Quantum fields on curved momentum space

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MARIE CURIE ACTIONS

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Observers in an expanding universe: *cosmological particle creation (Parker)*

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Less known context in which *free* QFT manifests non-trivial features:
field quantization on *group manifold* **momentum space**

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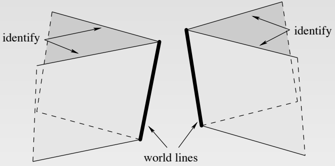

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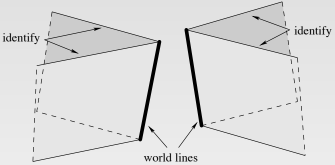

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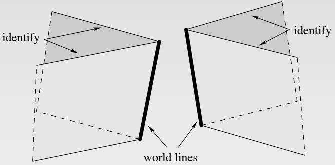

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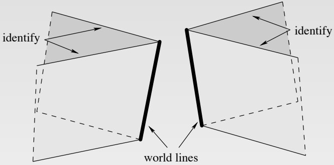

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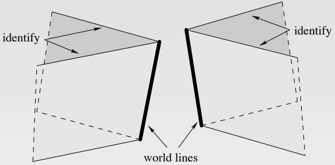

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Momenta become coordinate functions on a non-abelian group!

Outline

- **Warm up: free field quantization and observables**
- **“Bending” phase space in 3d: running spectral dimension**
- κ**-quantum fields: two-point function and a new quantization ambiguity**

Field quantization

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Classical fields	Quantum fields
state $=$ *point in phase space* $\phi \in \mathcal{S}$	**state** $=$ *ray in complex Hilbert space* $\mathcal{H}$
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- Fock space $\mathcal{F}_s(\mathcal{H}) = \bigoplus_{n=0}^{\infty} S_n \mathcal{H}^{\otimes n}$

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- The coproduct Δ extends the action of elements of $\mathcal{P}$ to multiparticle states

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define *group Fourier transform* (Freidel and Majid, hep-th/0601004)

$$\mathcal{F}(f)(x) = \int d\mu_H(g) f(g)\, e_g(x)\,,$$

maps fields *on the group manifold* to fields on a *dual "spacetime"*...

Group-valued plane waves and deformed symmetries

...the group structure induces a non-commutative $\star$-**product** for plane waves

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non-abelian composition of momenta = **non-trivial coproduct**

$$\boxed{\Delta P_a = P_a \otimes \mathbf{1} + \mathbf{1} \otimes P_a + G\,\epsilon_{abc} P_b \otimes P_c + \mathcal{O}(G^2)}$$

the *smoking gun* of symmetry deformation...P_a belong to a non-trivial Hopf algebra with G as a deformation parameter!

An application: heath kernel and spectral dimension

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$$K(x, x'; s) = \int d\mu_H(g) \, e^{-s(\mathbf{p}^2(g)+m^2)} e_g(x) e_g(x')$$

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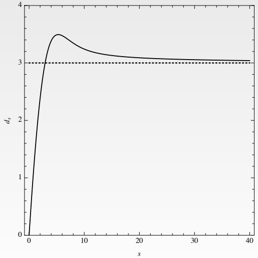
and calculate the *spectral dimension* $d_s = -2\frac{\partial \log \tilde{Tr}K}{\partial \log s}$... (plot for $G = 1$, $m = 0$)



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$$-\eta_0^2 + \eta_1^2 + \eta_2^2 + \eta_3^2 + \eta_4^2 = \kappa^2 \, ; \qquad \eta_0 + \eta_4 > 0$$

    with $\kappa \sim E_{Planck}$
  - dual Lie algebra "space-time" coordinates

$$[x_\mu, x_\nu] = -\frac{i}{\kappa}(x_\mu \delta^0_\nu - x_\nu \delta^0_\mu) \, .$$

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- consider a one-parameter group splitting of B, $0 \leq |\beta| \leq 1$

$$e_p \equiv e^{-i\frac{1-\beta}{2}p^0 x_0} e^{ip^j x_j} e^{-i\frac{1+\beta}{2}p^0 x_0}\,.$$

with momentum composition rules and "antipodes"

$$p \oplus_\beta q = (p^0 + q^0;\, p^j\ e^{\frac{1-\beta}{2\kappa}q^0} + q^j\ e^{-\frac{1+\beta}{2\kappa}p^0})\,, \qquad \ominus_\beta p = (-p^0;\, -e^{\frac{-\beta}{\kappa}p^0} p^i)\,.$$

each choice of β corresponds to a *choice of coordinates* on the group manifold.

κ-Poincaré II

for $\beta = 1$ we have "flat slicing" coordinates

$$
\begin{aligned}
\eta_0(p_0, \mathbf{p}) &= \kappa \sinh p_0/\kappa + \frac{\mathbf{p}^2}{2\kappa}\, e^{p_0/\kappa}, \\
\eta_i(p_0, \mathbf{p}) &= p_i\, e^{p_0/\kappa}, \\
\eta_4(p_0, \mathbf{p}) &= \kappa \cosh p_0/\kappa - \frac{\mathbf{p}^2}{2\kappa}\, e^{p_0/\kappa}.
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("bicrossproduct basis" introduced in Majid-Ruegg ('94) to prove that $\mathcal{P}_\kappa = U(\mathfrak{so}(3,1)) \bowtie \mathbb{C}(B)$)

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$$[N_j, P_l] = i\delta_{lj}\Big(\tfrac{\kappa}{2}\left(1 - e^{-\frac{2P_0}{\kappa}}\right) + \tfrac{1}{2\kappa}\vec{P}^2\Big) + \tfrac{i}{\kappa} P_l P_j$$

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- **deformed mass Casimir** $\Rightarrow$ Lorentz invariant hyperboloid on B: $\eta_4 = \mathrm{const.}$

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in the limit $\kappa \longrightarrow \infty$ recover ordinary Poincaré algebra

A new quantization ambiguity

Functions on the deformed mass-shell $\phi \in C^\infty(M_m^\kappa)$ defined by the "wave equation"

$$C_1^\kappa(P)\,\phi = m^2\phi$$

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No preferred choice of translation generators from which we can define an **energy** coordinate on M_m^κ and thus **no preferred choice of J and P^+** to define one-particle Hilbert space.

κ-particle Hilbert space and two-point function

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- Hilbert space = $C^\infty(M^\kappa)$ functions on deformed mass shell $\omega_\kappa^\pm(\mathbf{p}) = -\kappa \log\left(1 \mp \frac{|\mathbf{p}|}{\kappa}\right)$ equipped with inner product

$$(\phi_1, \phi_2)_\kappa = \int_{M_m^{\kappa+}} \frac{d\tilde{\mu}(\mathbf{p})}{2|\mathbf{p}|}\ \bar{\phi}_1(\mathbf{p})\, \phi_2(\mathbf{p})$$

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group Fourier transform $\Rightarrow$ *NC space-time* counterparts of functions on the mass-shell

$$\hat{\phi}(x) = \int_B d\mu(p)\ \delta(\mathcal{C}_1^\kappa(p))\,\tilde{\phi}(p)\,e_p(x)$$

field mode operators (MA, Phys. Rev. **D83**, 025025 (2011)): $\hat{\phi}_\kappa(\mathbf{p}) \equiv \frac{1}{2|\mathbf{p}|}(a(\mathbf{p}) + \mathcal{J}_\ominus(\mathbf{p})\,a^\dagger(\ominus\mathbf{p}))$

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Fundamental building block of κ-QFT: the two-point function

$$G_+(\mathbf{p}_1, t; \mathbf{p}_2, s) \equiv \langle 0|\hat{\phi}_\kappa(\mathbf{p}_1, t)\hat{\phi}_\kappa(\mathbf{p}_2, s)|0\rangle = \frac{\delta^3(\mathbf{p}_1 \oplus \mathbf{p}_2)}{2|\mathbf{p}_1|} \mathcal{J}_\ominus(\mathbf{p}_1)\exp(-i\omega_\kappa(\mathbf{p}_1)(t-s))$$

work in progress (with J. Kowalski-Glikman and T. Trzesniewski) with Feynman propagator and "zoology" of Green functions...

connection with field theories on *multifractal* spacetimes see M. Scalisi's talk this Friday

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is NOT an eigenstate of P_μ due to the role of **non-trivial coproduct**

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$$\sigma^\kappa(|\mathbf{k_1}\rangle \otimes |\mathbf{k_2}\rangle) = |(1-\epsilon_1)\,\mathbf{k_2}\rangle \otimes |(1-\epsilon_2)^{-1}\,\mathbf{k_1}\rangle\,, \qquad \epsilon_i = \frac{|\mathbf{k}_i|}{\kappa}$$

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$$\sigma^\kappa(|\mathbf{k_1}\rangle \otimes |\mathbf{k_2}\rangle) = |(1-\epsilon_1)\,\mathbf{k_2}\rangle \otimes |(1-\epsilon_2)^{-1}\,\mathbf{k_1}\rangle\,, \qquad \epsilon_i = \frac{|\mathbf{k}_i|}{\kappa}$$

- E.g. there will be **two** 2-particle states

$$|\mathbf{k_1}\mathbf{k_2}\rangle_\kappa = \frac{1}{\sqrt{2}}\left[|\,\mathbf{k_1}\rangle \otimes |\,\mathbf{k_2}\rangle + |\,(1-\epsilon_1)\mathbf{k_2}\rangle \otimes |\,(1-\epsilon_2)^{-1}\mathbf{k_1}\rangle\right]$$
$$|\mathbf{k_2}\mathbf{k_1}\rangle_\kappa = \frac{1}{\sqrt{2}}\left[|\,\mathbf{k_2}\rangle \otimes |\,\mathbf{k_1}\rangle + |\,(1-\epsilon_2)\mathbf{k_1}\rangle \otimes |\,(1-\epsilon_1)^{-1}\mathbf{k_2}\rangle\right]$$

  with **same energy** and different linear momentum

$$\mathbf{K}_{12} = \mathbf{k_1} \oplus \mathbf{k_2} = \mathbf{k_1} + (1-\epsilon_1)\mathbf{k_2}$$
$$\mathbf{K}_{21} = \mathbf{k_2} \oplus \mathbf{k_1} = \mathbf{k_2} + (1-\epsilon_2)\mathbf{k_1}$$

κ-Fock space

In ordinary QFT the full (bosonic) **Fock space** is obtained from symmetrized tensor prods of $\mathcal{H}$

In the κ**-deformed case** try to proceed in an analogous way BUT...
the symmetrized state

$$1/\sqrt{2}\,(|\mathbf{k_1}\rangle \otimes |\mathbf{k_2}\rangle + |\mathbf{k_2}\rangle \otimes |\mathbf{k_1}\rangle)$$

is NOT an eigenstate of P_μ due to the role of **non-trivial coproduct**

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given n-different modes one has $n!$ different n-particle states, one for each permutation of the n modes $\mathbf{k_1}\,,\mathbf{k_2}\ldots\mathbf{k_n}$

Hidden entanglement at the Planck scale

The non-trivial algebraic structure of κ-translations endows the Fock space with a "fine structure"

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- the 2-mode Hilbert space becomes $\mathcal{H}^2_\kappa \cong \mathcal{S}_2\mathcal{H}^2 \otimes \mathbb{C}^2$, where $\mathcal{S}_2\mathcal{H}^2$ is the ordinary symmetrized 2-mode Hilbert space and our states can be written as

$$
\begin{aligned}
|\epsilon\rangle \otimes |\uparrow\rangle &= |\mathbf{k_1}\mathbf{k_2}\rangle_\kappa \\
|\epsilon\rangle \otimes |\downarrow\rangle &= |\mathbf{k_2}\mathbf{k_1}\rangle_\kappa
\end{aligned}
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with $\epsilon = \epsilon(\mathbf{k_1}) + \epsilon(\mathbf{k_2})$

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- e.g. the state superposition of two total "classical" energies $\epsilon_A = \epsilon(\mathbf{k_1}_A) + \epsilon(\mathbf{k_2}_A)$ and $\epsilon_B = \epsilon(\mathbf{k_1}_B) + \epsilon(\mathbf{k_2}_B)$ can be entangled with the additional hidden modes e.g.

$$|\Psi\rangle = 1/\sqrt{2}(|\epsilon_A\rangle \otimes |\uparrow\rangle + |\epsilon_B\rangle \otimes |\downarrow\rangle)$$

...possible consequences for phenomenology?

(MA., D. Benedetti, [arXiv:0809.0889 [hep-th]]. MA., A. Marciano, [arXiv:0707.1329 [hep-th]]. MA, A. Hamma, S. Severini, [arXiv:0806.2145 [hep-th]].)

Conclusions

- Relativistic symmetries can be deformed to allow **"curvature" for momentum space**

- Strong motivations to look at such deformations from **2+1 gravity coupled to relativistic particles**...application: appearance of *running spectral dimension*

- Quantization of (free) field theories with group valued momenta leads to **ambiguities** related to the different choices of translation generators...physical interpretation of such ambiguities?

- What role of **deformed 2-point functions** for "trans-planckian" issues in semiclassical gravity (BH evaporation, Inflation)??

- At the multiparticle level the non-trivial behaviour of field modes leads to a **fine structure of Fock space**: interesting **entanglement** phenomena can take place