Cosmology of multiscale spacetimes

based on
arXiv:1307.6382

Gianluca Calcagni

Instituto de Estructura de la Materia (IEM) – CSIC

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01/27– Dimensional flow in quantum gravity

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• Noncommutative geometry [Connes 2006; Benedetti 2008; Alesci & Arzano 2012]; • CDT [Ambjørn, Jurkiewicz & Loll 2005; Benedetti & Henson 2009]; • Spin foams [Modesto (et al.) 2008–10]; • AS [Lauscher & Reuter 2005; Reuter & Saueressig 2011; G.C. et al. 2013]; • HL gravity [Hořava 2008,2009].

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- Various works in 2009-2013 (also with M. Arzano, A. Eichhorn, J. Magueijo, G. Nardelli, D. Oriti, D. Rogríguez, F. Saueressig, M. Scalisi).

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4. Differs from ST theories (v is not a Lorentzian scalar field) and unimodular gravity (metric structure is fully dynamical).

04/27– Measures

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- Action measure $\mathrm{d}\varrho(x) = \mathrm{d}^D x\, v(x) = \mathrm{d}^D q(x)$. General factorizable measures:

$$v(\ell_n, x) = \prod_{\mu=0}^{D-1} v_\mu(\ell_n, x^\mu)\,, \qquad v_\mu(\ell_n, x^\mu) \geq 0\,.$$

05/27– Example 1: Fractional measure

Represents random fractals.

$$\mathrm{d}\varrho_\alpha(x) = \mathrm{d}^D x\, v_\alpha(x) = \mathrm{d}^D x \prod_\mu \frac{|x^\mu|^{\alpha-1}}{\Gamma(\alpha)}$$

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"Geometric" coordinates:

$$q^\mu := \varrho_\alpha(x^\mu) = \frac{\mathrm{sgn}(x^\mu)|x^\mu|^\alpha}{\Gamma(\alpha+1)} \qquad \Rightarrow \qquad \mathrm{d}\varrho_\alpha = \mathrm{d}^D q$$

06/27– Example 1: Hausdorff dimension

Scaling property:

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Same result obtained via self-similarity theorem or via operational definition as the scaling of the volume of a D-ball of radius R: $\mathcal{V}^{(D)}(R) = \int_{D\text{-ball}} d\varrho_\alpha(x) \propto R^{D\alpha}$.

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Scale-dependent Hausdorff dimension. Simplest (but not toy) model, two terms (binomial measure):

$$I_D = I_D^{\alpha_1} + \ell_*^{D(\alpha_1-\alpha_2)} I_D^{\alpha_2} , \qquad [I_D] = -D\alpha_1 , \qquad \frac{1}{2} \leq \alpha_1 < \alpha_2 \leq 1 .$$

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$$\mathcal{V}^{(D)}(R) = \ell_*^{D\alpha_1} \left[\Omega_{D,\alpha_1} \left(\frac{R}{\ell_*}\right)^{D\alpha_1} + \Omega_{D,\alpha_2} \left(\frac{R}{\ell_*}\right)^{D\alpha_2}\right] .$$

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$$\mathcal{V}^{(D)}(R)=\ell_*^{D\alpha_1}\left[\Omega_{D,\alpha_1}\left(\frac{R}{\ell_*}\right)^{D\alpha_1}+\Omega_{D,\alpha_2}\left(\frac{R}{\ell_*}\right)^{D\alpha_2}\right].$$

$R\ll\ell_*:\qquad \mathcal{V}^{(D)}\sim R^{D\alpha_1}$

$R\gg\ell_*:\qquad \mathcal{V}^{(D)}\sim\tilde{R}^{D\alpha_2},\qquad \tilde{R}=R\ell_*^{-1+\alpha_1/\alpha_2}$

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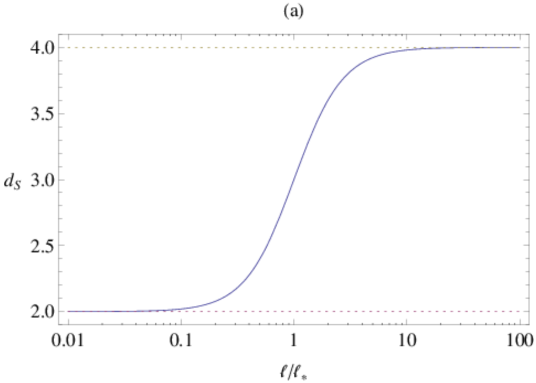

08/27– Example 3: Log-oscillating measure

$$\varrho_\alpha(x) \to \varrho_{\alpha,\omega} = c_+|x|^{\alpha+\mathrm{i}\omega} + c_-|x|^{\alpha-\mathrm{i}\omega}, \qquad \omega \geq 0\,.$$

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Summing over α, ω and imposing S to be real,

$$S = \int d\varrho(x)\,\mathcal{L}\,, \qquad d\varrho(x) = \prod_\mu \left[\sum_n g_n \sum_\omega d\varrho_{\alpha_n,\omega}(x^\mu)\right]$$

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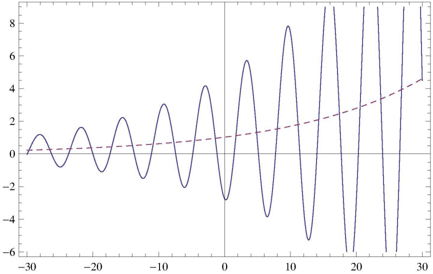
where

$$\boxed{\varrho_{\alpha,\omega}(x) = \frac{x^\alpha}{\Gamma(\alpha+1)}\left[1 + A_{\alpha,\omega}\cos\left(\omega \ln \frac{|x|}{\ell_\infty}\right) + B_{\alpha,\omega}\sin\left(\omega \ln \frac{|x|}{\ell_\infty}\right)\right]}$$

$A_{\alpha,\omega}$ and $B_{\alpha,\omega} \in \mathbb{R}$. Form of measure also dictated by fractal geometry arguments.

08/27– Example 3: Log-oscillating measure

Represents deterministic (multi)fractals.



09/27– Example 3: Discrete scale invariance

Oscillatory part of ϱ log-periodic under the transformation

$$\ln\frac{|x|}{\ell_\infty}\to\ln\frac{|x|}{\ell_\infty}+\frac{2\pi n}{\omega}\,,\qquad n=0,1,2,\ldots$$

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DSIs appear in chaotic systems [Sornette 1998].

10/27– Example: scalar field

$$S=\int \mathrm{d}^D x\, v(x)\,\left[-\frac{1}{2}\phi\,\mathcal{K}\,\phi-V(\phi)\right]\,.$$

The symmetries of $\mathcal{L}$ determine the Laplace–Beltrami operator $\mathcal{K}$

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- Weighted Laplacian: $\mathcal{K} = \mathcal{D}^2 := \frac{1}{\sqrt{v}}\,\partial_\mu\partial^\mu[\sqrt{v}\,\cdot\,] = \mathcal{K}^\dagger$. Dual to previous model (share the *same* diffusion equation). Not a fractal ($d_\mathrm{W} \neq 2d_\mathrm{H}/d_\mathrm{S}$).

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- q-Laplacian: $\mathcal{K}=\Box_q:=\eta^{\mu\nu}\frac{\partial}{\partial q^\mu(x^\mu)}\frac{\partial}{\partial q^\nu(x^\nu)}=\eta^{\mu\nu}\frac{1}{v_\mu}\partial_\mu\left[\frac{1}{v_\nu}\partial_\nu\,\cdot\,\right]=\mathcal{K}^\dagger$. Not trivial (physical momenta conjugate to x, not q!). It is a fractal.

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- Multifractional Laplacian: $\mathcal{K}=\mathcal{K}_*:=\eta^{\mu\nu}\left[\frac{1}{2}(\partial^\gamma_\mu\partial^\gamma_\nu+{}_\infty\bar{\partial}^\gamma_\mu{}_\infty\bar{\partial}^\gamma_\nu)\,\cdot\,\right]=\mathcal{K}_*^\dagger$.

12/27– Scales hierarchy

- *Boundary-effect regime ($\ell \sim \ell_\infty$).* $|x|/\ell_\infty \sim 1$, $\varrho(x) \sim \ln|x|$, natural relation with κ-Minkowski noncommutative spacetimes ($\ell_\infty = \ell_{\rm Pl}$).
- *Oscillatory transient regime ($\ell_\omega = \lambda_\omega \ell_\infty < \ell \ll \ell_*$).* Notion of dim. and vol. ambiguous unless averaged. DSI.
- *Multifractional regime ($\ell_\omega \ll \ell \lesssim \ell_*$).* Mesoscopic scales, average of the measure:
  $\varrho_\alpha(x) := \langle \varrho_{\alpha,\omega}(x) \rangle \propto |x|^\alpha$, $\qquad d\varrho(x) \sim \sum_\alpha g_\alpha d\varrho_\alpha(x)$.
  Dimensional flow. **Continuous symmetries (affinities) emerge**.
- *Classical regime ($\ell \gg \ell_*$).* Ordinary Poincaré-invariant field theory on Minkowski spacetime recovered, $\varrho(x) \sim \varrho_1(x) = x$. Dim. of spacetime is $d_{\rm H} = d_{\rm S} = 4 - \epsilon$.

13/27– Status

	$\Box, \Box^\dagger$	$\mathcal{D}^2$	$\Box_q$	$\mathcal{K}_*$
Momentum transform	✗?	✓	✓	?
Relativistic mechanics	✓	✓	✓	?
Perturbative field theory	✓?	✓	✓?	✓?
Symmetries and dynamics of scalar (Q)FT	?	✓	✓	?
Scalar QFT propagator	?	✓	✓?	✓?
Electrodynamics	?	✓	✓	?
Perturbative renormalizability	?	✗	✗	✓?
Avoids Collins *et al.*	?	✓	✓	?
Phenomenology (obs. constraints)	?	✓	?	?
Gravity	✓	?	?	?
Cosmology	✓?	✓	✓	?

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- Geometry of multiscale manifolds not Riemannian: possesses a nontrivial structure given a priori. Theory with weighted derivatives: Weyl integrable spacetimes.
- Only theory with q derivatives: What expected in a "covariant" description of a fractal: nontrivial geometric and differential structure at all points. Vielbeins move the measure "hole" around and maintain the anomalous scaling properties.
- Should be able to describe a sensible cosmology in an economic way: bounce and alternatives to inflation, big-bang and Λ problems reinterpreted.

15/27– Action: Will o' the WIST

$$ {}^{\beta}\Gamma^{\rho}_{\mu\nu}[g] := \frac{1}{2} g^{\rho\sigma} \left({}_{\beta}\mathcal{D}_{\mu} g_{\nu\sigma} + {}_{\beta}\mathcal{D}_{\nu} g_{\mu\sigma} - {}_{\beta}\mathcal{D}_{\sigma} g_{\mu\nu} \right) , \qquad {}_{\beta}\mathcal{D} = v^{-\beta} \partial [v^{\beta} \cdot] $$

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$$ \begin{aligned} S &:= \frac{1}{2\kappa^2}\int d^D x\, v\sqrt{-g}\,[\mathcal{R} - \omega\mathcal{D}_\mu v\mathcal{D}_\nu v - U(v)] + S_\mathrm{m} \\ &= \frac{1}{2\kappa^2}\int d^D x\, e^{\frac{1}{\beta}\Phi}\sqrt{-g}\left(\mathcal{R} - \frac{9\omega}{4\beta^2}e^{\frac{2}{\beta}\Phi}\partial_\mu\Phi\partial^\mu\Phi - U\right) + S_\mathrm{m}\,. \end{aligned} $$

16/27– $D=4$ Einstein and Friedmann equations

Einstein frame: $\bar{g}_{\mu\nu}=\mathrm{e}^{\Phi}g_{\mu\nu}$. $\Omega=(9\omega/4)\mathrm{e}^{2\Phi}-3/2$.

$$\kappa^2\bar{T}_{\mu\nu}=\bar{R}_{\mu\nu}-\frac{1}{2}\bar{g}_{\mu\nu}(\bar{R}-\mathrm{e}^{-\Phi}U)-\Omega\left(\partial_\mu\Phi\partial_\nu\Phi+\frac{1}{2}\bar{g}_{\mu\nu}\partial_\sigma\Phi\bar{\partial}^\sigma\Phi\right)$$

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Cosmology:

$$H^2 = \frac{\kappa^2}{3}\bar{\rho} + \frac{\Omega}{2}\frac{\dot{v}^2}{v^2} + \frac{U(v)}{6v} - \frac{\mathsf{K}}{a^2}\,,$$

$$2\dot{H} - \frac{2\mathsf{K}}{a^2} + \kappa^2(\bar{\rho} + \bar{P}) = -\Omega \frac{\dot{v}^2}{v^2}\,.$$

17/27– Flat vacuum solution ($\rho = P = \mathsf{K} = 0 = \omega$)

Binomial measure

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Scale factor (always superacceleration)

$$a(t) = \left(1 + \sqrt{\frac{t_*}{t}}\right)^{\frac{3}{8}} \exp\left\{\frac{9}{8}\left[H_0 \frac{t}{t_*} + \sqrt{\frac{t}{t_*}} - \frac{t}{t_*} \ln\left(1 + \sqrt{\frac{t_*}{t}}\right)\right]\right\}.$$

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Minimum of the 'potential' U at late times
$U_{\rm min} = U(v = 1) = 6H_0^2$.

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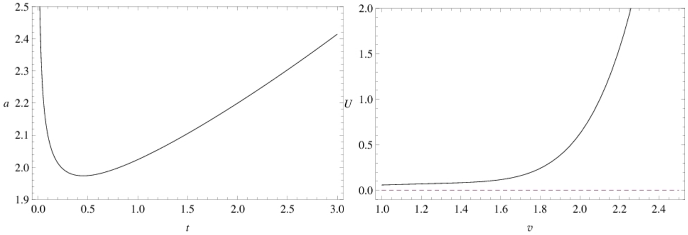

18/27– Frames and fractals

- Inertial frames clearly interpreted: Multiscale frames map a curvilinear coordinate system to the Cartesian one *with the same measure structure*.

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$$\mathrm{d}s^2 := g_{\mu\nu}\,\mathrm{d}q^\mu \otimes \mathrm{d}q^\nu , \qquad g_{\mu\nu} := \eta_{IJ} e_\mu^{\ I} e_\nu^{\ J} \not\propto \bar{g}_{\mu\nu} .$$

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- Gravity and cosmology easy to work out via $x \to q(x)$ mapping.

19/27– Action and Einstein equations

$$
{}^q\Gamma^\rho_{\mu\nu} := \tfrac{1}{2} g^{\rho\sigma}\left(\frac{1}{v_\mu}\partial_\mu g_{\nu\sigma} + \frac{1}{v_\nu}\partial_\nu g_{\mu\sigma} - \frac{1}{v_\sigma}\partial_\sigma g_{\mu\nu}\right) ,
$$

$$
{}^q R^\rho{}_{\mu\sigma\nu} := \frac{1}{v_\sigma}\partial_\sigma {}^q\Gamma^\rho_{\mu\nu} - \frac{1}{v_\nu}\partial_\nu {}^q\Gamma^\rho_{\mu\sigma} + {}^q\Gamma^\tau_{\mu\nu}\, {}^q\Gamma^\rho_{\sigma\tau} - {}^q\Gamma^\tau_{\mu\sigma}\, {}^q\Gamma^\rho_{\nu\tau} ,
$$

Action:

$$
S = \frac{1}{2\kappa^2}\int \mathrm{d}^D x\, v\, \sqrt{-g}\, ({}^q R - 2\Lambda) + S_\mathrm{m} ,
$$

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$$ S = \frac{1}{2\kappa^2}\int \mathrm{d}^D x\, v\, \sqrt{-g}\, ({}^q R - 2\Lambda) + S_\mathrm{m} , $$

Einstein equations:

$$ {}^q R_{\mu\nu} - \frac{1}{2} g_{\mu\nu}({}^q R - 2\Lambda) = \kappa^2\, {}^q T_{\mu\nu} . $$

Cosmology

$$\frac{H^2}{v^2} = \frac{\kappa^2}{3}\rho + \frac{\Lambda}{3} - \frac{\mathsf{K}}{a^2}\,,$$

$$\dot{\rho} + 3H(\rho + P) = 0\,.$$

Ordinary slow-roll approximation unnecessary.

20/27– Cosmology

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Ordinary slow-roll approximation unnecessary.
Geometric time coordinate

$$q(t) = \int^t \mathrm{d}t'\, v(t')\,.$$

21/27– “Power-law” solutions

$$\rho = \rho_0\, a^{-\frac{2}{p}}\,, \qquad a(t) = \left[\frac{q(t)}{t_*}\right]^p\,, \qquad p := \frac{2}{(D-1)(1+w)}\,,$$

Hubble parameter

$$H = p\frac{\dot{q}(t)}{q(t)} = p\frac{v(t)}{q(t)}\,.$$

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Hubble parameter

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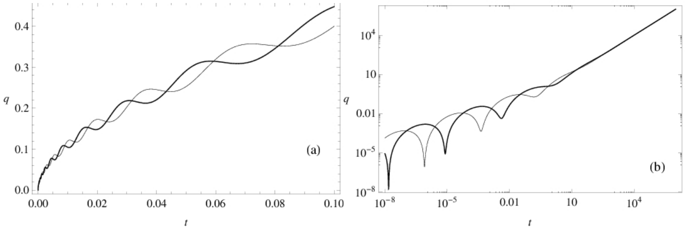
From now on choose log-oscillating measure

$$\begin{aligned} q(t) &= t + t_*\left(\frac{t}{t_*}\right)^\alpha F_\omega(\ln t)\,, \\ F_\omega(\ln t) &= 1 + A\cos\left[\omega\ln\left(\frac{t}{t_{\rm Pl}}\right)\right] + B\sin\left[\omega\ln\left(\frac{t}{t_{\rm Pl}}\right)\right]. \end{aligned}$$

22/27– Geometric coordinate



$H = 0$ at peaks and troughs, log-oscillations end after some time.

23/27– e-folds and cycles

Fully analytic properties.

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Slope of an expanding/contracting phase:

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Average slope of the trend of minima (or maxima):

$$\delta_{\uparrow\downarrow}=\frac{\ln(a_{m+1}/a_m)}{\ln(t_{m+1}/t_m)}=\mathcal{N}_{\uparrow\downarrow}\frac{\omega}{2\pi}\approx p\alpha\,.$$

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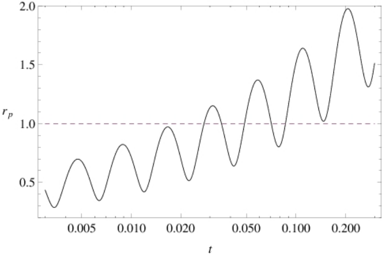
Net number of e-foldings per cycle:

$$\mathcal{N}_{\uparrow\downarrow} := \ln\frac{a_{m+1}}{a_m} = (\delta_\uparrow + \delta_\downarrow)\frac{\pi}{\omega} \approx \frac{2\pi\alpha p}{\omega}\,.$$

24/27– Alternative to inflation? $(0 < p < 1)$

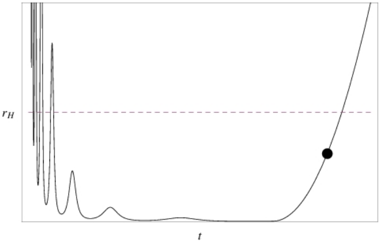
Horizon problem solved without invoking acceleration-inducing matter: particle horizon shrinks because of geometry!



Flatness problem not solved, unfortunately.

25/27– Cyclic mild inflation ($p \gtrsim 1$)

Horizon and flatness problem solved if mildly inflating matter is added.



26/27– Inflationary spectrum, big bang

Scalar power spectrum:

$$P_{\rm s} \sim \mathcal{A}\left(\frac{k}{k_*}\right)^{n_{\rm eff}-1}[F_\omega(\ln k)]^{1-n_{\rm s}}\,, \qquad n_{\rm eff}-1=\alpha(n_{\rm s}-1)\,.$$

Scale invariance without slow-roll approximation and log-oscillating pattern!!

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Scale invariance without slow-roll approximation and log-oscillating pattern!!

Big bang removed by a homogeneous contribution to the measure (integration constant):

$$q(t)\to t_{\rm bb}+q(t)\,.$$

27/27– Discussion

- Assuming the integro-differential structure of spacetime behaves as a multifractal leads to novel scenarios in particle physics and cosmology.
- Analytic cosmological solutions to be studies (stability, cosmic evolution with matter and radiation, etc.).
- Intriguing features purely generated by geometry (big bounces, acceleration without inflaton, alternative to inflation, mild inflation, cyclic cosmology).
- Power spectra at hand!

27/27– Discussion

$\varepsilon\upsilon\chi\alpha\rho\iota\sigma\tau\acute{\omega}$ (thank you)!