Introduction to Quantum Gravity

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A Pedagogical perspective on the challenges, ideas and results in non-perturbative of Quantum Gravity. Should also serve as an introduction to other related lectures, particularly by Bojowald, Mena and Rovelli.

Organization:

I. Gravity & the Quantum: Four Views of Reality

2. Overview of Approaches, Quantum Geometry and Black Holes

3. Implications of Quantum Geometry for Cosmology: Some Highlights.

Gravity & the Quantum: 4 Views of Physical Reality

• Goal: To illustrate through an exactly soluble midi-superspace how the new scales provided by $\hbar$ and G can change physics in rather dramatic ways. Focus: Conceptual Issues; opening of new potentials with the introduction of new fundamental constants.

• Idea: Shine laser light and describe what various theories claim happens in 2+1 dimensions. Rather dramatic differences.
Exactly soluble model; Also symmetry reduction on 4-d Einstein-Rosen Waves. Thus, a midi-superspace in 4-d Gravity. But will emphasize the 2+1 dimensional picture.
Qualitatively similar surprises in 4-d quantum gravity (Lectures II and III.)

Based on a decade old work by AA+ Varadarajan, AA, AA+Pierri, Gambini & Pullin, Barbero, Mena, et al (Madrid Group).

Organization:
1. Classical Physics View $(\hbar = 0, G = 0)$
2. Quantum Field Theory View $(\hbar \neq 0, G = 0)$
3. General Relativity View $(\hbar = 0, G \neq 0)$
4. Quantum gravity View $(\hbar = 0, G = 0)$
5. Summary.

1. Reality a la Classical Physics

- Useful duality in 2+1 dimensions: Maxwell $\leftrightarrow$ Klein-Gordon

$\mathrm{d}\,{}^\star F = 0 \Leftrightarrow F_{ab} = \epsilon_{ab}{}^{c}\partial_c\phi \qquad \mathrm{d}F = 0 \Leftrightarrow \nabla^a\nabla_a\phi = 0$.

Will work with the KG field for simplicity.

- Model: Shining a laser light $\Rightarrow$ Axi-symmetric Fields.

Classical Physics $\Leftrightarrow$ 3-d Minkowski space (M, η) with Field Eq. $\Box_\eta\phi = 0$. Propagation causal but not sharp along null characteristics.

- Solutions:

$\tilde{C}(R,T) = \int_o^\infty \mathrm{d}k[f_k^+(R,T)\,C(k) + f_k^-(R,T)\,\bar{C}(k)]$, where $f^+(k) \approx J_o(kR)\,e^{ikT}$

Energy: $H_o(\tilde{C}) = \int_o^\infty \mathrm{d}k\,k\,|C(k)|^2$ dimensions: $[C^2] \sim ML^2$

Laser beam $\Rightarrow$ Profile $C(k)$ Sharply peaked at some frequency k_o. .

2. Reality a la QFT/Quantum Optics

- $G = 0,\ [\hbar] \sim ML \neq 0$.

Again Minkowski space (M, η) now with $\Box_\eta \hat{\phi} = 0$.

- Operators:

$$\hat{\phi}(R,T) = \int_0^\infty \mathrm{d}k\ [f_k^+(R,T)\,\hat{a}(k) + f_k^-(R,T)\,\hat{a}^\dagger(k)]$$

$$[\hat{a}(k),\ \hat{a}^\dagger(k')] = \hbar\,\delta(k,k') \quad \Leftrightarrow \quad \text{CCR}$$

$$\hat{H}_o = \int_o^\infty \mathrm{d}k\ [k\ \hat{a}^\dagger(k)\,\hat{a}(k)] \quad \text{(normal ordering.)}$$

- States:

$|0\rangle$: Poincaré invariant, Cyclic.

Each classical solution $\tilde{C}(R,T)$ or equivalently, profile $C(k)$ defines a 1-particle state: $|C(k)\rangle = [\,\frac{1}{\hbar}\int \mathrm{d}k\ C(k)\,\hat{a}^\dagger(k)\,]\,|0\rangle$

Norm: $||C(k)||^2 = \frac{1}{\hbar}\int \mathrm{d}k\ |C(k)|^2$.

Note: $\hbar$ essential for dimensional reasons; Recall: $C^2(k) \sim ML^2$.

Interested in states that best approximate the classical field $\tilde{C}(R,T)$ obtained by shining laser light: Coherent states $|C(k)\rangle$ associated with the profile $C(k)$.

Quantum Optics

- Coherent states well suited to compare and contrast the classical description of part 1. $|\Psi_C\rangle := e^{\frac{1}{2}||C||^2}\ e^{\frac{1}{\hbar}\int \mathrm{d}k C(k)\hat{a}^\dagger(k)}\ \ |0\rangle$

Peaked at the classical solution $\tilde{C}(R,T)$ for all times. Relation to Part 1?

- Properties:

$\star$ $\langle\Psi_C|\,\hat{\Phi}(R,T)\,|\Psi_C\rangle = \tilde{C}(R,T)$; $\quad\langle\Psi_C|\,\hat{H}_o\,|\Psi_C\rangle H_o(\tilde{C})$

$\star$ Product of Uncertainty, $(\Delta\hat{\varphi})(\Delta\hat{\pi})$ saturated and furthermore distributed "equally", for all $\tilde{C}(R,T)$

$\star$ No frequency scale; ($[\hbar] \sim ML$) $|\Psi_C\rangle$ exists if $||C(k)|| < \infty$.

$\star$ Expected number of photons: $\mathcal{N} := \langle\Psi_C|\,\hat{\mathcal{N}}\,|\Psi_C\rangle = ||C(k)||^2 \equiv \frac{1}{\hbar}\int \mathrm{d}k|C(k)|^2$

- Uncertainties: $(\Delta\hat{\phi}(f))^2/\langle\hat{\phi}(f)\rangle^2 \gtrsim 1/4\mathcal{N}$; $\quad(\Delta\hat{H}_o)^2/\langle\hat{H}_o\rangle^2 \sim 1/\mathcal{N}$

Although conceptually very different from part 1, classical Physicist's view of reality becomes an excellent approximation if (and only if) $\mathcal{N} \gg 1$. So, fainter the laser beam, the classical description becomes less and less reliable.

3. Reality a la General Relativity

- Now $\hbar = 0$, $[G] \sim M^{-1} \neq 0$ Space-time $M = \mathbb{R}^3$ as before but g_{ab} curved & dynamical.

$\Box_g \, \phi = 0$; $G_{ab} = 8\pi G \;\, T_{ab}$;   (and, $F_{ab} = \epsilon_{ab}{}^{c}\partial_c\phi \Rightarrow T_{ab}(F) = T_{ab}(\phi)$!)
Axi-symmetry renders the problem exactly soluble.

- One can gauge fix using preferred coordinates θ, R, T. Killing vector $\partial/\partial\theta$; Its norm is given by R^2 and T uniquely determined up to additive constant by the form of the metric:

$$ds^2 = e^{G\Gamma(R,T)} \; (-\mathrm{d}T^2 + \mathrm{d}R^2) + R^2\mathrm{d}\theta^2 \;\; (\star)$$

$\Box_g \, \phi = 0 \quad \Leftrightarrow \quad \Box_\eta \, \phi = 0$, where η is obtained from g by setting $\Gamma(R,T) = 0$.
Einstein's Eqs $\Rightarrow \Gamma(R_o, T) = \frac{1}{2} \int_0^{R_o} \mathrm{d}R \, R \, [\, (\partial_T \phi)^2 + (\partial_R \phi)^2 \,](T)$
= Energy of ϕ in a box of radius R_o at time T in Minkowski space (M, η).

- Decoupling!. Solve $\Box_\eta \, \phi = 0$ in Minkowski space; calculate $\Gamma(R,T)$. Define g_{ab} given by $(\star)$. Then (ϕ, g) satisfies the Einstein-KG equation and is furthermore the general solution.

Note: (1) No gravitational collapse in 2+1 gravity because there is no length scale: [GM] $\sim L^0$! (2) This is precisely the KK reduction of the Einstein-Rosen cylindrical waves with respect to z-directional translation.

Notable features of the GR description

- Physical geometry quite different from Minkowskian:

$$ds^2 = e^{G\Gamma(R,T)}\,(-\mathrm{d}T^2 + \mathrm{d}R^2) + R^2\mathrm{d}\theta^2 \quad (\star)$$

Light cones open up. If $G\Gamma \gg 1$, large deviations from the classical physicist's description based on η!

- Deviations extend also outside the support of ϕ. There; metric is flat because in 3-d, $T_{ab} = 0, \Leftrightarrow R_{ab} = 0 \Leftrightarrow R_{abcd} = 0$. But because in this region $ds^2 = e^{GH_o}\,(-\mathrm{d}T^2 + \mathrm{d}R^2) + R^2\mathrm{d}\theta^2 \quad (\star)$

(Recall $\Gamma(R.T)$ = energy contained in a box of radius R w.r.t. Minkowskian metric),
There is a deficit angle at infinity. g does not approach η even at infinity!

- The total (ADM-type) Hamiltonian is bounded above!

$$H = \frac{1}{4G}\,(1 - e^{-4GH_o}) \approx H_o - 2(GH_o)H_o + \ldots$$

even though each term in the"perturbative' expansion in powers of G is unbounded. A genuinely non-perturbative effect.

- Exterior geometry determined by $H_o = \int_o^\infty \mathrm{d}k\,k|C(k)|^2$; Again no frequency scale. But there is a mass scale which makes the view of reality very different from the last two!

4. Reality a la Quantum Gravity

• $G \neq 0,\, \hbar \neq 0 \;\; \Rightarrow$ a new length scale: Planck length $\ell_{\rm Pl} = G\hbar$.
Recall: True degree of freedom in the scalar field; Metric a derived quantity. Systematic canonical quantization leads to the same scenario.

• Operators $\hat\phi(x)$ defined on the Minkowskian Fock space as in quantum optics. $\hat g$(x): 'derived'/secondary construct. Outside the laser beam,

$$\hat g_{ab} {\rm d}x^a {\rm d}x^b = e^{G\hat H_o}\,(-{\rm d}T^2 + {\rm d}R^2) + R^2 {\rm d}\theta^2 .$$

Framework almost the same as in the Quantum Optics of part 2 but physics now lies in operators not normally considered in quantum optics:

$$\hat g_{RR} = \hat g_{TT} = e^{-G\hat H_o} \quad \text{and} \quad \hat H := (1/4G)\,(1 - e^{-4G\hat H_o}).$$

• In General Relativity state determined by $\tilde C(R,T) \;\sim\; C(k)$
The corresponding Quantum Gravity State: $|\Psi_C\rangle$

$$\langle \Psi_C \mid \hat\phi(x) \mid \Psi_C\rangle = \tilde C(R,T) \quad \text{and sharply peaked.}$$

• Expectation values and fluctuations can be calculated exactly since $|\Psi_C\rangle$ is a coherent state.

Contrasting Classical & Quantum Geometries

- For simplicity let us focus on geometry outside the laser beam.

Classically, $g_{RR} = g_{TT} = e^{G \int_0^\infty \mathrm{d}k\ k\,|C(k)|^2} \equiv e^{GH_o(C)}$.
What happens in quantum gravity?

- Quantum Theory:

$$\langle \Psi_C \mid \hat{g}_{RR} \mid \Psi_C \rangle = e^{\frac{1}{\hbar} \int \mathrm{d}k\ |C(k)|^2\,(e^{G\hbar k} - 1)}.$$

Note: $\hbar$ appears unlike in GR or even flat space quantum theory where there is no $\hbar$ in the expectation values of $\hat{\phi}(x)$ or of $\hat{H}_o$.
Situation similar for the non-perturbative Hamiltonian:

$$\langle \Psi_C \mid \hat{H} \mid \Psi_C \rangle = \frac{1}{4G}\left[1 - e^{\frac{1}{\hbar} \int \mathrm{d}k\ |C(k)|^2\,(e^{-4G\hbar k} - 1)}\right].$$

Thus, the expectation values of the asymptotic metric and total (ADM-like) Hamiltonian very different from the classical theory.

- Low Energy limit: $G\hbar k_o \ll 1$,

$$\langle \hat{g}_{RR} \rangle \approxeq e^{G \int \mathrm{d}k\ |C(k)|^2\,(k + \frac{G\hbar k^2}{2} + \ldots)}$$

$$\approx g^{\mathrm{cl}}_{RR}\,(1 + \mathcal{N}\frac{(G\hbar k_o)^2}{2} + \ldots), \qquad \text{if } \mathcal{N}(G\hbar k_o)^2 \ll 1$$

Recovery of classical limit subtle. Requires: $\mathcal{N} \gg 1$ to recover ϕ^{cl} from $\langle \hat{\phi}(x) \rangle$, and $G\hbar k_o \ll 1$ & $\mathcal{N}(G\hbar k_o)^2 \ll 1$ to recover g^{cl} from $\langle \hat{g} \rangle$.

High Frequency Limit & Fluctuations

- High frequency $\Leftrightarrow$ $G\hbar k_o \gg 1$

$$\langle \hat{g}_{RR} \rangle \approxeq e^{\frac{1}{\hbar} \int \mathrm{d}k\, |C(k)|^2\, e^{G\hbar k}} \approx e^{\mathcal{N} e^{G\hbar k_o}} \gg e^{\mathcal{N} G\hbar\, k_o} = g^{\mathrm{cl}}_{RR}$$

⋆ $\hbar$ remains in the leading term!

⋆ Deviation worse if $\mathcal{N} \gg 1$; striking contrast with quantum optics view; Gross departures from the classical theory.

- Fluctuations: $(\Delta \hat{g}_{RR})^2 = \langle \hat{g}^2_{RR} \rangle - \langle \hat{g}_{RR} \rangle^2$.

  Exact Result: $(\Delta \hat{g}_{RR}/\langle \hat{g}_{RR} \rangle)^2 = e^{\frac{1}{\hbar} \int \mathrm{d}k\, |C(k)|^2\, (1-e^{G\hbar k})^2} - 1$.

Low frequency limit: $G\hbar k_o \ll 1$.

$(\Delta \hat{g}_{RR}/\langle \hat{g}_{RR} \rangle)^2 \approx e^{\mathcal{N}\, (G\hbar k_o)^2} - 1$. So, fluctuations are small only if $\mathcal{N}(G\hbar k_o)^2 \ll 1$. Same condition on k_o as was required to recover g^{cl}_{RR} from the expectation value.

High frequency limit: $G\hbar k_o \gg 1$:

$(\Delta \hat{g}_{RR}/\langle \hat{g}_{RR} \rangle)^2 \approx e^{\mathcal{N}\, e^{2G\hbar k_o}}$. HUGE! $\mathcal{N} \approx 1$ and $G\hbar k_o \approx 1$ $\Rightarrow$ Relative fluctuation $\sim 10^3$! So, even a 'blip' at (trans-)Planckian frequency photon in the profile would give large fluctuations in quantum geometry well away from the laser beam.

- Similar results also for the total, non-perturbative Hamiltonian.

Highlights of Quantum Gravity Findings

- Minkowski space an exact solution: $C(k) = 0$, $|\Psi_C\rangle = |0\rangle$; Eigenstate of $\hat{g}_{RR}$ and $\hat{H}$; No fluctuations. Furthermore, there is an infinite dimensional sector of “good” classical solutions, i.e., solutions recovered from the full quantum gravity description. But the sector is subtle. One has to have $\mathcal{N} = (1/\hbar) \int \mathrm{d}k |C(k)|^2 \gg 1$ (for small fluctuations in the Maxwell field) and $\mathcal{N} (G\hbar k_o)^2 \ll 1$ for geometry fluctuations to be small .

- If $|C(k)|$ significant in the trans-Plackian regime (e.g. ~ 1 photon of Planck frequency) classical solution $(\phi = \tilde{C}(R,T); g_{ab}(R,T))$ is “spurious” because of large fluctuations in the metric and full Hamiltonian operators. Unforeseen limitation of both classical GR and quantum optics.

- Mechanism: Non-linearities of Einstein’s equations magnify small fluctuations in matter to huge metric fluctuations. Specific non-linearities important. For example, the QED analog does not exhibit such large quantum fluctuations.

- Start with ‘semi-classical states associated with geometry’ ? Large fluctuations transferred to matter (Gambini & Pullin).

4. Summary

• Used an exactly soluble model in 2+1 dimensions (also $\sim$ 4-d Einstein-Rosen waves). Three strikingly different notions of physical reality for what happens in a simple thought experiment. $\hbar, G$ and $(\hbar, G)$ bring in new scales. Intuition based on QFT and GR was often quite wrong!

• QFT/Quantum Optics ($\hbar \neq 0, G = 0$): New operator $\hat{\mathcal{N}}$. Classical intuition OK if $\mathcal{N} = (1/hbar) \int \mathrm{d}k |C(k)|^2 \gg 1$.

• GR ($\hbar = 0, G \neq 0$): New effects significant if $GH_o \gg 1$.
⋆ Light cones open up. Geometry flat but non-Minkowskian near infinity.
⋆ Total Hamiltonian $H = (1/4G)(1 - e^{-4GH_o})$ is bounded from above although Minkowskian energy H_o is not.

• Quantum Gravity ($\hbar \neq 0, G \neq 0$): New length scale $[G\hbar] \sim L$. Unforeseen effects even far from the laser beam:
⋆ $\mathcal{N} \gg 1$ no longer sufficient for fluctuations to be small.
Exponentially large $(\Delta \hat{g}_{RR}/\langle \hat{g}_{RR} \rangle)$ if $\mathcal{N}(G\hbar k_o)^2 > 1$.
⋆ If $C(K)$ has a blip at $k \geq (G\hbar)^{-1}$ (even ~ 1 photon at (trans-) Planck frequency), $\langle \hat{g}_{RR} \rangle$ very different from g^{cl}_{RR} and huge fluctuations.

• Non-perturbative effects can lead to unforeseen limitations both of QFT and GR. But the situation in full 4-d is much more subtle.