Bjorken Flow from an AdS Schwarzschild Black Hole

James Alsup

University of Tennessee
Department of Physics and Astronomy
High Energy Theory Group

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J. Alsup and G. Siopsis, Phys. Rev. Lett. **101** (2008) arXiv:0712.2164
J. Alsup and G. Siopsis, Phys. Rev. D **79**, 066011 (2009) arXiv:0812.1818

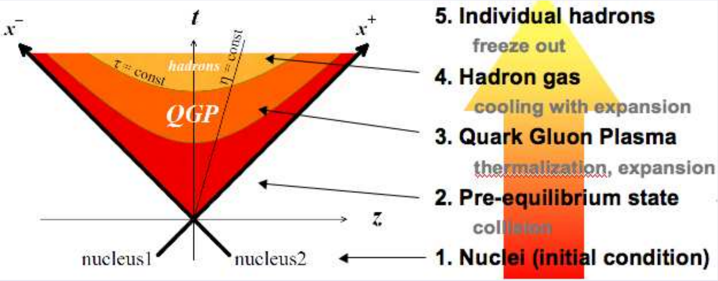
Outline

RHIC seen as Hydrodynamics

Experiment seems to suggest QGP is strongly coupled

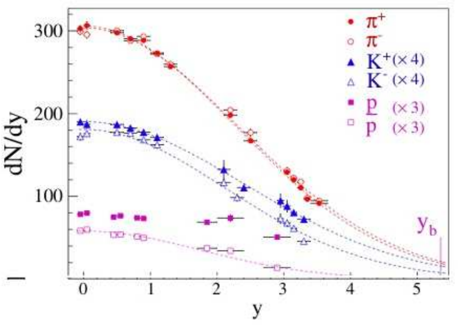
- creating hydrodynamic behavior *[Starinets Lectures, . . .]*

Hierarchy of times, *[diagram from Itakura talk]*



RHIC seen as Hydrodynamics

Bjorken suggested to study the central rapidity region



[Murray, BRAHMS Collaboration]

- "plateau" for particle production, $\frac{dN}{dy} = constant, |y| < 1$
- all particles share the same proper time, τ, and independent of Lorentz frame

RHIC seen as Hydrodynamics

Initial conditions

Time	Energy Density	Temperature	C.O.M Energy
τ_0	ϵ_0	T_0	$\sqrt{s}$

Hydrodynamic equations

- respect symmetry of initial conditions (boost invariance)
- simple solutions from conservation and conformality

$$\nabla_\mu T^{\mu\nu} = 0\,, \quad T^\mu_\mu = 0\,, \quad T_{\mu\nu} = (\varepsilon + p) u_\mu u_\nu + p g_{\mu\nu} + t^{(diss)}_{\mu\nu}$$

$$\varepsilon = 3p = \frac{\varepsilon_0}{\tau^{4/3}} - \frac{2\eta_0}{\tau^2} + \dots,\ T = T_0 \left(\frac{1}{\tau^{1/3}} - \frac{\eta_0}{2\varepsilon_0 \tau} + \dots \right),\ \dots$$

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AdS$_5$ Black Hole

Duality

At $T \sim T_C$ $\mathcal{N} = 4$ SYM and QGP become more similar

- For Bjorken hydrodynamics dual description $\rightarrow$ introduce time dependence and same symmetries into bulk metric

- AdS$_5$ Schwarzschild black hole approximate solution for large longitudinal proper time, τ *[Janik and Peschanski, Janik Lectures]*

$$ds^2 = \frac{1}{\tilde{z}^2}\left(-(1-\frac{2\mu\tilde{z}^4}{\tau^{4/3}})d\tau^2 + \tau^2 dy^2 + (d\tilde{x}^{\perp})^2 + \frac{d\tilde{z}^2}{1-\frac{2\mu\tilde{z}^4}{\tau^{4/3}}}\right)$$

▸ Dual CFT stress-energy tensor follows Bjorken!

- Holographic Renormalization $g_{\mu\nu} \Rightarrow \langle T_{\mu\nu}\rangle$ *[Skenderis Lectures]*

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Time Dependence

Thermodynamics

- Temperature and entropy are well understood for a static black hole
- Concepts change with time dependence

Several approximate solutions have been found *[Heller, Janik, Sin, Nakamura, Kim, Buchel, Benincasa . . .]*

- higher orders in τ *[Janik Lectures]*
  - $\eta/s = 1/4\pi$, relaxation time, . . .
  - break down at 3rd order
- Eddington-Finklestein coordinates *[Hubeny Lectures]*
  - redefinition of expansion parameter τ
  - good to all orders
  - use of apparent horizon

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Static to Dynamic solutions

Exact Solution

Time dependent metric is known exactly in 3 dimensions

- equivalent to static AdS Schwarzschild black hole
  [Kajantie, Louko, Tahkokallio]

Gives rise to 2-dim Bjorken hydrodynamics

- Temperature, entropy are understood from conformal transformation

$\Rightarrow$ but 3-dim gravity is special

Can this be done in other than 3 dimensions? 5-D?

JA and G. Siopsis

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AdS_5 static black hole

AdS Schwarzschild black hole

- Large AdS_5 Schwarzschild black hole exact solution

$$R_{\mu\nu} - \left(\frac{1}{2}R + \Lambda_5\right) g_{\mu\nu} = 0, \quad \Lambda_5 = -6$$

$$ds^2 = \frac{1}{z^2}\left(-(1-2\mu z^4)dt^2 + d\vec{x}^2 + \frac{dz^2}{1-2\mu z^4}\right)$$

- the horizon occurs at

$$z_H = (2\mu)^{-1/4}, \quad \vec{x} \in \mathcal{R}^3$$

- with temperature

$$T_H = \frac{1}{\pi z_+}$$

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AdS_5 dynamic black hole

AdS_5 boundaries

- Two types of boundaries

$$ds^2_{\text{b.h.}} \;\rightarrow\; \frac{1}{z^2}\left(-dt^2 + d\vec{x}^{\,2} + dz^2\right)$$

$$ds^2_{\text{Bjorken}} \;\rightarrow\; \frac{1}{\tilde{z}^2}\left(-d\tau^2 + \tau^2 dy^2 + (d\tilde{x}^{\perp})^2 + d\tilde{z}^2\right)$$

- Instead of $z = \text{const.}$ hypersurfaces at the boundary, slice with $\tilde{z} = \text{const.}$
$\Rightarrow$ gives rise to flowing hydrodynamics

AdS$_5$ dynamic black hole

- For $\tau \to \infty$ with $\tilde{x}^{\perp}$, τy, and $\frac{\tilde{z}}{\tau^{1/3}}$ fixed

$$t = \frac{3}{2}\tau^{2/3}, \quad x^1 = \tau^{2/3} y, \quad x^{\perp} = \frac{\tilde{x}^{\perp}}{\tau^{1/3}}, \quad z = \frac{\tilde{z}}{\tau^{1/3}}$$

Transformed Metric

$$ds^2_{\text{b.h.}} = \frac{1}{\tilde{z}^2}\left[-\left(1 - \frac{2\mu\tilde{z}^4}{\tau^{4/3}}\right)d\tau^2 + \frac{d\tilde{z}^2}{1 - \frac{2\mu\tilde{z}^4}{\tau^{4/3}}} + \tau^2 dy^2 + (d\tilde{x}^{\perp})^2\right] + \mathcal{O}(\tau^{-4/3})$$

$\Rightarrow$ Bjorken dual metric in large τ limit

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Boundary Theory

Thermodynamics

- Flowing thermo. may be calculated at boundaries

$$ds^2_{z\to 0} = \tau^{-2/3}\left[ds^2_{\tilde{z}\to 0} + \mathcal{O}\left(\frac{\tilde{x}^i\tilde{x}^j}{\tau^2}\right)\right]$$

- For conformal transformation in 4 dimensions

$$\tilde{g}^{(boundary)}_{\mu\nu} = e^{-2\phi} g^{(boundary)}_{\mu\nu}$$

$$\Rightarrow \tilde{T} = e^{\phi} T \quad , \quad \tilde{s} = e^{3\phi/2} s$$

- Temperature and entropy of fluid are found as

$$\tilde{T} = \frac{T_H}{\tau^{1/3}} \quad , \quad \tilde{s} = \frac{dp}{dT} = \frac{s_0}{\tau}$$

▶ Bjorken Flow from exact solution of Einstein equations!

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Next-to-leading order

Can viscosity be understood from a Schwarzschild BH?

JA and G. Siopsis

Higher orders

- Respect boost and transverse coordinates invariance
- Introduce terms $\mathcal{O}(1/\tau)$
  - systematically done with Mathematica

$$t = \frac{3}{2}\tau^{2/3} - \mathcal{C}_1 \ln\tau + \frac{f_1(v)}{\tau^{2/3}} \ , \ \ z = \tilde{z}\left(\frac{1}{\tau^{1/3}} - \frac{\mathcal{C}_1}{\tau}\right)$$

$$x^1 = \tau y\left(\frac{1}{\tau^{1/3}} - \frac{\mathcal{C}_1 + b_1(v)}{\tau}\right), x^\perp = \tilde{x}^\perp\left(\frac{1}{\tau^{1/3}} - \frac{\mathcal{C}_1 + c_1(v)}{\tau}\right)$$

- with $v = \tilde{z}/\tau^{1/3}$ kept fixed and $\mathcal{C}_1,\ b_1(v),\ c_1(v)$ to be determined

Next-to-leading order

Dual conformal field theory

- $b_1(v),\ c_1(v)$ can be found with next-to-leading order Einstein equations
  - possible divergences
- Stress-energy tensor and thermo can then be calculated

▶ viscous Bjorken hydrodynamics with

$$\eta_0 = 2\mathcal{C}_1\varepsilon_0 \longrightarrow \eta/s = \frac{3\mathcal{C}_1}{2\pi}(2\mu)^{1/3}$$

No constraint on viscosity transport coefficient
$\Rightarrow$ at next-to-next-to-leading order

Issue of Divergences

Identification of Bulk Divergence

Linear combinations of Riemann tensor components may be formed with a vielbein e_a^A

$$\mathcal{R}_{abcd} = e_a^A e_b^B e_c^C e_d^D R_{ABCD}$$

Divergence is indication of geometry's singular nature

Flowing metric up to the $\mathcal{O}(1/\tau)$ terms $b_1(v), \ldots$ is divergent

$$\mathcal{R}_{0101} = 1 + 2\mu v^4 + \frac{32\mathcal{C}_1\mu^2 v^8}{\tau^{2/3}} \frac{1}{1 - 2\mu v^4}$$

Terms $\mathcal{O}(1/\tau^{4/3})$ contribute to $\mathcal{R}_{abcd}$ which give regularity

- dependent on y and $x^\perp$

$$\Rightarrow \mathcal{R}_{0101} = 1 + 2\mu v^4 - \frac{8\mathcal{C}_1\mu v^4}{\tau^{2/3}}$$

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Next-to-next-to-leading order

Transformation

Alter the transformation to include next-to-next-to-leading order, $\mathcal{O}(\tau^{4/3})$

- introduce $a_2(v), b_2(v), c_2(v), f_2(v)$ and $\mathcal{C}_2$
- y and $x^\perp$ dependence is unavoidable

$\Rightarrow$ Flowing metric must be perturbed by power law perturbation to produce boost invariant flow

$$ds^2_{\text{perturbed}} = ds^2_{\text{b.h.}} - \frac{1}{\tilde{z}^2}\left[\frac{v^2\mathcal{A}(v)}{\tau^{4/3}}d\tilde{z}^2 + 2\mathcal{A}_\mu d\tilde{x}^\mu d\tilde{z}\right]$$

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Solution

- The Einstein equations allow for the solutions to $b_2(v), c_2(v), f_2(v)$
  - $a_2(v)$ remains arbitrary due to $\mathcal{A}(v)$
- The solution has a divergent curvature invariant $\mathcal{R}^2 = R_{ABCD}R^{ABCD}$ at the horizon

$\Rightarrow$ Constraint on $\mathcal{C}_1$

Nonsingular for only

$$\mathcal{C}_1 = \frac{1}{6(2\mu)^{1/4}} \quad \Rightarrow \quad \frac{\eta}{s} = \frac{1}{4\pi}$$

- Equivalent to AdS perturbations and subleading approximate solutions

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Summary

- Ideal Bjorken hydrodynamics found by slicing near the boundary of a large AdS Schwarzschild black hole
- Viscous dissipation accounted for as well
  - Exact solution that asymptotically approaches Bjorken Dual at NLO
  - Has been generalized to d-dim AdS space
- Value for flowing system's viscosity found by perturbing AdS black hole at NNLO

Future study of QNMs around flowing geometry
$\Rightarrow$ thermalization, ellipticity, ...