class-II solutions : Constant curvature

For $h^{(4)}_{\mu\nu} = k^{(4)}_{\mu\nu} = \frac{\delta_{\mu\nu}}{1+\frac{\kappa}{4}\delta_{\mu\nu}x^\mu x^\nu}$, a constant curvature space (sphere, plane, hyperboloid), we have a black hole solution,

$$ds^2 = -V(r)\,dt^2 + \frac{dr^2}{V(r)} + r^2 k^{(4)}_{\mu\nu}(x)\,dx^\mu dx^\nu$$

with lapse function,

$$V(r) = \kappa + \frac{r^2}{12\alpha}\left(1 \pm \sqrt{1 + \frac{12}{5}\alpha\Lambda + 24\frac{\alpha M}{r^5}}\right)$$

with $\kappa = 0, \pm 1$ the horizon curvature and M related to mass (Boulware and Deser, Cai)

- For small α and large r we pick up the GR static black hole with cosmological constant for the "-" branch.
- Similar black hole solutions to GR with a possible additional branch singularity where the square root hits zero.
- Setting the mass parameter $M = 0$ gives us the vacua of the theory.

$$V(r) = \kappa + \frac{r^2}{12\alpha}\left(1 \pm \sqrt{1 + \frac{12}{5}\alpha\Lambda}\right)$$

- α couples to the cosmological constant and shifts its value. There are two branches, a GR branch and a pure GB branch with no GR limit.

Class II solutions: Gleiser&Dotti condition

When however the horizon metric, $h^{(4)}_{\mu\nu}$ is arbitrary,

$$ds^2 = -V(r)\,dt^2 + \frac{dr^2}{V(r)} + r^2 h^{(4)}_{\mu\nu}(x)\,dx^\mu dx^\nu$$

solving the field equation we find:

$$V(r) = \frac{\mathcal{R}^{(4)}}{12} + \frac{r^2}{12\alpha}\left(1 - \sqrt{1 + \frac{12}{5}\alpha\Lambda + \frac{\alpha^2\left(\mathcal{R}^{(4)} - 6\hat{\mathcal{G}}^{(4)}\right)}{r^4} + 24\frac{\alpha M}{r^5}}\right)$$

- where $\mathcal{R}^{(4)}$, $\mathcal{R}^{(4)} - 6\hat{\mathcal{G}}^{(4)}$ are constant horizon quantities
- and $h^{(4)}_{\mu\nu}$ is an Einstein space.
- Therefore, the horizon is an Einstein space ($\mathcal{R}^{(4)} = \frac{\kappa}{12}$) with constant and positive Weyl square ($C^{\lambda\mu\nu\rho}C_{\sigma\mu\nu\rho} = 4\Theta\delta^\lambda_\sigma$ (Gleiser-Dotti condition)).

$$V(r) = \kappa + \frac{r^2}{12\alpha}\left(1 - \sqrt{1 + \frac{12}{5}\alpha\Lambda - 24\frac{\alpha^2\Theta}{r^4} + 24\frac{\alpha M}{r^5}}\right)$$

- Example: $\mathcal{H} = S^2 \times S^2$: $ds^2 = \rho^2\left(d\theta_1^2 + \sin^2\theta_1 d\phi_1^2 + d\theta_2^2 + \sin^2\theta_2 d\phi_2^2\right)$.

$$\kappa = \frac{1}{3\rho^2},\ \Theta = \frac{4}{3\rho^4}$$

Example: S^2XS^2 horizon black hole

The lapse function is

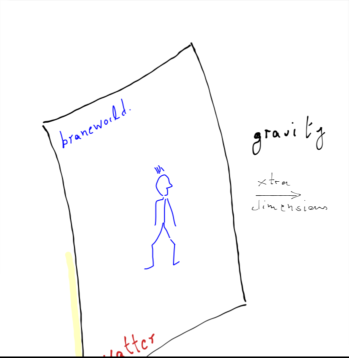
$$V(r) = \kappa + \frac{r^2}{12\alpha}\left(1 - \sqrt{1 + \frac{12}{5}\alpha\Lambda - 24\frac{\alpha^2\Theta}{r^4} + 24\frac{\alpha M}{r^5}}\right)$$

- For the horizon geometry we have $\kappa = \frac{1}{3\rho^2}$, $\Theta = \frac{4}{3\rho^4}$ where ρ is the sphere radius.
- M is an integration constant related to the mass of the solution, Θ stems from the non-zero Weyl curvature of the horizon. For GR only κ appears.
- Since $\kappa \neq 1/\rho^2$ we have a solid angle deficit. Similar to the asymptotics of a global monopole.
- For $M = 0$, and $\alpha = 0$ the solution is singular at $r = 0$. The GR solution is singular. But, for $\alpha \neq 0$ even the solution without mass can have a horizon formed due to the geometrical term Θ [Bogdanos et al.].
- The higher order term cloaks the spacetime singularity by an event

Braneworld

Central idea

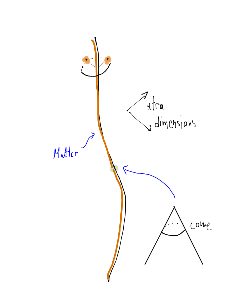
Matter lives on a distributional brane
gravity lives in a higher dimensional space-time



Codimension 2 braneworld

Codimension 2

Analogue of a 2 dimensional cosmic string living in 4 dimensions



$ds^2 = -dt^2 + dz^2 + d\rho^2 + \rho^2\beta^2 d\theta^2$ with tension related to conical deficit $8\pi G_4 T = -2\pi(1-\beta)$

- In GR only a pure tension distributional brane can be described

Matching conditions in Lovelock theory

Codimension 2

Higher order terms in Lovelock theory render possible the description of a cosmological braneworld of codimension 2

- Junction conditions read, $8\pi G_6 T_{\mu\nu} = 2\pi(1-\beta)[-k^{(4)}_{\mu\nu} + \frac{2\alpha}{3}(G_{\mu\nu} + W_{\mu\nu})]$ where
- $W^{\nu}_{\mu} = K^{\lambda}_{\mu}K^{\nu}_{\lambda} - KK^{\nu}_{\mu} + \frac{1}{2}\delta^{\nu}_{\mu}(K^2 - K_{\mu\nu}K^{\mu\nu})$ brane bending terms and $G_{\mu\nu}$ is the induced brane Einstein tensor [Bostock et al., CC and Zegers].
- Hence (in principle) we can now consider a perfect fluid induced matter tensor
- Notice that GR 4 dimensional behavior is provided by the GB term of the bulk.
- The deficit angle is now a function of time. $\beta = \beta(t)$ in accord with the space-time symmetries

In fact, for $\beta = constant$, we can write an effective action similar to effective

Codimension 2 brane cosmology

Codimension 2 cosmology

Solve at brane location matching conditions and EOM [CC,Papazoglou&Kofinas]

for distributional perfect fluid matter.

- We have geometric acceleration due to EH term of the bulk
- Effectively we have a scalar tensor theory with β scalar dof. Time varying $G_{(4)}$
- Generically energy is not conserved.

$$\dot{\rho} + 3H(\rho + P) + \frac{\dot{\beta}}{\beta(1-\beta)}\rho = 0$$

  The conical deficit angle is a free function of time $\beta = \beta(t)$ that has to be fixed by the bulk solution at infinity. In other words "dark radiation" constant of codimension 1 is now promoted to a free function.
- For codimension 1 we were helped by Birkhoff's theorem here this is no longer true.

Codimension 2 brane cosmology

Codimension 2 cosmology

Solve locally at brane location matching conditions and EOM

- We have geometric acceleration due to EH term of the bulk
- We find FRW cosmology supplemented by an extra fluid (from brane bending terms)
- Example: Setting zero brane tension and admitting constant β we have

$$H^2+\frac{\kappa}{a^2}=-\frac{1}{2\alpha}+\frac{k_6^2}{3\alpha(1-\beta)}\rho(t)+\frac{c^2}{a^{2(1-3w)}}$$

$$\frac{\ddot{a}}{a}=-\frac{1}{2\alpha}+3w\frac{c^2}{a^{2(1-3w)}}-\frac{(1+3w)k_6^2}{6\alpha(1-\beta)}\rho(t)$$

- Ordinary perfect fluid yields a two-component perfect fluid. For example $w=1/3$ yields cosmological constant, or $w=0$ yields curvature.

What effect has Lovelock theory to 4 dim gravity?

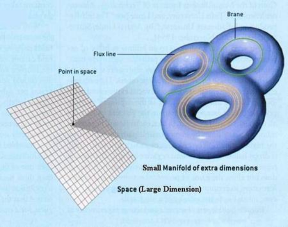
Scalar tensor theories: galileon symmetry

- Start with 5 or 6 dimensional Lovelock theory.

$$\int d^4x\, d^N X \sqrt{-g}\left[R + \alpha \hat{\mathcal{G}}\right]$$

- Consider a toroidal compactification

$$ds^2 = g^{(4)}_{ab}(x)dx^a dx^b + e^{\phi(x)/N}\delta_{AB}dX^A dX^B$$



- Question: What does Kaluza-Klein reduction of Lovelock theory give?
- Higher order scalar-tensor-(EM) theory [Madore, Mueller-Hoysen, Horndeski, Kerner, Binetruy et al.]

  For explicit calculation see [S C Davis], and for full blast see [K V Acoleyen et al.]
- Field equations are of second order! Covariant galileon terms [K V Acoleyen et al.]

Kaluza-Klein reduction

Start from Einstein → Lovelock theory in vacuum [CC, Gouteraux and Kiritsis]

$$S_{(4+n)} = \int d^{4+n}x\sqrt{-g^{(4+n)}}\left[R - 2\Lambda + \alpha\hat{\mathcal{G}}\right], \hat{\mathcal{G}} = R_{ABCD}R^{ABCD} - 4R_{AB}R^{AB} + R^2$$

Consider a simple metric Anzatz and show it is a consistent truncation (see eg. [Kanitscheider and Skenderis]),

$$\mathrm{d}s^2_{(4+n)} = \mathrm{d}\bar{s}^2_{(p+1)} + \mathrm{e}^{\phi}\mathrm{d}\tilde{K}^2_{(n)}\,.$$

Compactify on some curved n dimensional constant curvature manifold $\tilde{K}$.

$$\bar{S}_{galileon} = \int \mathrm{d}^{p+1}x\,\sqrt{-\bar{g}}\,e^{\frac{n}{2}\phi}\Big\{\bar{R} - 2\Lambda + \hat{\alpha}\bar{\mathcal{G}} + \frac{n}{4}(n-1)\partial\phi^2 - \hat{\alpha}n(n-1)\bar{G}^{\mu\nu}\partial_\mu\phi\partial_\nu\phi$$
$$-\frac{\hat{\alpha}}{4}n(n-1)(n-2)\partial\phi^2\Box\phi + \frac{\hat{\alpha}}{16}n(n-1)^2(n-2)\left(\partial\phi^2\right)^2$$
$$+e^{-\phi}\tilde{R}\left[1 + \hat{\alpha}\bar{R} + \hat{\alpha}4(n-2)(n-3)\partial\phi^2\right] + \hat{\alpha}\tilde{\mathcal{G}}e^{-2\phi}\Big\}\,,$$

- Tilded quantities are related to compactified $\tilde{K}$ geometry and are constants. They yield 4 dim potentials.
- Barred quantities are 4 dimensional. Notice that Lovelock densities interact with the scalar field.
- Coefficient n is extended to the real line.

Take Lovelock black hole with m 2-spheres $H = S^{(2)} X S^{(2)} .. X S^{(2)}$.
Compactify on $m-1$ of these keeping one $S^{(2)}$ in 4 dims.
We obtain,

$$
\begin{aligned}
\mathrm{d}\bar{s}^2_{(4)} &= -V(R)\mathrm{d}t^2 + \frac{\mathrm{d}R^2}{V(R)} + \frac{R^2}{n+1}\mathrm{d}S^2\,, \\
V(R) &= \kappa + \frac{R^2}{\tilde{\alpha}_r}\left[1 \mp \sqrt{1 - \frac{2\tilde{\alpha}_r^2\kappa^2}{(n-1)R^4} + \frac{4\tilde{\alpha}_r \mathfrak{m}}{R^{3+n}}}\right], \\
\tilde{\alpha}_r &= 2\hat{\alpha} n(n+1), \qquad \kappa = 1 \\
e^{\phi/n} &= \frac{R^2}{n+1}\,,
\end{aligned}
$$

- Taking $n \to 0$ gives us Schwarzchild.
- Taking $\tilde{\alpha}_r \to 0$ gives Einstein Dilaton solution [Chan Horne and Mann]
- Solution has solid deficit angle. Solution is similar to the external field of the gravitational monopole.
- Solution for zero mass is therefore singular and has non-trivial topology discinguishing it from GR.
- For large R and small $\tilde{\alpha}_r$ we have $V(R) \sim 1 + \frac{\tilde{\alpha}_r}{(n-1)R^2} - \frac{2\mathfrak{m}}{R^{n+1}} + \ldots$
- Higher order terms give rise to an extra horizon, a bit like in RN geometry.
- For $\mathfrak{m} = 0$ and $0 < n < 1$ we hide the singularity at $R = 0$
- Higher order term cloaks an otherwise naked singularity

Universe is accelerating → Enter the cosmological constant

Easiest way out: Assume a tiny cosmological constant $\rho_\Lambda = \frac{\Lambda_{obs}}{8\pi G} = (10^{-3} eV)^4$, ie modify Einstein's equation by,

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- $\sqrt{\Lambda}$ is as tiny as the inverse size of the Universe today, $r_0 = H_0^{-1}$
- In fact had we used planetary trajectories ,SchdS rather than Sch, we would get, $\Lambda_{local} < 10^{11} \Lambda_{obs}$
- Note that $\frac{\textbf{Solar system scales}}{\textbf{Cosmological Scales}} \sim \frac{\textbf{10 A.U.}}{H_0^{-1}} = 10^{-14}$
- Typical mass scale for neutrinos... In fact Λ for GR is a bit like m_I and m_G in Newtonnian theory...
- But things get worse...
- Theoretically, the size of the Universe would not even include the island of Naxos!

Cosmological constant problem, [S Weinberg Rev. Mod. Phys. 1989]

Cosmological constant behaves as vacuum energy which according to the strong equivalence principle gravitates,

- Vacuum energy fluctuations are at the UV cutoff of the QFT $\Lambda_{vac}/8\pi G \sim m_{Pl}^4 \ldots$
- Vacuum potential energy from spontaneous symmetry breaking $\Lambda_{EW} \sim (200 GeV)^4$
- Bare gravitational cosmological constant Λ_{bare}

$$\Lambda_{obs} \sim \Lambda_{vac} + \Lambda_{pot} + \Lambda_{bare}$$

Enormous Fine-tuning inbetween theoretical and observational value

- Why such a discrepancy between theory and observation? Weinberg no-go theorem big CC
- Why is Λ_{obs} so small and not exactly zero? small cc
- Why do we observe it now ?

The Big CC problem

- Assume an effective, conserved and covariant 4 dimensional theory
  Consider gravity action including all contributions of cosmological constant in the scalar potential term,

$$S[\pi_1, ..., \pi_N, g_{\mu\nu}] = \int d^4x \sqrt{-g} R + L(\pi_1, ..., \pi_N, g_{\mu\nu}, \partial^m) + Matter$$

  If $g_{\mu\nu} = \eta_{\mu\nu}, \pi^i = constant$. Then $\Lambda = 0$ (Weinberg no-go)
- It is impossible to find trivial solutions to Einstein's field equations (flat spacetime) without fine tuning the cosmological constant to zero.
- Any dynamical solution will fine tune the parameters of my theory in the action.

Clearly to have any chance here we must go beyond GR and break symmetries...

Self-Tuning: general idea

Weinberg's no go theorem tells us that in GR we cannot have a Poincare invariant vacuum with $\Lambda \neq 0$

Question: What if we break Poincare invariance but, at the level of the scalar field?

Keep $g_{\mu\nu} = \eta_{\mu\nu}$ locally but allow for $\phi \neq \mathit{constant}$.

Can we have a portion of flat spacetime whatever the value of the cosmological constant and without fine-tuning any of the parameters of the theory?

Toy model theory of self-adjusting scalar field.

- Solving this problem classically means that vacuum energy does not gravitate and we break SEP not EEP.
- Beyond leading order $O(\Lambda^4)$, radiative corrections $O(\Lambda^6/M_{Pl^2})$ may spoil self-tuning.

We need:

1. A cosmological background
2. A sufficiently general theory to work with

A general scalar tensor theory

- Consider ϕ and $g_{\mu\nu}$ as gravitational DoF.
- Consider $\mathcal{L} = \mathcal{L}(g_{\mu\nu}, g_{\mu\nu,i_1}, ..., g_{\mu\nu,i_1...i_p}, \phi, \phi_{,i_1}, ..., \phi_{,i_1...i_q})$ with $p, q \geq 2$ but finite
- $\mathcal{L}$ has higher than second derivatives

What is the most general scalar-tensor theory giving second order field equations?

Similar to Lovelock's theorem but for the presence of higher derivatives in $\mathcal{L}$.
Here second order field equations in principle protect vacua from ghost instabilities.

The Horndeski action [Horndeski 1974, Int. J. Theor. Phys.], [Deffayet et al.]

$$\begin{aligned}
\mathcal{L} &= \kappa_1(\phi,\rho)\delta^{ijk}_{\mu\nu\sigma}\nabla^\mu\nabla_i\phi R_{jk}{}^{\nu\sigma} - \frac{4}{3}\kappa_{1,\rho}(\phi,\rho)\delta^{ijk}_{\mu\nu\sigma}\nabla^\mu\nabla_i\phi\nabla^\nu\nabla_j\phi\nabla^\sigma\nabla_k\phi \\
&+\kappa_3(\phi,\rho)\delta^{ijk}_{\mu\nu\sigma}\nabla_i\phi\nabla^\mu\phi R_{jk}{}^{\nu\sigma} - 4\kappa_{3,\rho}(\phi,\rho)\delta^{ijk}_{\mu\nu\sigma}\nabla_i\phi\nabla^\mu\phi\nabla^\nu\nabla_j\phi\nabla^\sigma\nabla_k\phi \\
&+F(\phi,\rho)\delta^{ij}_{\mu\nu}R_{ij}{}^{\mu\nu} - 4F(\phi,\rho)_{,\rho}\delta^{ij}_{\mu\nu}\nabla_i\phi\nabla^\mu\phi\nabla^\nu\nabla_j\phi \\
&-3[2F(\phi,\rho)_{,\phi} + \rho\kappa_8(\phi,\rho)]\nabla_\mu\nabla^\mu\phi + 2\kappa_8(\phi,\rho)\delta^{ij}_{\mu\nu}\nabla_i\phi\nabla^\mu\phi\nabla^\nu\nabla_j\phi \\
&+\kappa_9(\phi,\rho), \\
\rho &= \nabla_\mu\phi\nabla^\mu\phi,
\end{aligned}$$

where $\kappa_i(\phi,\rho)$, $i = 1,3,8,9$ are 4 arbitrary functions of the scalar field ϕ and its kinetic term denoted as ρ and

$$F_{,\rho} = \kappa_{1,\phi} - \kappa_3 - 2\rho\kappa_{3,\rho}$$

$$\delta^{i_1\ldots i_h}_{j_1\ldots j_h} = h!\delta^{i_1}_{[j_1}\ldots\delta^{i_h}_{j_h]}$$

Field equations are second order in metric $g_{\mu\nu}$ and ϕ and theory is unique

Cosmological field equations

Consider cosmological background:

1. Assume, $ds^2 = -dt^2 + a^2(t)\left[\frac{dr^2}{1-kr^2} + r^2(d\theta^2 + \sin^2\theta\, d\phi^2)\right]$, $\phi = \phi(t)$
2. Modified Friedmann eq (with some matter source).

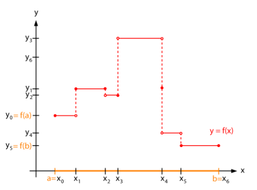
$$\mathcal{H}(a, \dot{a}, \phi, \dot{\phi}) = -\rho_m$$

   Third order polynomial in $H = \frac{\dot{a}}{a}$ with coeffs depending on the Horndeski functionals. Up to first derivatives present.
3. Scalar eq.

$$\mathcal{E}(a, \dot{a}, \ddot{a}, \phi, \dot{\phi}, \ddot{\phi}) = 0$$
$$\ddot{\phi} f(\phi, \dot{\phi}, a, \dot{a}) + g(\phi, \dot{\phi}, a, \dot{a}, \ddot{a}) = 0$$

   Linear in $\ddot{\phi}$ and $\ddot{a}$.
   Also have 2nd Friedmann equation or usual energy conservation.

Main Assumptions

- Vacuum energy does not gravitate.
- Assume that $\rho_m = \rho_\Lambda$, a piecewise discontinuous step function of time t. Discontinuous points, $t = t_\star$, are phase transitions which are point like and arbitrary in time.
- 

  $x = time$, and $y = \rho_\Lambda$.
- Assume that spacetime is flat or a flat portion for all t
- $H^2 + \frac{\kappa}{a^2} = 0$, with $\kappa = 0$, or $\kappa = -1$ Milne spacetime $(a(t) = t)$
- ϕ not constant but in principle a function of time t!

Milne spacetime

Milne space-time is the cosmological version of flat space-time,

$$ds^2 = -dt^2 + \frac{t^2}{l^2}\left(\frac{d\chi^2}{1+\chi^2} + \chi^2 d\Omega^2\right)$$

Riemann curvature is everywhere zero and the slicing of space is hyperbolic ie., of negative scalar curvature, $1/l^2$

Hubble is $H = 1/t$

The self tuning filter

Mathematical regularity imposed by a distributional source

1. We are going to set $H^2 + \frac{\kappa}{a^2}=0$, with $\rho(\Lambda)$ piecewise discontinuous. Then
2. $$\mathcal{H}(a, \phi, \dot{\phi}) = -\rho_\Lambda$$
   $a(t)$, $\dot{a}$ and $\phi(t)$ are continuous whereas $\dot{\phi}$ is discontinuous at $t = t_\star$.
   $\mathcal{H}$ has to depend on $\dot{\phi}$
3. Scalar eq. on shell is
   $$\mathcal{E}(a, \phi, \dot{\phi}, \ddot{\phi}) = \ddot{\phi} f(\phi, \dot{\phi}, a) + g(\phi, \dot{\phi}, a) = 0$$
   $\ddot{\phi}$ has a $\delta(t - t_\star)$ singularity at $t = t_\star$
4. Hence on shell, $\mathcal{E}$ has no dependance on $\ddot{\phi}$. $\dot{\phi}$ fixed by Friedmann eq.
5. In the presence of matter cosmology must be non trivial. Hence $\mathcal{E}$ must depend on $\ddot{a}$

Fab 4

Putting it all together

from Horndeski s general action,

$$\begin{aligned}
\mathcal{L} \quad = \quad & \kappa_1(\phi,\rho)\delta^{ijk}_{\mu\nu\sigma}\nabla^\mu\nabla_i\phi R_{jk}{}^{\nu\sigma} - \frac{4}{3}\kappa_{1,\rho}(\phi,\rho)\delta^{ijk}_{\mu\nu\sigma}\nabla^\mu\nabla_i\phi\nabla^\nu\nabla_j\phi\nabla^\sigma\nabla_k\phi \\
& +\kappa_3(\phi,\rho)\delta^{ijk}_{\mu\nu\sigma}\nabla_i\phi\nabla^\mu\phi R_{jk}{}^{\nu\sigma} - 4\kappa_{3,\rho}(\phi,\rho)\delta^{ijk}_{\mu\nu\sigma}\nabla_i\phi\nabla^\mu\phi\nabla^\nu\nabla_j\phi\nabla^\sigma\nabla_k\phi \\
& +F(\phi,\rho)\delta^{ij}_{\mu\nu}R_{ij}{}^{\mu\nu} - 4F(\phi,\rho)_{,\rho}\delta^{ij}_{\mu\nu}\nabla_i\phi\nabla^\mu\phi\nabla^\nu\nabla_j\phi \\
& -3[2F(\phi,\rho)_{,\phi} + \rho\kappa_8(\phi\ \rho)]\nabla_\mu\nabla^\mu\phi + 2\kappa_8\delta^{ij}_{\mu\nu}\nabla_i\phi\nabla^\mu\phi\nabla^\nu\nabla_j\phi \\
& +\kappa_9(\phi,\rho)
\end{aligned}$$

Self-tuning filter

$$\begin{aligned}
\mathcal{L}_{john} &= \sqrt{-g}\,V_{john}(\phi)G^{\mu\nu}\nabla_\mu\phi\nabla_\nu\phi \\
\mathcal{L}_{paul} &= \sqrt{-g}\,V_{paul}(\phi)(*R*)^{\mu\nu\alpha\beta}\nabla_\mu\phi\nabla_\alpha\phi\nabla_\nu\nabla_\beta\phi \\
\mathcal{L}_{george} &= \sqrt{-g}\,V_{george}(\phi)R \\
\mathcal{L}_{ringo} &= \sqrt{-g}\,V_{ringo}(\phi)\hat{G}
\end{aligned}$$

The double dual tensor

- In 4 dimensions we can define a dual of the curvature tensor by dualising each pair of indices much like the Faraday tensor in EM

$$*F^{ab} = \frac{1}{2}\varepsilon^{abcd}\ F_{cd}$$

- Double Dual $(*R*)$

$$(*R*)_{\mu\nu\sigma\lambda} = -\frac{1}{4}\varepsilon_{\mu\nu}{}^{ij}\ R_{ijkl}\ \varepsilon_{\sigma\lambda}{}^{kl} = \tfrac{1}{4}\delta^{ijkl}_{\mu\nu\sigma\lambda}\ R_{ijkl}$$

As appearing in the Horndeski action

1. Same index properties as the Riemann tensor
2. But also divergence free:

$$\nabla_i(*R*)_{jkl}{}^{i} = 0$$

3. Simple trace is Einstein

$$(*R*)^{ik}{}_{jk} = -G^i_j,$$

Paul

- Last term is not recognisable,
- $$\begin{aligned}\mathcal{L}_{paul} &= \sqrt{-g}V_{Paul}(\phi)\left[R^{\mu\nu\alpha\beta}\nabla_\mu\phi\nabla_\alpha\phi\nabla_\nu\nabla_\beta\phi + \right.\\ &\quad +G^{\mu\nu}(\nabla_\mu\phi\nabla_\alpha\phi - g_{\mu\alpha}(\nabla\phi)^2)\nabla^\alpha\nabla_\nu\phi \\ &\quad +R^{\mu\nu}(\nabla_\mu\nabla_\alpha\phi - g_{\mu\alpha}\Box\phi)\nabla^\alpha\phi\nabla_\nu\phi]\end{aligned}$$
- ???However,
$$(*R*)^{\mu\nu\alpha\beta} = R^{\mu\nu\alpha\beta} + 2R^{\nu[\alpha}g^{\beta]\mu} - 2R^{\mu[\alpha}g^{\beta]\nu} + Rg^{\mu[\alpha}g^{\beta]\nu} \ ,$$
- Therefore
$$\mathcal{L}_{paul} = \sqrt{-g}V_{paul}(\phi)(*R*)^{\mu\nu\alpha\beta}\nabla_\mu\phi\nabla_\alpha\phi\nabla_\nu\nabla_\beta\phi$$
- Also a higher KK Lovelock density [K V Akoleyen]

Cosmology equations and self tuning

- Friedmann equation reads $\mathcal{H} = -\rho_\Lambda$
-

$$
\begin{aligned}
\mathcal{H}_{john} &= 3V_{john}(\phi)\dot{\phi}^2\left(H^2+\frac{\kappa}{a^2}\right)+6V_{john}(\phi)\dot{\phi}^2H^2 \\
\mathcal{H}_{paul} &= -9V_{paul}(\phi)\dot{\phi}^3H\left(H^2+\frac{\kappa}{a^2}\right)-6V_{paul}(\phi)\dot{\phi}^3H^3 \\
\mathcal{H}_{george} &= -6V_{george}(\phi)\left[\left(H^2+\frac{\kappa}{a^2}\right)+H\dot{\phi}\frac{V'_{george}}{V_{george}}\right] \\
\mathcal{H}_{ringo} &= -24V'_{ringo}(\phi)\dot{\phi}H\left(H^2+\frac{\kappa}{a^2}\right)
\end{aligned}
$$

- First find self tuning vacuum setting $H^2+\frac{\kappa}{a^2}=0$
- Algebraic equation with respect to $\dot{\phi}$. Hence ϕ is a function of time t with discontinuous first derivatives at $t=t_*$
- Ringo cannot self-tune without a little help from his friends.

A simple self-tuning solution

Consider a slowly varying scalar field for late time behavior:

- Milne metric $ds^2 = -dt^2 + t^2 \left(\frac{d\chi^2}{1+\chi^2} + \chi^2 d\Omega^2 \right)$

$$V_{john} = C_j \,,\, V_{paul} = C_p \,,\, V_{george} = C_g + C_g^1\,\phi \,,\, V_{ringo} = C_r + C_r^1\,\phi - \frac{1}{4} C_j\,\phi^2$$

- Friedmann equation reads,

$$c_j(\dot{\phi}H)^2 - c_p(\dot{\phi}H)^3 - c_g^1(\dot{\phi}H) + \rho_\Lambda = 0$$

  with $\rho_\Lambda = \Lambda$, vacuum cosmological constant. Note that $\dot{\phi}H$ appear as homogeneous powers of time.
- Hence since $H = 1/t$ for Milne, taking $\phi = \phi_0 + \phi_1 t^2$ gives $c_j(\phi_1)^2 - c_p(\phi_1)^3 - c_g^1(\phi_1) + \rho_\Lambda = 0$ a constant constraint.
- Integration constant ϕ_1 is fixed by the cosmological constant for arbitrary values of the theory potentials. Self tuning.
- Rindler metric $ds^2 = dr^2 - r^2 \left(-dT^2 + \cosh^2(t) d\Omega^2 \right)$ with $\phi = \phi_0 + \phi_1 r^2$

Fab 4 black hole

Fab 4 theory reads.

$$\mathcal{L}_{Fab4} = \sqrt{-g}(V_{john}(\phi)G^{\mu\nu}\nabla_\mu\phi\nabla_\nu\phi + (*R*)^{\mu\nu\alpha\beta}\nabla_\mu\phi\nabla_\alpha\phi\nabla_\nu\nabla_\beta\phi + V_{george}(\phi)R + V_{ringo}($$

All terms are scalar-tensor interaction terms. There are no pure kinetic terms for the scalar field. This enables the scalar field equation to be redundant for flat space-time. We would like to be able to find black hole solutions that self-tune the cosmological constant. In other words solutions to the above with a cosmological constant where space-time will be asymptotically flat! A bit of work shows that such a black hole must be radiating.

Conclusions

- Modifying Einstein s theory of GR is an interesting but difficult task.
- Have presented two well-defined theories of gravity modification: Lovelock theory and Horndeski theory
- Have presented a novel black hole solution of Horndeski theory constructed from Lovelock theory
- We have filtered out the theory with self-tuning properties starting from Horndeski's general theory
- Theory has enchanting geometrical properties which we need to understand
- Still have 4 free functions which parametrise the theory. These need to be fixed by cosmology, stability and local constraints.

Many questions unanswered:

1. What is the Fab 4 cosmology? In other words for which of the potentials do we get usual Hot Big Bang cosmology?
2. What does perturbation theory around a self-tuning vacuum give?

Cosmology equations and self tuning

- Scalar equation, $E_\phi = E_{john} + E_{paul} + E_{george} + E_{ringo} = 0$
-

$$E_{john} = 6\frac{d}{dt}\left[a^3 V_{john}(\phi)\dot{\phi}\Delta_2\right] - 3a^3 V'_{john}(\phi)\dot{\phi}^2\Delta_2$$

$$E_{paul} = -9\frac{d}{dt}\left[a^3 V_{paul}(\phi)\dot{\phi}^2 H\Delta_2\right] + 3a^3 V'_{paul}(\phi)\dot{\phi}^3 H\Delta_2$$

$$E_{george} = -6\frac{d}{dt}\left[a^3 V'_{george}(\phi)\Delta_1\right] + 6a^3 V''_{george}(\phi)\dot{\phi}\Delta_1 + 6a^3 V'_{george}(\phi)\Delta_1^2$$

$$E_{ringo} = -24V'_{ringo}(\phi)\frac{d}{dt}\left[a^3\left(\frac{\kappa}{a^2}\Delta_1 + \frac{1}{3}\Delta_3\right)\right]$$

- where

$$\Delta_n = H^n - \left(\frac{\sqrt{-\kappa}}{a}\right)^n$$

- which vanishes on shell as it should
- For non trivial cosmology need