Lattice Spinor Gravity

Quantum gravity

- Quantum field theory
- Functional integral formulation

Symmetries are crucial

- Diffeomorphism symmetry

  (invariance under general coordinate transformations)

- Gravity with fermions : local Lorentz symmetry

Degrees of freedom less important :

metric, vierbein , spinors , random triangles , conformal fields…

Graviton , metric : collective degrees of freedom in theory with diffeomorphism symmetry

Regularized quantum gravity

① For finite number of lattice points : functional integral should be well defined

② Lattice action invariant under local Lorentz-transformations

③ Continuum limit exists where gravitational interactions remain present

④ Diffeomorphism invariance of continuum limit , and geometrical lattice origin for this

Spinor gravity

is formulated in terms of fermions

Unified Theory of fermions and bosons

Fermions fundamental

Bosons collective degrees of freedom

- Alternative to supersymmetry
- Graviton, photon, gluons, W-,Z-bosons , Higgs scalar : all are collective degrees of freedom (composite)
- Composite bosons look fundamental at large distances, e.g. hydrogen atom, helium nucleus, pions
- Characteristic scale for compositeness : Planck mass

Massless collective fields or bound states – familiar if dictated by symmetries

for chiral QCD :

Pions are massless bound states of massless quarks !

for strongly interacting electrons :

antiferromagnetic spin waves

Gauge bosons, scalars …

from vielbein components
in higher dimensions
(Kaluza, Klein)

→

concentrate first on gravity

Geometrical degrees of freedom

- $\Psi(x)$: spinor field (Grassmann variable)
- vielbein : fermion bilinear

$$\tilde{E}_\mu^m = i\bar{\psi}\gamma^m\partial_\mu\psi$$

$$E_\mu^m(x) = \langle \tilde{E}_\mu^m(x) \rangle$$

Possible Action

$$S_E \sim \int d^d x \det\left(\tilde{E}^m_\mu(x)\right)$$

$$\tilde{E} = \frac{1}{d!}\epsilon^{\mu_1\ldots\mu_d}\epsilon_{m_1\ldots m_d}\tilde{E}^{m_1}_{\mu_1}\ldots\tilde{E}^{m_d}_{\mu_d} = \det(\tilde{E}^m_\mu)$$

contains 2d powers of spinors

d derivatives contracted with ε - tensor

$$\tilde{E}^m_\mu = i\bar{\psi}\gamma^m\partial_\mu\psi$$

Symmetries

- General coordinate transformations (diffeomorphisms)
- Spinor $\psi(x)$: transforms as scalar
- Vielbein $\tilde{E}_\mu^m = i\bar{\psi}\gamma^m\partial_\mu\psi$: transforms as vector
- Action S : invariant

K.Akama,Y.Chikashige,T.Matsuki,H.Terazawa (1978)
K.Akama (1978)
D.Amati, G.Veneziano (1981)
G.Denardo,E.Spallucci (1987)

Lorentz- transformations

Global Lorentz transformations:

- spinor ψ
- vielbein transforms as vector
- action invariant

Local Lorentz transformations:

- vielbein does **not** transform as vector
- inhomogeneous piece, missing covariant derivative

$$\tilde{E}_\mu^m = i\bar{\psi}\gamma^m\partial_\mu\psi$$

Two alternatives :

1) Gravity with **global** and not local Lorentz symmetry ?
Compatible with observation !

2) Action with
local Lorentz symmetry ?
Can be constructed !

Spinor gravity with local Lorentz symmetry

Spinor degrees of freedom

- Grassmann variables ψ_γ^a
- Spinor index $\gamma = 1 \dots 8$
- Two flavors $a = 1, 2$
- Variables at every space-time point

$$x^\mu = (x^0, x^1, x^2, x^3)$$

- Complex Grassmann variables

$$\varphi_\alpha^a(x) = \psi_\alpha^a(x) + i\psi_{\alpha+4}^a(x)$$

Action with local Lorentz symmetry

$$S = \alpha \int d^4x A^{(8)} D + c.c.$$

A : product of
all eight spinors ,
maximal number ,
totally antisymmetric

$$\begin{aligned} A^{(8)} &= \frac{1}{8!}\epsilon_{\epsilon_1\epsilon_2\dots\epsilon_8}\varphi_{\epsilon_1}\dots\varphi_{\epsilon_8} \\ &= \frac{1}{(24)^2}\epsilon_{\alpha_1\alpha_2\alpha_3\alpha_4}\varphi^1_{\alpha_1}\dots\varphi^1_{\alpha_4}\epsilon_{\beta_1\beta_2\beta_3\beta_4}\varphi^2_{\beta_1}\dots\varphi^2_{\beta_4} \end{aligned}$$

$$= \varphi^1_1\varphi^1_2\varphi^1_3\varphi^1_4\varphi^2_1\varphi^2_2\varphi^2_3\varphi^2_4$$

D : antisymmetric product
of four derivatives ,
L is totally symmetric
Lorentz invariant tensor

$$D = \epsilon^{\mu_1\mu_2\mu_3\mu_4}\partial_{\mu_1}\varphi_{\eta_1}\partial_{\mu_2}\varphi_{\eta_2}\partial_{\mu_3}\varphi_{\eta_3}\partial_{\mu_4}\varphi_{\eta_4}L_{\eta_1\eta_2\eta_3\eta_4}$$

Double index $\eta = (\beta, b)$

Symmetric four-index invariant

Symmetric invariant bilinears

$$S^{\pm}_{\eta_1\eta_2} = (S^{\pm})^{b_1 b_2}_{\beta_1\beta_2} = \mp(C_{\pm})_{\beta_1\beta_2}(\tau_2)^{b_1 b_2}$$

Lorentz invariant tensors

$$\begin{aligned} C_+ &= \frac{1}{2}(C_1 + C_2) = \frac{1}{2}C_1(1+\bar{\gamma}) = \begin{pmatrix} \tau_2 & , & 0 \\ 0 & ; & 0 \end{pmatrix}, \\ C_- &= \frac{1}{2}(C_1 - C_2) = \frac{1}{2}C_1(1-\bar{\gamma}) = \begin{pmatrix} 0 & , & 0 \\ 0 & ; & -\tau_2 \end{pmatrix} \end{aligned}$$

Symmetric four-index invariant

$$L_{\eta_1\eta_2\eta_3\eta_4} = \frac{1}{6}(S^+_{\eta_1\eta_2}S^-_{\eta_3\eta_4} + S^+_{\eta_1\eta_3}S^-_{\eta_2\eta_4} + S^+_{\eta_1\eta_4}S^-_{\eta_2\eta_3}$$

$$+S^+_{\eta_3\eta_4}S^-_{\eta_1\eta_2} + S^+_{\eta_2\eta_4}S^-_{\eta_1\eta_3} + S^+_{\eta_2\eta_3}S^-_{\eta_1\eta_4})$$

Two flavors needed in four dimensions for this construction

Weyl spinors

$$\varphi_+ = \frac{1}{2}(1+\bar{\gamma})\varphi \ , \ \varphi_- = \frac{1}{2}(1-\bar{\gamma})\varphi$$

$$\bar{\gamma} = -\gamma^0\gamma^1\gamma^2\gamma^3$$ = diag (1 , 1 , -1 , -1)

$$\gamma^0 = \tau_1 \otimes 1 \ , \ \gamma^k = \tau_2 \otimes \tau_k .$$

Action in terms of Weyl - spinors

$$S = \alpha \int d^4x \epsilon^{\mu_1\mu_2\mu_3\mu_4} F^+_{\mu_1\mu_2} F^-_{\mu_3\mu_4} + c.c.$$

$$F^\pm_{\mu_1\mu_2} = A^\pm D^\pm_{\mu_1\mu_2}$$

$$A^+ = \varphi^1_{+1}\varphi^1_{+2}\varphi^2_{+1}\varphi^2_{+2}$$

$$D^\pm_{\mu_1\mu_2} = \partial_{\mu_1}\varphi_{\eta_1} S^\pm_{\eta_1\eta_2} \partial_{\mu_2}\varphi_{\eta_2}$$

Relation to previous formulation

$$A^{(8)} = A^+ A^-$$

$$D = \epsilon^{\mu_1\mu_2\mu_3\mu_4} D^+_{\mu_1\mu_2} D^-_{\mu_3\mu_4}$$

$$S = \alpha \int d^4x A^{(8)} D + c.c.$$

SO(4,C) - symmetry

$$\delta\varphi^a_\alpha(x) = -\frac{1}{2}\epsilon_{mn}(x)(\Sigma_E^{mn})_{\alpha\beta}\varphi^a_\beta(x)$$

$$\Sigma_E^{mn} = -\frac{1}{4}[\gamma_E^m, \gamma_E^n] \;,\; \{\gamma_E^m, \gamma_E^n\} = 2\delta^{mn}$$

Action invariant for arbitrary
complex *transformation parameters ε !*

Real ε : SO (4) - transformations

Signature of time

Difference in signature between
space and time :

only from spontaneous symmetry breaking ,
e.g. by
expectation value of vierbein – bilinear !

Minkowski - action

$$S = -iS_M \;,\; e^{-S} = e^{iS_M}$$

Action describes **simultaneously** euclidean and Minkowski theory !

SO (1,3) transformations : $\epsilon_{0k} = -i\epsilon_{0k}^{(M)}$ $\epsilon_{kl}^{(M)} = \epsilon_{kl}$

$$\delta\varphi = -\frac{1}{2}\epsilon_{mn}^{(M)}\Sigma_M^{mn}\varphi,$$

$$\Sigma_M^{mn} = -\frac{1}{4}[\gamma_M^m, \gamma_M^n] \;,\; \{\gamma_M^m, \gamma_M^n\} = \eta^{mn}$$

$$\gamma_M^0 = -i\gamma_E^0, \gamma_M^k = \gamma_E^k$$

Emergence of geometry

Euclidean vierbein bilinear

$$\tilde{E}^m_\mu = \varphi^a C \gamma^m \partial_\mu \varphi^b V^{ab} = -\partial_\mu \varphi^a C \gamma^m \varphi^b V^{ab}$$

Minkowski -
vierbein bilinear

$$\tilde{E}^{(M)m}_\mu = \varphi V C \gamma^m_M \partial_\mu \varphi$$

$$\tilde{E}^{(M)0}_\mu = -i \tilde{E}^0_\mu \ , \ \tilde{E}^{(M)k}_\mu = \tilde{E}^k_\mu .$$

Global
Lorentz - transformation

$$\delta \tilde{E}^{(M)m}_\mu = -\tilde{E}^{(M)n}_\mu \epsilon^{(M)m}_n$$

vierbein $\langle \tilde{E}^{(M)m}_\mu \rangle = \langle (\tilde{E}^{M)m}_\mu)^* \rangle =$ e^m_μ / Δ

metric $g_{\mu\nu} = e^m_\mu e^n_\nu \eta_{mn}$

Can action can be reformulated in terms of vierbein bilinear ?

$$S = \alpha \int d^4 x W \det(\tilde{E}_\mu^m) + c.c.,$$

No suitable W exists

How to get gravitational field equations ?

How to determine geometry of space-time, vierbein and metric ?

Functional integral formulation of gravity

- Calculability
  (at least in principle)
- Quantum gravity
- Non-perturbative formulation

$$Z = \int \mathcal{D}\psi g_f \exp(-S) g_{in},$$

$$\int \mathcal{D}\psi = \prod_x \prod_{a=1}^{2} \{ \int d\psi_1^a(x) \dots \int d\psi_8^a(x) \}$$

$$\langle \mathcal{A} \rangle = Z^{-1} \int \mathcal{D}\psi g_f \mathcal{A} \exp(-S) g_{in}.$$

Vierbein and metric

$$E_\mu^m(x) = \langle \tilde{E}_\mu^m(x) \rangle$$

$$g_{\mu\nu}(x) = E_\mu^m(x) E_{\nu m}(x)$$

Generating functional

$$Z[J] = \int \mathcal{D}\psi \exp\left\{ - (S + S_J) \right\}$$

$$S_J = - \int d^d x J_m^\mu \tilde{E}_\mu^m$$

$$E_\mu^m(x) = \langle \tilde{E}_\mu^m(x) \rangle = \frac{\delta \ln Z}{\delta J_m^\mu(x)}$$

If regularized functional measure can be defined (consistent with diffeomorphisms)

Non- perturbative definition of quantum gravity

$$Z[J] = \int \underline{\mathcal{D}\psi} \exp\left\{ -\left(S + S_J\right)\right\}$$

Effective action

$$\Gamma[E_\mu^m] = -W[J_m^\mu] + \int d^dx\, J_m^\mu E_\mu^m$$

W=ln Z

Gravitational field equation for vierbein

$$\frac{\delta\Gamma}{\delta E_\mu^m} = J_m^\mu$$

similar for metric

Symmetries dictate general form of effective action and gravitational field equation

diffeomorphisms !

Effective action for metric :
curvature scalar R + additional terms

Lattice spinor gravity

Lattice regularization

- Hypercubic lattice
- Even sublattice $y^\mu = \tilde{y}^\mu \Delta, \; \tilde{y}^\mu \text{ integer}, \; \Sigma_\mu \tilde{y}^\mu \text{ even}$
- Odd sublattice $z^\mu = \tilde{z}^\mu \Delta \; , \; \tilde{z}^\mu \text{ integer}, \; \Sigma_\mu \tilde{z}^\mu \text{ odd}$

- Spinor degrees of freedom on points of odd sublattice

Lattice action

- Associate cell to each point y of even sublattice
- Action: sum over cells

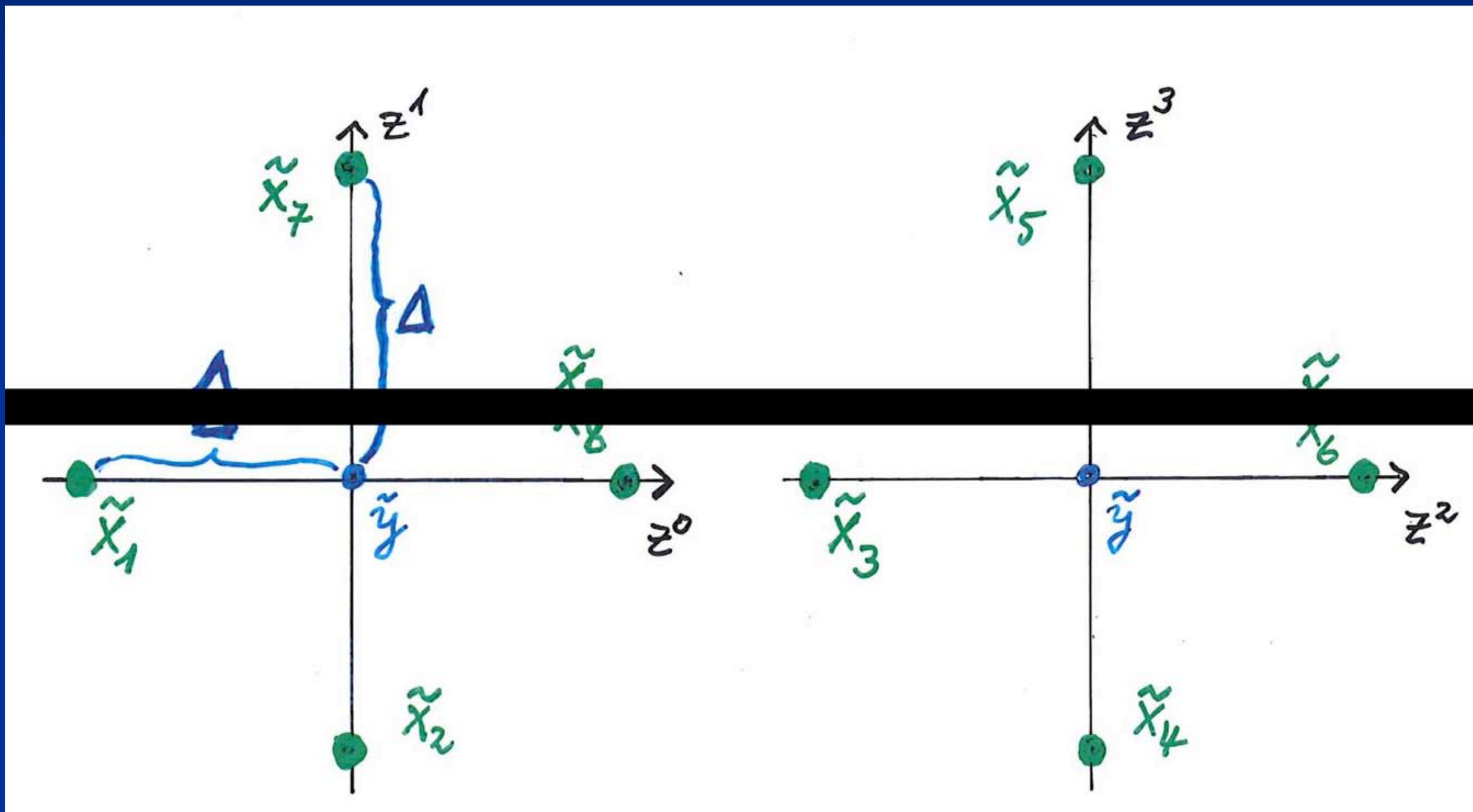
$$S = \tilde{\alpha} \sum_y \mathcal{L}(y) + c.c.$$

- For each cell : twelve spinors located at nearest neighbors of y (on odd sublattice)

$$\tilde{z}^\mu \left(\tilde{x}_j (\tilde{y}) \right) = \tilde{y}^\mu + V_j^\mu$$

$$\begin{aligned}
V_1 &= (-1,0,0,0) \quad , \quad V_5 = (0,0,0,1) \\
V_2 &= (0,-1,0,0) \quad , \quad V_6 = (0,0,1,0) \\
V_3 &= (0,0,-1,0) \quad , \quad V_7 = (0,1,0,0) \\
V_4 &= (0,0,0,-1) \quad , \quad V_8 = (1,0,0,0)
\end{aligned}$$

cells



z^1
$\tilde{x}_7$
Δ
Δ
[redacted]
z^0
$\tilde{x}_1$
$\tilde{y}$
$\tilde{x}_2$
z^3
$\tilde{x}_5$
$\tilde{x}_3$
$\tilde{y}$
z^2
$\tilde{x}_4$

Local SO (4,C) symmetry

Basic SO(4,C) invariant building blocks

$$\tilde{\mathcal{H}}^k_\pm(\tilde{x}) = \varphi^a_\alpha(\tilde{x})(C_\pm)_{\alpha\beta}(\tau_2\tau_k)^{ab}\varphi^b_\beta(\tilde{x})$$

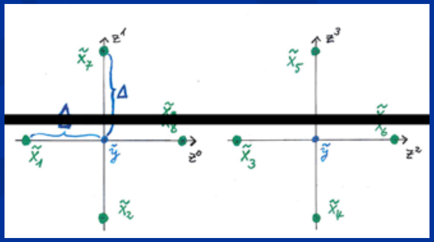
Lattice action

$$\begin{aligned}\mathcal{L}(y) &= \frac{1}{6}\{\mathcal{F}_+^{1,2,8,7}\mathcal{F}_-^{3,4,6,5} + \mathcal{F}_+^{1,3,8,6}\mathcal{F}_-^{7,4,2,5} \\ &\quad +\mathcal{F}_+^{1,4,8,5}\mathcal{F}_-^{3,7,6,2} + (\mathcal{F}_+ \leftrightarrow \mathcal{F}_-)\}.\end{aligned}$$

$$\mathcal{F}^{abcd}_\pm = \frac{1}{24}\epsilon^{klm}[\tilde{\mathcal{H}}^k_\pm(\tilde{x}_a)\tilde{\mathcal{H}}^l_\pm(\tilde{x}_b)\tilde{\mathcal{H}}^m_\pm(\tilde{x}_c)$$

$$+\tilde{\mathcal{H}}^k_\pm(\tilde{x}_b)\tilde{\mathcal{H}}^l_\pm(\tilde{x}_c)\tilde{\mathcal{H}}^m_\pm(\tilde{x}_d) + \tilde{\mathcal{H}}^k_\pm(\tilde{x}_c)\tilde{\mathcal{H}}^l_\pm(\tilde{x}_d)\tilde{\mathcal{H}}^m_\pm(\tilde{x}_a)$$

$$+\tilde{\mathcal{H}}^k_\pm(\tilde{x}_d)\tilde{\mathcal{H}}^l_\pm(\tilde{x}_a)\tilde{\mathcal{H}}^m_\pm(\tilde{x}_b)]$$



Lattice symmetries

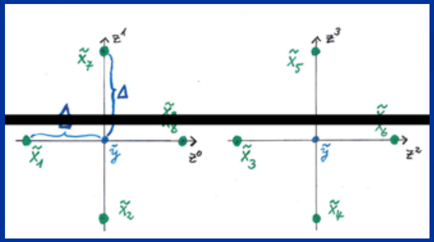
- Rotations by $\pi/2$ in all lattice planes

$$\mathcal{F}_{\pm}^{abcd} = \mathcal{F}_{\pm}^{bcda} = \mathcal{F}_{\pm}^{cdab} = \mathcal{F}_{\pm}^{dabc}$$

- Reflections of all lattice coordinates

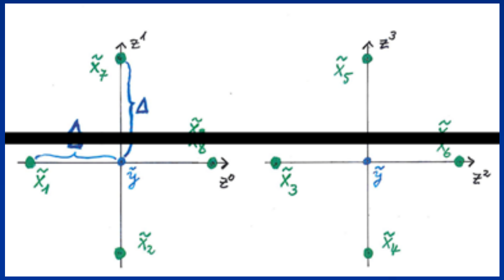
$$\mathcal{F}_{\pm}^{cbad} = \mathcal{F}_{\pm}^{adcb} = -\mathcal{F}_{\pm}^{abcd}$$

- Diagonal reflections e.g $z_1 \leftrightarrow z_2$



Lattice derivatives

$$
\begin{aligned}
\hat{\partial}_0\varphi(y) &= \frac{1}{2\Delta}\left(\varphi(\tilde{x}_8)-\varphi(\tilde{x}_1)\right)\\
\hat{\partial}_1\varphi(y) &= \frac{1}{2\Delta}\left(\varphi(\tilde{x}_7)-\varphi(\tilde{x}_2)\right)\\
\hat{\partial}_2\varphi(y) &= \frac{1}{2\Delta}\left(\varphi(\tilde{x}_6)-\varphi(\tilde{x}_3)\right)\\
\hat{\partial}_3\varphi(y) &= \frac{1}{2\Delta}\left(\varphi(\tilde{x}_5)-\varphi(\tilde{x}_4)\right)
\end{aligned}
$$



and cell averages

$$
\begin{aligned}
\bar{\varphi}_0(y) &= \frac{1}{2}\left(\varphi(\tilde{x}_1)+\varphi(\tilde{x}_8)\right)\,,\ \bar{\varphi}_1(y)=\frac{1}{2}\left(\varphi(\tilde{x}_2)+\varphi(\tilde{x}_7)\right)\\
\bar{\varphi}_2(y) &= \frac{1}{2}\left(\varphi(\tilde{x}_3)+\varphi(\tilde{x}_6)\right)\,,\ \bar{\varphi}_3(y)=\frac{1}{2}\left(\varphi(\tilde{x}_4)+\varphi(\tilde{x}_5)\right)
\end{aligned}
$$

express spinors in derivatives and averages

$$\varphi(\tilde{x}_j)=\sigma_j^\mu\bar{\varphi}_\mu+V_j^\mu\Delta\hat{\partial}_\mu\varphi$$

$$\sigma_j^\mu=(V_j^\mu)^2$$

Bilinears and lattice derivatives

$$\mathcal{H}^k_\pm(\tilde{x}_j) = \sigma^\mu_j \bar{\mathcal{H}}^k_{\pm\mu}(y) + 2\Delta V^\mu_j \tilde{\mathcal{D}}^k_{\pm\mu}(y) + \Delta^2 \sigma^\mu_j \mathcal{G}^k_{\pm\mu}(y).$$

$$\tilde{\mathcal{D}}^k_{\pm\mu} = (\bar{\varphi}_\mu)^a_\alpha (C_\pm)_{\alpha\beta} (\tau_2\tau_k)^{ab} \hat{\partial}_\mu \varphi^b_\beta$$

$$\tilde{\mathcal{G}}^k_{\pm\mu} = \hat{\partial}_\mu \varphi^a_\alpha (C_\pm)_{\alpha\beta} (\tau_2\tau_k)^{ab} \hat{\partial}_\mu \varphi^b_\beta$$

$$\hat{\mathcal{H}}^k_{\pm\mu} = \bar{\mathcal{H}}^k_{\pm\mu} + \Delta^2 \tilde{\mathcal{G}}^k_{\pm\mu}\ ,\ \mathcal{H}^k_{\pm ab} = \frac{1}{2}(\hat{\mathcal{H}}^k_{\pm a} + \hat{\mathcal{H}}^k_{\pm b}).$$

Action in terms of lattice derivatives

$$\mathcal{F}_{+}^{1,2,8,7} = \frac{2\Delta^2}{3}\epsilon^{klm}\mathcal{H}^k_{+01}(\tilde{\mathcal{D}}^l_{+0}\tilde{\mathcal{D}}^m_{+1} - \tilde{\mathcal{D}}^l_{+1}\tilde{\mathcal{D}}^m_{+0}).$$

$$\mathcal{F}^{\pm}_{01} = -\mathcal{F}^{\pm}_{10} = \mathcal{F}^{1,2,8,7}_{\pm}$$

$$\mathcal{F}^{\pm}_{\mu\nu} = \frac{2\Delta^2}{3}\epsilon^{klm}\mathcal{H}^k_{\pm\mu\nu}(\tilde{\mathcal{D}}^l_{\pm\mu}\tilde{\mathcal{D}}^m_{\pm\nu} - \tilde{\mathcal{D}}^l_{\pm\nu}\tilde{\mathcal{D}}^m_{\pm\mu})$$

$$\mathcal{L}(y) = \frac{1}{24}\epsilon^{\mu_1\mu_2\mu_3\mu_4}\mathcal{F}^{+}_{\mu_1\mu_2}\mathcal{F}^{-}_{\mu_3\mu_4}$$

$$\tilde{\mathcal{D}}^k_{\pm\mu} = (\bar{\varphi}_\mu)^a_\alpha (C_\pm)_{\alpha\beta}(\tau_2\tau_k)^{ab}\hat{\partial}_\mu\varphi^b_\beta$$

Continuum limit

$$\mathcal{L}(y) \to \frac{32}{3}\Delta^4 \epsilon^{\mu_1\mu_2\mu_3\mu_4} F^+_{\mu_1\mu_2} F^-_{\mu_3\mu_4}$$

$$\Delta^4 \Sigma_y = \frac{1}{2}\int_y$$

Lattice distance Δ drops out in continuum limit !

$$S = \frac{16}{3}\tilde{\alpha}\int_y \epsilon^{\mu_1\mu_2\mu_3\mu_4} F^+_{\mu_1\mu_2} F^-_{\mu_3\mu_4} + c.c$$

$$\tilde{\alpha} = 3\alpha/16$$

Regularized quantum gravity

- For finite number of lattice points : functional integral should be well defined
- Lattice action invariant under local Lorentz-transformations
- Continuum limit exists where gravitational interactions remain present
- Diffeomorphism invariance of continuum limit , and geometrical lattice origin for this

Lattice diffeomorphism invariance

- Lattice equivalent of diffeomorphism symmetry in continuum
- Action does not depend on positioning of lattice points in manifold , once formulated in terms of lattice derivatives and average fields in cells
- Arbitrary instead of regular lattices
- Continuum limit of lattice diffeomorphism invariant action is invariant under general coordinate transformations

Lattice action and functional measure of spinor gravity are lattice diffeomorphism invariant !

Gauge symmetries

Proposed action for lattice gravity has also chiral SU(2) x SU(2) local gauge symmetry in continuum limit , acting on flavor indices.

Lattice action :
only global gauge symmetry realized

Next tasks

- Compute effective action for composite metric
- Verify presence of Einstein-Hilbert term (curvature scalar)

Conclusions

- Unified theory based only on fermions seems possible
- Quantum gravity –

  functional measure can be regulated
- Does realistic higher dimensional unified

  model exist ?

end

Gravitational field equation and energy momentum tensor

$$\frac{\delta \Gamma}{\delta E_{\mu}^{m}} = J_{m}^{\mu}$$

$$T^{\mu\nu} = E^{-1} E^{m\mu} J_{m}^{\nu}$$

Special case : effective action depends only on metric

$$\Gamma_0'[E_\mu^m] = \Gamma_0'\Big[g_{\nu\rho}[E_\mu^m]\Big]$$

$$g_{\mu\nu} = E_\mu^m E_{\nu m}$$

$$T_{(g)}^{\mu\nu} = -\frac{2}{\sqrt{g}} \frac{\delta \Gamma_0'}{\delta g_{\mu\nu}}$$

$$T^{\mu\nu} = -E^{-1} E^{m\mu} \frac{\delta \Gamma_0'}{\delta g_{\rho\sigma}} \frac{\delta g_{\rho\sigma}}{\delta E_\nu^m} = T_{(g)}^{\mu\nu}$$

Unified theory in higher dimensions and energy momentum tensor

- Only spinors , no additional fields – no genuine source
- $J^{\mu}{}_{m}$: expectation values different from vielbein
  and incoherent fluctuations

- Can account for matter or radiation in effective four dimensional theory (including gauge fields as higher dimensional vielbein-components)

Time space asymmetry from spontaneous symmetry breaking

C.W. , PRL , 2004

Idea : difference in signature from spontaneous symmetry breaking

With spinors : signature depends on
signature of Lorentz group

- Unified setting with complex orthogonal group:
- Both **euclidean** orthogonal group and **minkowskian** Lorentz group are subgroups
- Realized signature depends on ground state !

Complex orthogonal group

d=16 , ψ : 256 – component spinor ,
real Grassmann algebra

$$\delta\psi = \begin{pmatrix} \rho, & -\tau \\ \tau, & \rho \end{pmatrix} \psi$$

$$\rho = -\frac{1}{2}\epsilon_{mn}\hat{\Sigma}^{mn} \ , \ \tau = \frac{1}{2}\bar{\epsilon}_{mn}\hat{\Sigma}^{mn}$$

$$\Sigma_E^{mn} = \hat{\Sigma}^{mn}\,\mathbb{1}\ , \ B^{mn} = -\hat{\Sigma}^{mn} I,$$
$$I = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix},\ I^2 = -1$$

SO(16,C)

ϱ ,τ :
antisymmetric
128 x 128 matrices

Compact part : **ϱ**
Non-compact part : **τ**

vielbein

$$\tilde{E}^0_\mu = \psi_\alpha \partial_\mu \psi_\alpha \ , \ \tilde{E}^k_\mu = \psi_\alpha (\hat{a}^k I)_{\alpha\beta} \partial_\mu \psi_\beta$$

$$\{\hat{a}^k, \hat{a}^l\} = -2\delta^{kl} \ , \ k, l = 1 \dots 15$$

$$\hat{\Sigma}^{kl} = \frac{1}{4}[\hat{a}^k, \hat{a}^l] \ , \ \hat{\Sigma}^{0k} = -\frac{1}{2}\hat{a}^k$$

$E_\mu{}^m = \delta_\mu{}^m$:

SO(1,15) - symmetry

however :

Minkowski signature not singled out in action !

Formulation of action invariant under SO(16,C)

- Even invariant under larger symmetry group SO(128,C)
- Local symmetry !

complex formulation

so far real Grassmann algebra

introduce complex structure by

$$\varphi_{\hat{\alpha}} = \psi_{\hat{\alpha}} + i\psi_{128+\hat{\alpha}}\ ,\ \varphi^*_{\hat{\alpha}} = \psi_{\hat{\alpha}} - i\psi_{128+\hat{\alpha}}$$

$$\delta\varphi_{\hat{\alpha}} = \sigma_{\hat{\alpha}\hat{\beta}}\varphi_{\hat{\beta}}\ ,\ \sigma = \rho + i\tau$$

σ is antisymmetric 128 x 128 matrix , generates SO(128,C)

Invariant action

(complex orthogonal group, diffeomorphisms)

$$S = \alpha \int d^d x W[\varphi] R(\varphi, \varphi^*) + c.c.,$$

$$W[\varphi] = \frac{1}{16!} \epsilon^{\mu_1 \dots \mu_{16}} \partial_{\mu_1} \varphi_{\hat{\alpha}_1} \dots \partial_{\mu_{16}} \varphi_{\hat{\alpha}_{16}} L^{\hat{\alpha}_1 \dots \hat{\alpha}_{16}}$$

$$L^{\hat{\alpha}_1 \dots \hat{\alpha}_{16}} = sym \left\{ \delta^{\hat{\alpha}_1 \hat{\alpha}_2} \delta^{\hat{\alpha}_3 \hat{\alpha}_4} \dots \delta^{\hat{\alpha}_{15} \hat{\alpha}_{16}} \right\}$$

$$R(\varphi, \varphi^*) = T(\varphi) + \tau T(\varphi^*) + \kappa T(\varphi) T(\varphi^*),$$

$$T(\varphi) = \frac{1}{128!} \epsilon^{\hat{\beta}_1 \dots \hat{\beta}_{128}} \varphi_{\hat{\beta}_1} \dots \varphi_{\hat{\beta}_{128}}$$

invariants with respect to
SO(128,C)
and therefore also
with respect to subgroup
SO (16,C)

contractions with
δ and ε – tensors

no mixed terms φ φ*

For τ = 0 : **local** Lorentz-symmetry !!

Generalized Lorentz symmetry

- Example d=16 : SO(128,C) instead of SO(1,15)
- Important for existence of chiral spinors in effective four dimensional theory after dimensional reduction of higher dimensional gravity

S.Weinberg

Unification in d=16 or d=18 ?

- Start with irreducible spinor
- Dimensional reduction of gravity on suitable internal space
- Gauge bosons from Kaluza-Klein-mechanism
- 12 internal dimensions : SO(10) x SO(3) gauge symmetry : unification + generation group
- 14 internal dimensions : more U(1) gener. sym.

(d=18 : anomaly of local Lorentz symmetry)

L.Alvarez-Gaume,E.Witten

Ground state with appropriate isometries:
guarantees massless gauge bosons and graviton in spectrum

Chiral fermion generations

- Chiral fermion generations according to chirality index
  C.W. , Nucl.Phys. B223,109 (1983) ;
  E. Witten , Shelter Island conference,1983
- Nonvanishing index for brane geometries (noncompact internal space)
  C.W. , Nucl.Phys. B242,473 (1984)
- and wharping
  C.W. , Nucl.Phys. B253,366 (1985)
- d=4 mod 4 possible for 'extended Lorentz symmetry' (otherwise only d = 2 mod 8)

Rather realistic model known

- d=18 : first step : brane compactifcation

- d=6, SO(12) theory : (anomaly free)
- second step : monopole compactification

- d=4 with three generations,
  including generation symmetries
- SSB of generation symmetry: realistic mass and mixing hierarchies for quarks and leptons
  (except large Cabibbo angle)

C.W. , Nucl.Phys. B244,359(1984) ; B260,402 (1985) ; B261,461 (1985) ; B279,711 (1987)

Comparison with string theory

	SStrings	Sp.Grav.
Unification of bosons and fermions	ok	ok
Unification of all interactions (d >4)	ok	ok
Non-perturbative (functional integral) formulation	-	ok
Manifest invariance under diffeomophisms	-	ok

Comparison with string theory

	SStrings	Sp.Grav.
Finiteness/regularization	ok	ok
Uniqueness of ground state/ predictivity	-	?
No dimensionless parameter	ok	?