Fractal and noncommutative spacetimes: a connection

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work in collaboration with
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based on [arXiv:1107.5308]

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Before entering into the talk...

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Fractal spacetime?

Noncommutative spacetime?

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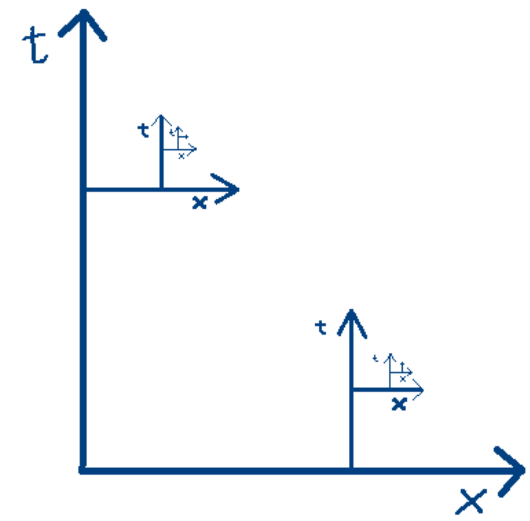
Fractal spacetime?

-a space where the ordinary differential calculus is not valid

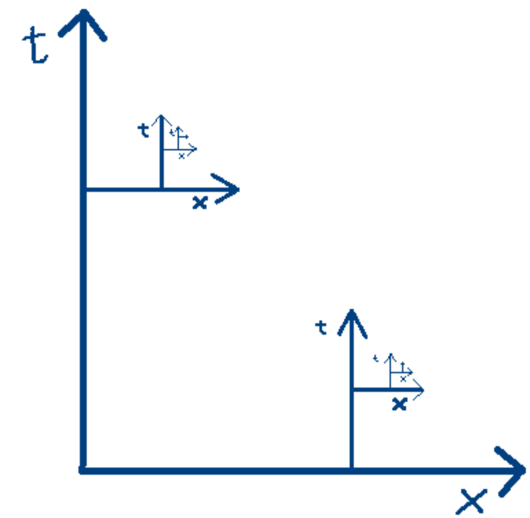
-a space with *dimensional flow*
[dimension changes together with the scale]



Noncommutative spacetime?

Before entering into the talk...

Fractal spacetime?



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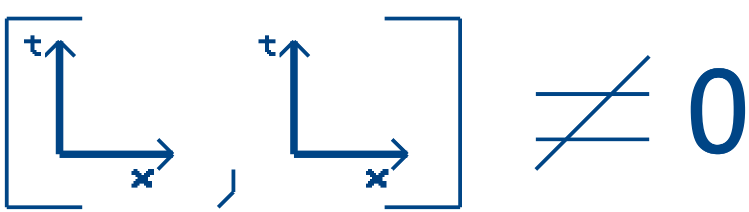
Noncommutative spacetime?



-a space with noncommuting spacetime coordinates such as

$$[x_\mu, x_\nu] = \theta_{\mu\nu}(x)$$

Starting points

1-Dimensional flow in QG

2-Fractional field theories

3-Noncommutative spacetimes

Starting points Dimensional flow in QG

- Independent approaches to quantum gravity predict that physics is **lower (~2) dimensional at short scales**
    - *CDT* [Ambjørn et al. 2005].
    - *QEG* (Asymptotic safety) [Lauscher & Reuter 2005].
    - *HL gravity* [Horava 2008, 2009].
    - *LQG and spin foams* [Modesto et al. 2008-2010].
- At lower energies the dimension runs to 4

Starting points Dimensional flow in QG

Which is the most natural space?

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Which is the most natural space? **MULTIFRACTAL**

we could think to live in a multifractal Universe

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WHICH MATHS?

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not very developed to construct physics on a fractal

in certain continuum approximation

the natural candidate is **FRACTIONAL CALCULUS**

[Calcagni, arXiv:1106.5787]

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- **Key point**: replacing the standard Lebesgue measure in the action with a **Lebesgue-Stieltjies measure** with **anomalous scaling**

$$d^D x \to d\varrho(x)\,, \qquad [\varrho] = -D\alpha \neq -D$$

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- **Multifractal action**

$$S = \int d\varrho(x)\,\mathcal{L}\,, \qquad d\varrho(x) = \sum_\alpha g_\alpha \sum_\omega \prod_\mu d\varrho_{\alpha,\omega}(x^\mu)$$

where

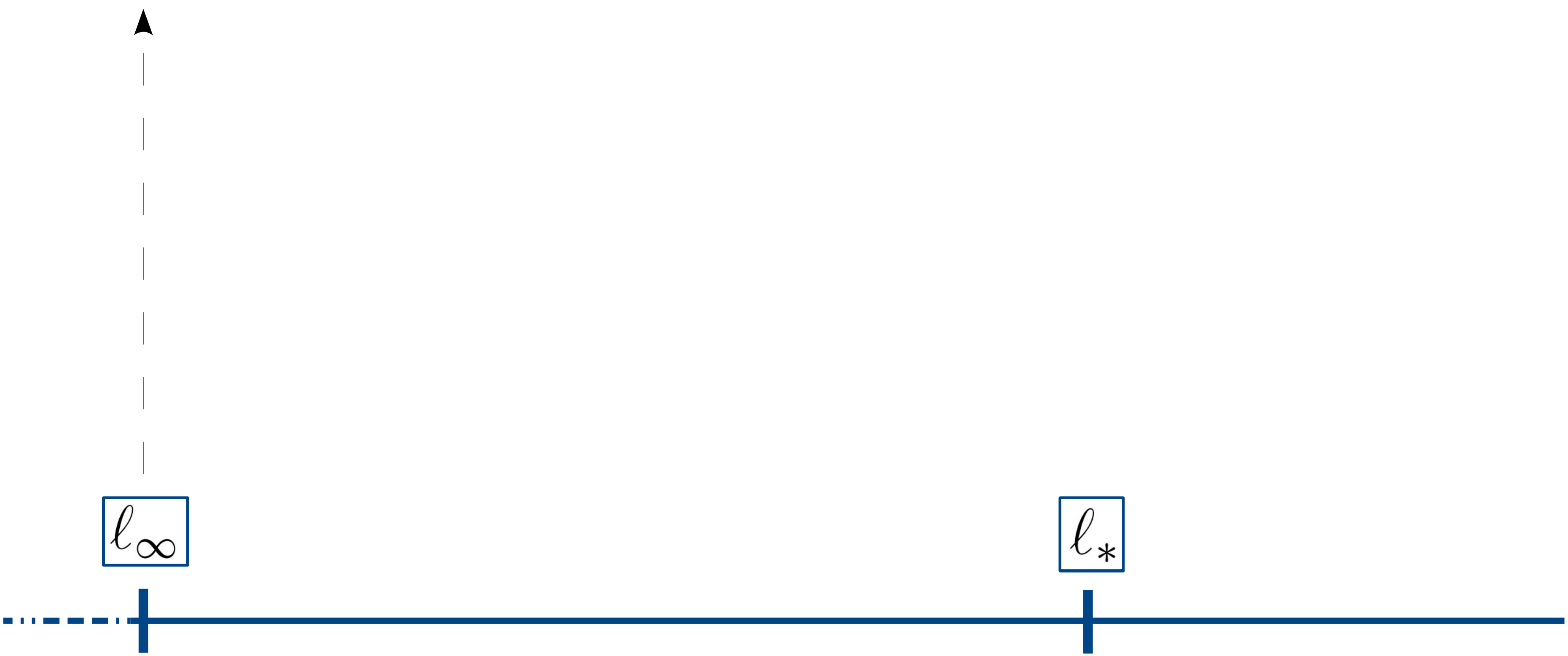
$$\varrho_{\alpha,\omega}(x) = \frac{x^\alpha}{\Gamma(\alpha+1)}\left[1 + A_{\alpha,\omega}\cos\left(\omega \ln \frac{x}{\ell_\infty}\right) + B_{\alpha,\omega}\sin\left(\omega \ln \frac{x}{\ell_\infty}\right)\right]$$

Starting points Fractional field theories [Calcagni 2010-2011]

$\ell \sim \ell_\infty$

FUNDAMENTAL REGIME

$$\varrho(x) \sim \ln x$$



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$\ell_\infty < \ell \ll \ell_*$

OSCILLATORY TRANSIENT REGIME

- Oscillatory measures.
- Notions of volume and dimension are ambiguos.

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MULTIFRACTIONAL REGIME

$$\varrho(x) \sim \sum_\alpha \langle \varrho_{\alpha,\omega}(x) \rangle \propto \sum_\alpha x^\alpha$$

One can take the average of the measure. The scaling of the volume is anomalous and changes with resolution.

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CLASSICAL REGIME

$$\varrho(x) \sim \varrho_1(x) = x$$

Ordinary Poincaré-invariant field theory on Minkowski spacetime is recovered.

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- They could play a role in the description of spacetime at Plack scale [Doplicher et al. 1995, Szabo 2003, Amelino-Camelia 2002]

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- **Nontrivial measures**

  from the **cyclicity requirement** for the NC action
  [Dimitrijevic et al. 2003, Agostini et al. 2006]

$$\mathcal{I}([\hat{f},\hat{g}]) := \int d^D x\, v(x)[f,g]_* = 0$$

Motivations

- Similarities between measures of the action functional in both theories

Fractional measure

$$d\varrho \sim d^D x \prod_\mu (x^\mu)^{\alpha-1}$$

κ-Minkowski measure

$$d\varrho \sim d^D x \prod_\mu (x^\mu)^{-1}$$

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$$d_s = \begin{cases} 4 & \text{for } s \to \infty \\ 3 & \text{for } s \to 0 \end{cases}$$

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- Poincaré algebra is deformed in both theories

The connection

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κ-Minkowski cyclicity-inducing measure !

The fundamental scale of κ-Minkowski (Planck scale ℓ_{Pl}) is identified with ℓ_∞

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Can it come from some noncommutative algebra ?

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MULTIFRACTIONAL REGIME $\ell_\omega \ll \ell \lesssim \ell_*$

It comes from the following nonlinear **fractional algebra**

$$[X_i, T] = i\lambda X_i^{1-\alpha}\,, \qquad [X_i, X_j] = 0$$

found by means of

<u>Jacobian method</u>

coordinate transformation

$$Q_i := \frac{X_i^\alpha}{\alpha}$$

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Interpolating spacetime structure between κ-Minkowski and canonical noncommutativity

-for $\alpha = 0$ κ-Minkowski spacetime

-for $\alpha = 1$ canonical spacetime

found by means of

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Physical picture

mapping at different regimes

$\ell \gg \ell_*$

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Physical picture

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The geometry of spacetime is **κ-Minkowski**

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thank you!

more about...

Dimensional flow in QG	S. Carlip [arXiv::0909.3329]
Fractional field theories	G. Calcagni [arXiv:1107.5041] [arXiv:1106.5787] [arXiv:1106.0295]
NC spacetimes and cyclic properties	A. Agostini et al [hep-th/0407227]
Our work	M. Arzano, G. Calcagni, D. Oriti and MS [arXiv:1107.5308]