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# Anti de Sitter black holes

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Abstract: In these introductory notes we define constant curvature spacetimes and discuss their symmetries, basic properties and the construction of their spacetime diagram. We then discuss the important properties of anti de Sitter spacetime giving global and local parametrisations. We study the static black holes and then discuss their basic properties and novel topological effects due to the presence of a negative cosmological constant. Finally we discuss via the Euclidean path integral approach their thermodynamic properties in the canonical ensemble with a heat bath of constant temperature.

## Contents



# 1. Spacetimes of constant curvature

## 1.1 Spaces of maximal symmetry and constant curvature

The simplest vacuum solutions of Einstein's equations with cosmological constant,

$$G_{AB} + \Lambda\, g_{AB} = 0 \tag{1.1}$$

are spacetimes of constant curvature. They are locally characterised by the condition,

$$R_{ABCD} = \frac{R}{(d-1)d}(g_{AC}\, g_{BD} - g_{AD}\, g_{BC}) \tag{1.2}$$

where $d$ is the spacetime dimension. Using (1.1) we see that,

$$G_{AB} = -R\, g_{AB}\frac{d-2}{2d} = -\Lambda\, g_{AB}$$

ie $R = \frac{2d}{d-2}\Lambda$ are the constant curvature solutions of the Einstein equations with cosmological constant. In particular for $\Lambda = 0$ we have flat-Minkowski spacetime, for $\Lambda > 0$ positively curved, de-Sitter spacetime and for $\Lambda < 0$ anti-de-Sitter spacetime. The above three spacetimes are of maximal symmetry and therefore admit the maximal number of Killing vectors. Flat spacetime for example is isometric under the group of Poincaré transformations, in other words

$d$-dimensional Lorentz coordinate transformations plus $d$ translations. This is obvious in the inertial Cartesian system of coordinates. Therefore we have $d + \frac{d(d-1)}{2} = \frac{d(d+1)}{2}$ Killing vectors as generators of these symmetries all together. Since this is the maximal number of Killing vectors (see exercise 1) this spacetime is maximally symmetric. Flat spacetime is defined as the unique space of zero Riemann curvature.

Before moving on to non zero constant curvature spacetimes it is useful to classify the maximally symmetric $n = d-1$-spacelike sections. They are respectively representing locally Euclidean, spherical and hyperbolic sections and their line element can be written in a compact fashion as,

$$dK_n^2 = \frac{d\chi^2}{1-\kappa\chi^2} + \chi^2 d\Omega_{n-1}^2 \tag{1.3}$$

for $\kappa = 0, 1, -1$ respectively. The spherical line element $d\Omega_{n-1}^2$ ($n > 1$) is given by the iterative relation,

$$d\Omega_k^2 = d\theta_k^2 + \sin^2(\theta_k) d\Omega_{k-1}^2, \ldots \quad , d\Omega_1 = d\theta_1, \quad \theta_k \in [0, \pi[, ..., \theta_1 \in [0, 2\pi[, \qquad k = 1, ..., n-1 \tag{1.4}$$

Setting $\chi = \sin\phi$ or $\chi = \sinh\phi$ in (1.3) for $\kappa = 1, -1$ respectively gives us the usual line element for the unit sphere and hyperboloid,

$$d\Omega_n^2 = d\phi^2 + \sin^2\phi d\Omega_{n-1}^2, \quad \phi \in [0, \pi[ \tag{1.5}$$
$$dH_n^2 = d\psi^2 + \sinh^2\psi d\Omega_{n-1}^2, \quad \psi \in [0, +\infty[ \tag{1.6}$$

## 1.2 Flat spacetime

### 1.2.1 Conformal space-time diagram

Let us now turn our attention to flat spacetime and in particular to its conformal spacetime diagram [1]. One can write $d$ dimensional Minkowski spacetime (1.3) as,

$$ds^2 = -dt^2 + dr^2 + r^2 d\Omega_{n-1}^2 \tag{1.7}$$

in a spherical coordinate system. Note that we have to take two copies of this to obtain the whole of (cartesian) Minkowski spacetime, $r > 0$ and $r < 0$. To obtain the structure of flat spacetime at infinity [1] we will go to a spacetime metric which is conformally equivalent,

$$g_{AB}(x) = \Omega^2(x) g_{AB}^{mink}$$

to the initial one (1.7). The structure we obtain at infinity will be common to all asymptotically flat spacetimes. This is due to the fact that a conformal transformation will shrink or stretch spacetime but will not alter the null cones and therefore the asymptotic properties of spacetime. The main idea therefore is to use conformal transformations in order to bring asymptotic infinities to finite values for the conformally transformed metric. This is the central idea of Carter-Penrose diagrams.

To achieve this for flat spacetime we first pass to null retarded and advanced coordinates, $u = t - r$, $v = t + r$, where $u$ or $v = constant$ correspond to null geodesics of (1.7). Now we can bring coordinate infinity to finite values via the coordinate transformation, $\tan\tilde{U} = u$, $\tan\tilde{V} = v$ with $-\frac{1}{2}\pi < \tilde{U}, \tilde{V} < \frac{1}{2}\pi$. Go once again to time-space like coordinates $\tilde{t} = \tilde{U} + \tilde{V}$, $\tilde{r} = \tilde{V} - \tilde{U}$. Then (1.7) is conformally equivalent to

$$d\tilde{s}^2 = -d\tilde{t}^2 + d\tilde{r}^2 + \sin^2\tilde{r} d\Omega_{n-1}^2$$

with conformal factor, $\Omega = \sec(\tilde{U})\sec(\tilde{V})$, *i.e.*,$ds^2 = \Omega^2 d\tilde{s}^2$. Its important to keep track of the coordinate ranges,

$$-\pi < t' + r' < \pi, \qquad -\pi < t' - r' < \pi, \qquad r' > 0 \tag{1.8}$$

We can now draw the Carter-Penrose spacetime diagram for flat spacetime. The domain defined by (1.8) is a triangle with its boundary defining asymptotic infinity of flat spacetime and with $r' = 0$ as a mirror for the other portion $r' < 0$. It will correspond to a diamond shaped region of the Einstein cylinder. In the diagram $\Im^-$ and $\Im^+$ stand for null past infinity and future null infinity respectively, whereas $i^-$, $i^+$ are the endpoints of timelike geodesics $r = constant$. $i^0$ is that of spacelike geodesics $t = constant$. A Cauchy surface is a spacelike surface intersecting all null and timelike inextensible geodesics and will inevitably touch $i^0$ asymptotically. A Cauchy surface is therefore a valid set of initial data. Note that a non geodesic curve can reach null infinity $\Im^+$ if it is uniformly accelerated approaching the speed of light as $t \to +\infty$. The tangent curve emanating from $C$ at $\Im^+$ is the acceleration horizon of the worldline $C$. No events above this line can be witnessed by the observer. Concrete simple examples are those involving Rindler and Milne spacetimes which we briefly turn to now.

### 1.2.2 Rindler and Milne spacetimes

The line element for Rindler spacetime is given by,

$$ds^2 = -x_R^2 dt_R^2 + dx_R^2 + x_R^2 \cosh^2 t_R d\Omega_{n-1}^2, \qquad x_R > 0 \tag{1.9}$$

and we immediately note that for $x_R = 0$ we have a singularity. Direct calculation of the Riemann tensor for this spacetime gives us identically zero *i.e.*,we have a coordinate (and not curvature) singularity and furthermore Rindler spacetime is just a coordinate patch of flat spacetime. Indeed the coordinate transformation relating it to (1.7) is given by,

$$\tanh t_R = \frac{t}{r} = \frac{\sin(\tilde{V} + \tilde{U})}{\sin(\tilde{V} - \tilde{U})}, \qquad x_R = \sqrt{r^2 - t^2} = \sqrt{-\tan\tilde{U}\tan\tilde{V}} \tag{1.10}$$

We can see that Rindler spacetime covers only part of the global spacetime diagram since $\tilde{U}\tilde{V} < 0$. The lines $\tilde{V} = 0$ and $\tilde{U} = 0$ corresponding to, $x_R = 0$ and $t_R \sim \pm\infty$, are event horizons for the observer with coordinate time $t_R$ and trajectory $x_R = constant$. An observer with worldline $x_R = constant$ is in uniform acceleration $a = \frac{1}{x_R}$ since his trajectory in the original inertial Minkowski coordinates are hyperbolas (1.10). The inverse transformation reads,

$$t = x_R \sinh t_R, \qquad r = x_R \cosh t_R$$

To get Milne spacetime we can consider,

$$x_R \to it_M, \quad t_R \to x_M + \frac{i\pi}{2},$$

This transformation, is one involving complex coordinates, however it maps us to a real metric which is a different portion of flat spacetime ($\tilde{U}\tilde{V} > 0$). It is Milne spacetime,

$$ds^2 = -dt_M^2 + t_M^2 dH_n^2, \qquad t_M > 0 \tag{1.11}$$

which is now a cosmological or time dependent type of metric. Notice it has no initial big bang singularity. An observer of Milne spacetime has rather a cosmological horizon at $t_M = 0$ and has Hubble expansion rate $H = \frac{1}{t_M}$ in proper time. Note that (just like in de Sitter spacetime) there is no horizon problem in this cosmological metric due to the fact that Milne spacetime is free of curvature singularities.

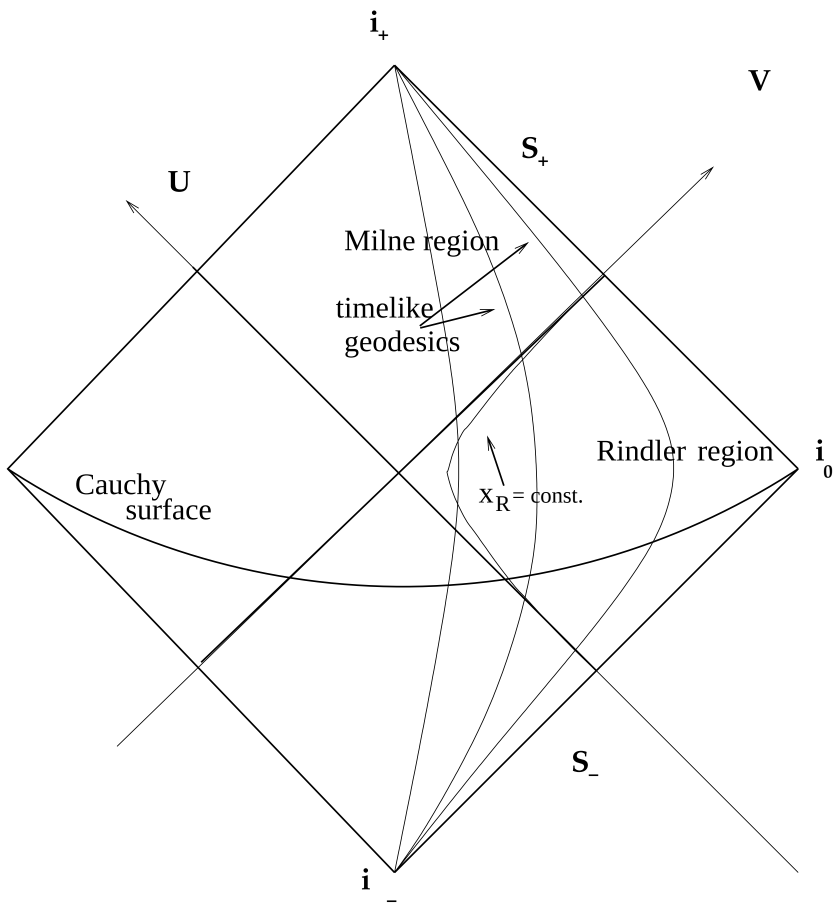


**Figure 1:** Spacetime diagram of Minkowski spacetime. The quarter diamonds are the regions of Rindler and Milne spacetimes.

### 1.3 Anti de Sitter spacetime

#### 1.3.1 Definition, boundary and isometries

Useful and elegant representations of de-Sitter and anti de Sitter spacetimes are obtained by embedding $d$ dimensional hypersurfaces in $d+1$ dimensional flat spacetime. Then de Sitter spacetime of curvature scale $a$ is defined as the hyperboloid,

$$-X_0^2 + \sum_{i=1}^{d} X_i^2 = a^2 \tag{1.12}$$

embedded in $d+1$ dimensional Minkowski spacetime. De-Sitter space is topologically $R \times S^{D-1}$ and thus its spatial sections are compact (for a full analysis see [1] or [3]). We will focus on adS space from now on.

AdS spacetime in turn is obtained by considering the hyperboloid,

$$X_0^2 + X_d^2 - \sum_{i=1}^{d-1} X_i^2 = l^2 \tag{1.13}$$

of radius of curvature $l > 0$ embedded in the spacetime,

$$ds^2 = -(dX_0^2 + dX_d^2) + \sum_{i=1}^{d-1} dX_i^2 \tag{1.14}$$

where note the double time coordinates. Clearly, any element of the Lorentz group $SO(2, d-1)$ will leave (1.14) and (1.13) unchanged (by construction). Also we see that translation invariance is broken by (1.13). Since $SO(2,d-1)$ has $\frac{d(d+1)}{2}$ Killing generators just like flat spacetime, which is of maximal symmetry, $SO(2,d-1)$ is the precise isometry group of adS. A second important point is that adS spacetime has a conformal boundary[1] at infinity. To see this we rescale all coordinates by $X_A \to X_A\lambda$ and take $\lambda \to \infty$. This limit defines the boundary as,

$$X_0^2 + X_d^2 - \sum_{i=1}^{d-1} X_i^2 = 0, \tag{1.15}$$

$$X_A = X_A\lambda, \qquad i = 1..d+1 \tag{1.16}$$

Suppose first that $X_0 \neq 0$: we can divide by $X_0$ and then rescale. We therefore have that the boundary verifies,

$$-X_d^2 + \sum_{i=1}^{d-1} X_i^2 = 1$$

which is a hyperboloid in $d-1$ dimensions. This is just $d-1$ dimensional de Sitter space according to (1.12). The topology is that of $R \times S^{d-2}$ Alternatively, if $X_0 = 0$ we have a sphere in $d-2$ dimensions (times a point). Adding the two spaces together the boundary is a maximally symmetric space $S^1 \times S^{d-2}$. Since the adS isometry acts on this space the boundary preserves $SO(1,d-2)$ symmetry. Note however that additionally we still have $d$ extra dilatation transformations (1.16) for the boundary metric. All in all the boundary of adS admits as many symmetries as adS space.

### 1.3.2 Parametrisations

Let us now construct line elements for adS space in $d$ dimensions. A global parametrisation is constructed as follows. Given the form of (1.13) we consider two spheres $X_0^2 + X_d^2 = r_1^2$, $\sum_{i=1}^{i=d-1} X_i^2 = r_2^2$ of radii $r_1$, $r_2$ such that

$$r_1^2 - r_2^2 = l^2 \tag{1.17}$$

This equation is solved setting $r_1 = l\cosh(u/l)$, $r_2 = l\sinh(u/l)$ where $u \in [0,+\infty[$. Now we just take the relevant parametrisations in polar spherical coordinates, (1.3) for $\kappa = 0$, and replace them in the line element (1.14),

$$\begin{aligned} ds^2 &= -(dr_1^2 + r_1^2 d\psi^2) + dr_2^2 + r_2^2 d\Omega_{d-2} = \\ &= -l^2\cosh^2(u/l)d\psi^2 + du^2 + l^2\sinh^2(u/l)d\Omega_{d-2}^2 \end{aligned} \tag{1.18}$$

This is the global parametrisation of adS since all points of the hyperboloid are taken into account exactly once. This metric is solution to the Einstein equations with cosmological constant,

$$2\Lambda = -\frac{(d-1)(d-2)}{l^2}. \tag{1.19}$$

Note that the timelike coordinate $\psi$ is an angular coordinate $\psi \in [-\pi,\pi[$. This signifies that adS is a spacetime with closed timelike curves! We can however get around this; since the space is not simply connected (ie the time circle cannot be topologically reduced to a point) we can unwrap the circle of the time coordinate and take a new coordinate $t \in ]-\infty,+\infty[$ with $\mathrm{t} \equiv \psi$ in each $2\pi$-interval. This means that we are effectively taking infinite copies of the hyperboloid. This is the universal covering of adS space,

$$ds^2 = -l^2\cosh^2(u/l)d\mathrm{t}^2 + du^2 + l^2\sinh^2(u/l)d\Omega_{D-2}^2 \tag{1.20}$$

[1]The boundary is called conformal for it admits not one but an equivalence class of metrics which are related via an overall conformal transformation, $g_{\mu\nu}^{boundary} = \Omega\gamma_{\mu\nu}^{boundary}$

which is however not a Cauchy space. In other words a Cauchy surface presents no longer sufficient initial data to describe the entire space. Data on the boundary of adS have to be specified. We will come back to this when studying the spacetime diagram. Note in passing that $\partial_u$ is not a Killing vector for adS. In adS space we miss a global spacelike Killing vector generating translational invariance in $u$ (just like in de-Sitter space we do not have a global timelike Killing vector). Taking $u = u_0$ constant with $u_0$ large we see that,

$$ds^2 \sim l^2 e^{\frac{2u_0}{l}} \left(-dt^2 + d\Omega_{d-2}^2\right)$$

that the geometry of the boundary is indeed topologically $R \times S^{d-2}$ for the Universal covering of adS. Had we considered $\psi$ we would have got, $S^1 \times S^{d-2}$.

Let us now consider a local parametrisation defined by,

$$\begin{aligned} y = l \ln \frac{X_d + X_{d-1}}{l}, \qquad & t = \frac{X_0}{X_d + X_{d-1}}, \\ x_i = \frac{X_i}{X_d + X_{d-1}}, \qquad & i = 1...d-2 \end{aligned} \tag{1.21}$$

This parametrisation covers only half of the hyperboloid since we must have $X_d + X_{d-1} > 0$ in order for (1.21) to be well defined. In order to find the relevant line element we must invert (1.21). Using (1.13) it is easy to show that,

$$X_d + X_{d-1} = l e^{\frac{y}{l}}, \qquad X_d - X_{d-1} = l e^{\frac{-y}{l}} \left(1 + e^{\frac{2y}{l}} \frac{\sum_{j=1}^{d-2} x_j^2 - t^2}{l^2}\right)$$

Taking the sum and the difference we then obtain the inverse transformation,

$$\begin{aligned} X_{d-1} &= l \sinh(\frac{y}{l}) - \frac{l}{2} e^{\frac{y}{l}} (\mathrm{x}^2 - t^2) \\ X_0 &= l t e^{\frac{y}{l}}, \qquad X_i = l x_i e^{\frac{y}{l}} \\ X_d &= l \cosh(\frac{y}{l}) + \frac{l}{2} e^{\frac{y}{l}} (\mathrm{x}^2 - t^2) \end{aligned} \tag{1.22}$$

Inserting into (1.14) we now get the desired line element,

$$ds^2 = l^2 e^{\frac{2y}{l}} (-dt^2 + d\mathrm{x}^2) + dy^2 \tag{1.23}$$

with $y \in ]-\infty, +\infty[$ measuring proper distance. Note that by rescaling the $(t, \mathrm{x})$ coordinates we can get rid of the $l^2$ factor. Poincaré coordinates are obtained by setting $r = l e^{\frac{y}{l}}$. We obtain,

$$ds^2 = r^2(-dt^2 + d\mathrm{x}^2) + \frac{l^2 dr^2}{r^2} \tag{1.24}$$

with $r \geq 0$. Again note from (1.21) that Poincaré coordinates cover only half of the hyperboloid. To cover all of $adS$ we take two portions $r > 0$ and $r < 0$. Note that the boundary is attained at $r \to \infty$ whereas we have a horizon at $r = 0$ due to the fact that we cannot cover all of adS space. This is a degenerate (no temperature) Killing horizon associated to the Poincaré flat slicing of adS space. Finally a proper time parametrisation of adS is given by the line element,

$$ds^2 = -d\tau^2 + l^2 \cos^2 \tau dH_{d-1}^2 \tag{1.25}$$

This coordinate system is only defined for $\tau \in ]-\frac{1}{2}\pi, \frac{1}{2}\pi[$. Keeping spatial sections constant it gives us the timelike geodesics of adS which are obviously going to be periodic in $\tau$.

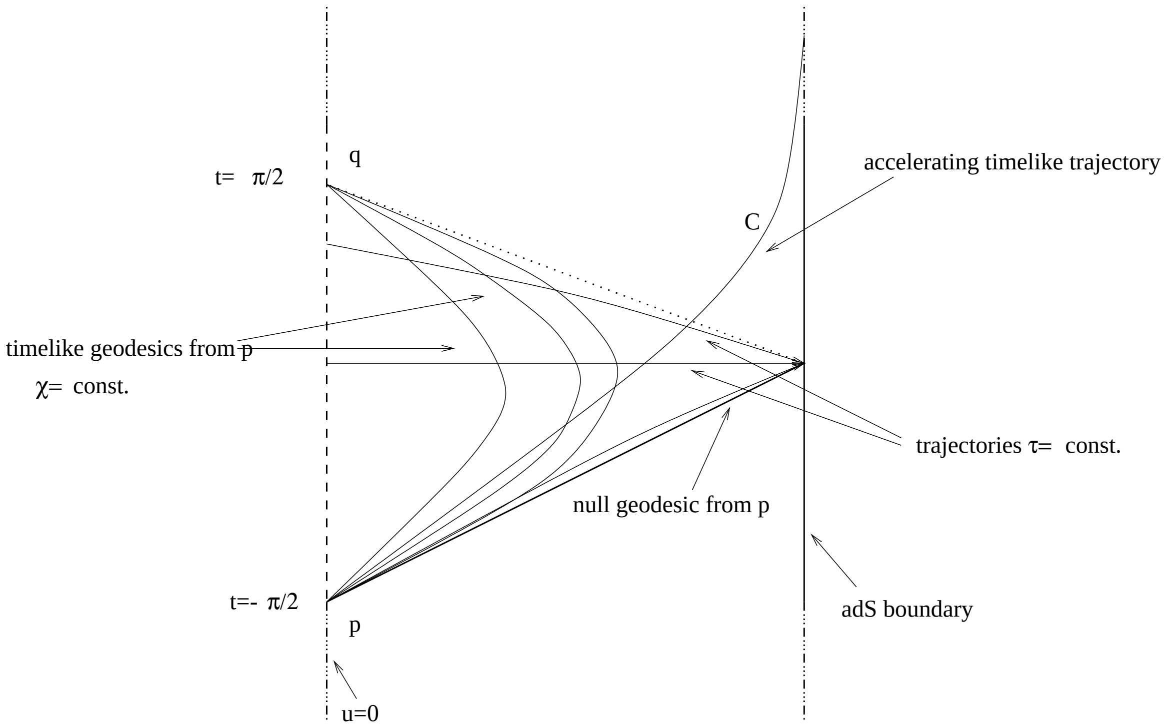


**Figure 2:** Spacetime diagram with timelike geodesics of adS spacetime (1.25). All timelike geodesics from $\tau = 0$ focus in the past at $p$ and and in the future at $q$. The resulting "geodesic" triangular region covers only part of the timelike future of $p$. Also a Cauchy surface at $\tau = 0$ covers only the triangular region in between $p$ and $q$ and their null past and future respectively. Any event beyond this triangle is not causally connected to $\tau = 0$ unless suitable boundary conditions are imposed at the adS boundary. The accelerating timelike curve $C$ gets to see beyond this region (and ends up at the boundary).

### 1.3.3 Spacetime diagram

Let us turn our attention now to the spacetime diagram for adS. Our starting point is the global coordinate system (1.20). As before we write the metric in a conformally flat form and define novel coordinates so as to bring infinity of the radial coordinate to a new finite coordinate value. Time however is either periodic in $\psi$ or infinite in $t$. This amounts to solving $d\theta = \frac{du}{l \cosh \frac{u}{l}}$ and therefore considering the coordinate transformation,

$$\tan(\frac{\theta}{2} + \frac{\pi}{4}) = e^{\frac{u}{l}} \tag{1.26}$$

with $\theta \in [0, \frac{\pi}{2}[$. The line element is,

$$ds^2 = \frac{l^2}{\cos^2 \theta} \left( -dt^2 + d\theta^2 + \sin^2 \theta d\Omega_{n-1}^2 \right) \tag{1.27}$$

and it is conformally equivalent to a quarter of the Einstein cylinder (1.7). The difference here is that $\theta \in [0, \frac{\pi}{2}[$ rather than $\theta \in [0, \pi[$ for flat space. Also note that we cannot make a conformal transformation which brings time infinities to finite values for then the conformal factor explodes. Therefore the spacetime diagram consists of an infinite strip of length $\pi$ (for the Universal covering). The boundary resides at $\theta = \frac{\pi}{2}$, the endpoint of both future and past null geodesics. Note that timelike infinity in the past $i^-$ or future $i^+$ is infinite. As a consequence there is no finite Cauchy spacelike surface at all in this space. This is because although any constant $t_0$ surface, $X$ say, covers the whole of adS at $t = t_0$, however, we can find null surfaces that never intersect $X$. This means that Universal adS is not globally hyperbolic: Cauchy data on arbitrary spacelike surface $X$, determines the system's evolution only in a region bounded by a null hypersurface (called a Cauchy horizon). Physics on adS depends also on the boundary conditions imposed at the boundary. Secondly all timelike geodesics emanating from $t=0$ focus at point $q$ and diverge at $p$ never reaching the boundary of adS. Unlike de-Sitter space which inflates spacetime anti-de-Sitter never allows free-falling particles to escape to the boundary. There are actually regions in the future of $p$ which can never be reached by any geodesic as is shown in the figure but only by accelerated observers as the curve $C$ in (2). This is a sign that the gravitational potential in adS will be

Poincaré coordinates cover the region $r=\frac{l}{\cos\theta}(\cos t+\sin\theta)>0$ which gives us the triangular region in the spacetime diagram. Note that $r=const$ worldlines are now accelerating trajectories and as a result we have the presence of an adS horizon at $r\sim\infty$. The Poincaré patch is in this sense similar to the Rindler patch in flat spacetime.

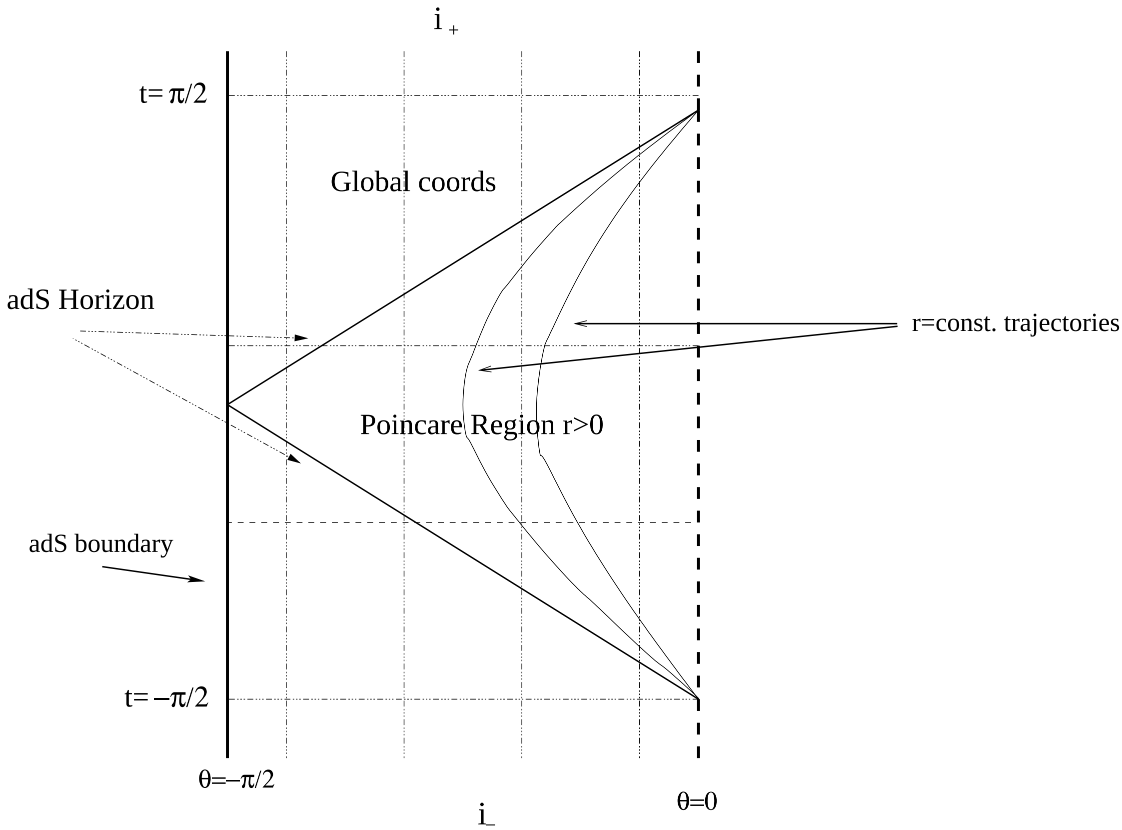


**Figure 3:** Spacetime diagram of adS spacetime

**Exercise 1**: Show that for a Killing vector we have,

$$\nabla_A\nabla_B\xi_C=-R_{BCA}{}^D\xi_D$$

Deduce that any Killing vector can be obtained by its values $\xi^A$ and $\nabla_A\xi_B$ at any point $P\in\mathcal{M}$. From Killing's relation deduce therefore that there are at most $\frac{d(d+1)}{2}$ linearly independent Killing vectors in $\mathcal{M}$ and hence at most $\frac{d(d+1)}{2}$ isometries of the metric (see [2]).

**Exercise 2** The constant curvature slicings of adS are given by the following line element,

$$ds^2=-(\kappa+\frac{r^2}{l^2})dt^2+\frac{dr^2}{\kappa+\frac{r^2}{l^2}}+r^2dK^2_{n-1} \qquad (1.28)$$

For $\kappa=0$ we obtain the flat slicing already encountered (1.24). Show that the spherical slicing is nothing but that of global coordinates (1.20). Find the coordinate transformation relating the flat and spherical parametrisations. Find then the region of validity of the Poincaré coordinates.

Show that the hyperbolic slicing gives a time dependent version of adS. Find its domain of validity.

**Exercise** Show that the global coordinate system for de-Sitter space is,

$$ds^2 = -dt^2 + a^2 \cosh^2(\frac{t}{a}) d\Omega_n^2 \tag{1.29}$$

Therefore there is no global timelike Killing vector in de-Sitter space. Construct the Poincaré slicing and the Carter-Penrose diagram for de-Sitter space (see [1]).

# 2. Static black holes

## 2.1 Basic properties

Assume a $d = n + 1$ dimensional spacetime such that it has $n - 1$ dimensional sections of constant curvature given by the line element (1.3). Then one can show that the general solution of Einstein's equations with cosmological constant admits a locally timelike Killing vector. This is a slightly generalised version of Birkhoff' s theorem for vacuum spacetime. The solution reads (see [8], [9])

$$ds^2 = -V_\kappa(r) dt^2 + \frac{dr^2}{V_\kappa(r)} + \frac{r^2}{l^2} dK_{n-1}^2 \ , \tag{2.1}$$

where

$$V_\kappa(r) = \kappa - \frac{\mu}{r^{n-2}} + \frac{r^2}{l^2} \ , \tag{2.2}$$

where the $(n-1)$ dimensional metric $dK_{n-1}^2$ is given by (1.3) and the adS curvature length $l$ is related to the cosmological constant (1.19). $\mu$ is an integration constant which, as we will see, is associated to the black hole mass [11].

Unlike de Sitter or flat spacetime, black holes in adS [4] will exist for all 3 values of $\kappa$. In other words black hole horizons do not have only spherical topology in adS, we can have toroidal or even hyperbolic black holes. The horizons have to undergo topological identifications so as to make the horizon surface compact. For flat horizons one gets a torus or non orientable surfaces such as the Klein bottle. For a locally $H^2$ horizon (in a 4 dimensional hyperbolic black hole) one considers a quotient space $\Sigma = H^2/\Gamma$ by a discrete subgroup $\Gamma$ made of discrete boosts of $SO(2,1)$ which is as we saw earlier on the symmetry group of $H^2$. Then $\Sigma$ is a compact space of genus $g$. The compact space has $4g$ sides and the sum of its angles has to give an overall angle of $2\pi$ in order to avoid conical singularities. The fundamental domain is a polygon whose edges are geodesics of $H^2$. The simplest case is a regular hyperbolic octagon with opposite edges identified [13]. Often these black holes are referred to as topological black holes [13], [8] due to the identifications one has to undergo in order to compactify the horizon geometries. In fact, any Einstein space of dimension $n - 1$ can form a horizon for an adS black hole [8].

The solutions for $k = +1$ are sometimes called "Schwarzschild-adS" solutions because they reduce to the standard Schwarzschild solution when the cosmological constant vanishes, $l \to \infty$, and to adS in global coordinates when $\mu = 0$. In fact, these are the real asymptotically adS solutions precisely because asymptotically we recover the full adS space. Moreover, their topology is $\mathbb{R}^2 \times S^{n-1}$, and the horizon is the sphere $S^{n-1}$, like that of the Schwarzschild solution. The case $\kappa = 0$ appears at the near–horizon limit of (non-dilatonic) $p$-branes and their horizon has the geometry of $\mathbb{R}^{n-1}$.

Given that these black holes are static their horizons are the zeros of the potential. We will denote their outermost event horizon by $r = r_+$, $V(r_+) = 0$. This event horizon has typically a non zero temperature which can be calculated the standard Euclidean way: Indeed consider $t \longrightarrow i\tau$. We have

$$ds^2 = V(r)d\tau^2 + \frac{dr^2}{V(r)} + r^2 dK_{n-1}^2 \tag{2.3}$$

and the metric is then of Euclidean signature for $r > r_+$. This can be seen by expanding around $r = r_+$,

$$ds^2 \sim \left(\tfrac{1}{4}V_{r_\pm}'^2\right)\rho_\pm^2 d\theta^2 + d\rho_\pm^2 + ... \tag{2.4}$$

with radial isotropic (or cylindrical) coordinate $\rho_\pm = \sqrt{\frac{2(r-r_\pm)}{|V_{r_\pm}'|}}$. Clearly in order to evade a conical singularity at the origin of the axis,$r = r_+$ we must impose the periodicity,

$$\beta = \frac{4\pi}{|V'(r_+)|} \tag{2.5}$$

As we will see, the Euclidean quantum field propagator, with the imposed periodic boundary conditions, describes a canonical ensemble of states in thermal equilibrium at a heat bath of temperature $T = \beta^{-1}$ [5] where,

$$\beta = \frac{4\pi l^2 r_+}{nr_+^2 + \kappa(n-2)l^2}\ , \tag{2.6}$$

This relation can be inverted to find

$$r_+ = \frac{2\pi l^2}{n\beta}\left[1 \pm \sqrt{1 - \kappa\frac{n(n-2)\beta^2}{4\pi^2 l^2}}\ \right] \tag{2.7}$$

which allows us to take $\beta$ as the parameter that determines the solution For $k = +1$ the minus branch for $r_+$ exists, which corresponds to small black holes in adS. Notice that in the limit where $r_+ \gg l$ the $k = \pm 1$ classes of solutions approach the planar black hole class $k = 0$. This admits an interpretation in terms of an "infinite volume" limit, in which the curvature radius of $S^{n-1}$ or $H^{n-1}$ is much larger than the thermal wavelength of the system [6].

Setting $\mu = 0$ we recover differing patches of adS space. For $\kappa = 0$ we obtain the Poincaré patch which covers only part of adS. In this case $r \longrightarrow \infty$ is the boundary of adS whereas $r = 0$ is a horizon. This is a degenerate Killing horizon and there is no temperature associated with it (This is also true for the $\kappa = 1$ case). They are then the ground states for their respective class of solutions parametrised by $\mu$. For $\kappa = -1$ however, we have a bifurcate Killing horizon at $r_+ = l$ with $r > l$. Again the hyperbolic slicing covers a yet smaller triangular portion of adS but the horizon in question has temperature, $\beta = 2\pi l$. This patch is very similar to the Rindler patch of flat spacetime. Therefore in this case it is not clear what is the ground state. We stress however that the $\mu = 0$ case is the only one which has no curvature singularity. This is an important question since when calculating the partition function it is important to specify the background solution with which to annihilate divergences. AdS/CFT is capital in resolving this and we will come back to this in a moment [11].

Furthermore, for the $\kappa = -1$ class of black holes [9], and in contrast to the $\kappa = +1, 0$ classes, the zero temperature solution exists and is different from the one that is isometric to adS. In fact, for $\kappa = -1$ there is a range of negative values for $\mu$ such that the solutions still possess regular horizons. The minimum values of $\mu$ and $r_+$ that are compatible with cosmic censorship, for which the horizon is degenerate, are

$$\mu_{\rm ext} = -\frac{2}{n-2}\left(\frac{n-2}{n}\right)^{n/2} l^{n-2}\ , \qquad r_{\rm ext} = \sqrt{\frac{n-2}{n}}\,l\ , \tag{2.8}$$

and, in particular,

$$\mu_{\rm ext} = -\frac{l^2}{4} \ , \qquad r_{\rm ext} = \frac{l}{\sqrt{2}}, \qquad \text{for } n=4 \ . \tag{2.9}$$

For these values of the parameters, the black hole is extremal. The Penrose diagram for a hyperbolic black hole with negative $\mu$ is like that of a Reissner-Nordström-adS black hole. For positive $\mu$ it is instead like that of a Schwarzschild-adS black hole [13].

### 2.2 Path integral formulation, thermodynamics and the Hawking-Page phase transition

The thermodynamics of adS black holes pocess many interesting properties which are rather different from their asymptotically flat or de Sitter cousins. We will use the path integral method developed by Hartle and Hawking [14] in order to calculate the partition function and then, using standard thermodynamic formulas, the basic thermodynamic quantities for adS black holes. Already as we mentioned above the $\kappa = -1$ case presents subtleties due to the fact that the background adS solution possesses a temperature at $r_+ = l$ whereas at the same time there is an extremal black hole of zero temperature. So which background should we use? The adS/CFT correspondence cures this ambiguity by providing through the CFT the correct geometric counter-terms. These are used to cancel out the infinities and provide a background independent way to calculate the partition function [10], [12] [11]. In order to avoid confusion we will concentrate on pre adS/CFT methods and leave this as one of the many motivations for adS/CFT to be studied later on. We therefore concentrate here on the $\kappa = 1$ case (for the conserved charges see [16]) giving in particular a summary of the Hawking-Page phase transition.

We will consider a canonical ensemble with a constant temperature heat bath. In asymptotically flat space although a black hole can be in equilibrium with thermal radiation at some constant temperature $T_0$ this equilibrium is unstable once gravitational corrections are taken into account. In other words, if the mass of the black hole increases its temperature decreases. This means that the canonical ensemble, where the black hole is in thermal equilibrium with a heat bath of constant temperature, is ill defined in this case. In adS space however the thermal radiation remains confined close to the black hole since the gravitational potential, $V \sim r^2/l^2$ increases for large $r$. Non-zero rest mass particles are confined and prevented from escaping to infinity and one can consider a canonical ensemble description for given temperature $T$. Effectively, though volume is infinite adS provides a gravitational box. The canonical ensemble partition function is defined via a path integral [15] that takes us from a given configuration $S$ $(S, g, \phi) \longrightarrow (S', g', \phi')$ to $S'$ via all possible paths. These paths represent matter fields and metrics flowing to zero and adS respectively in periodic time $\tau$ with period $\beta$ (2.5). Here $\phi$ represents collectively matter fields and $g$ Euclidean metrics.

$$\langle g', \phi', S' | g, \phi, S\rangle = \int D(\phi, g) \exp\left(-I[\phi, g]\right) \tag{2.10}$$

The path integral is taken in Euclidean signature in order for the amplitude to be an elliptic operator with an exponential damping factor rather than an oscillating one. All defined above possible configurations are allowed but one can expect that regular classical solutions, ie saddle points of the action, are going to give the dominant contributions to this integral and in particular to the partition function $Z = e^{-I}$ where,

$$I_{sol} = -\frac{1}{16\pi G}\int d^d x\sqrt{g}(R - 2\Lambda) + BT = \frac{d-1}{8\pi G}\int d^d x\sqrt{g} = \frac{d-1}{8\pi G}\, Vol \tag{2.11}$$

where the boundary term (BT) can be neglected for adS (unlike flat space). The volume integral is infinite, both for adS space and the black hole in question. We therefore resort to [7] considering an upper cut-off $r < R$, subtracting the two volume integrals and then taking the limit $R \to \infty$. Let us take a closer look [6]. For adS and the black hole we have respectively,

$$Vol_1(R) = \int_0^{\beta_0} d\tau \int_0^R dr \int_{S^{d-2}} d\Omega r^{d-2} \tag{2.12}$$

$$Vol_2(R) = \int_0^{\beta} d\tau \int_{r_+}^R dr \int_{S^{d-2}} d\Omega r^{d-2} \tag{2.13}$$

and note the boundary differences in the two integrals. Although the period $\beta$ for the black hole is fixed by (2.6) for adS space it is arbitrary, $\beta_0$. We therefore fix $\beta_0$ so that the temperature of both configurations is the same at $r = R$. Remember we are assuming a common heat bath for the canonical ensemble at temperature $T$. This sets,

$$\beta_0 = \frac{\beta\sqrt{1 + \frac{R^2}{l^2} - \frac{\mu}{R^{d-3}}}}{\sqrt{1 + \frac{R^2}{l^2}}} \tag{2.14}$$

After evaluating the action difference and taking the limit we finally obtain,

$$-log(Z) = I = \frac{Vol(S^{d-2})(l^2 r_+^{d-2} - r_+^{d+2})}{4G((d-1)r_+^2 + (d-3)l^2)} \tag{2.15}$$

We see that $I$ is positive for small $r_+$ which means that the tunneling probability from adS to a black hole is exponentially suppressed. We have thus semi-classical stability. On the contrary the sign is inverted for large $r_+$ which points to an instability physically favouring tunneling to black holes. Clearly in between we expect to find a phase transition. Let us now pursue to find in a standard way the thermodynamic quantities. The mean energy of thermal radiation is $E = \sum_{states} p_i E_i$, where $p_i = \frac{1}{Z} e^{-\beta E_i}$ and therefore,

$$E = -\frac{\partial}{\partial \beta} log Z = \frac{(d-2)Vol(S^{d-2})(l^{-2} r_+^{d-1} - r_+^{d-3})}{16\pi G} = M \tag{2.16}$$

where $M = \frac{d-2)Vol(S^{d-2})\mu}{16\pi G}$ is the gravitational mass of the black hole ([16] for $\kappa = 1$), (the general conserved charges for locally asymptotically adS spacetimes are given in [11]). The entropy $S = \sum_{states} p_i \log p_i$ is given by,

$$S = \beta E - I = \frac{1}{4G} r_+^{d-2} Vol(S^{d-2}) = \frac{A}{4G} \tag{2.17}$$

where $A$ is the volume of the horizon. Therefore is adS the entropy-area relation is verified. The heat capacity ie, the amount of heat energy required to increase the temperature by a unit quantity, is given by,

$$\begin{aligned} C &= \frac{\partial E}{\partial T} = \frac{\partial E}{\partial r_+}\frac{\partial r_+}{\partial T} = \\ &= \frac{(n-1)Vol(S^{d-2})r_+^{n-3}}{8n}(nr_+^2 + (n-2)l^2)\left[1 \pm \sqrt{1 - \kappa\frac{n(n-2)\beta^2}{4\pi^2 l^2}}\right] \end{aligned} \tag{2.18}$$

$C$, the second derivative of the action with respect to temperature, gives thermodynamic stability. The free energy is given by, $F = E - TS = IT$

First we note that $r_+$ is an increasing function of the mass $M$. Now it is easy to check that $0 \leq \beta < \beta_{max}$, where $\beta_{max} = \frac{2\pi l}{\sqrt{(d-3)(d-1)}}$. Hence for temperatures $T < T_{min} = \frac{1}{\beta_{max}}$ there is no black hole and we have a pure (adS) phase of thermal radiation with negative free energy. For $T > T_{min}$ there are two black holes with $M_- < M_+$ (2.7). The black hole with the smaller mass has negative specific heat. It is therefore unstable to decay either to a larger black hole or to pure thermal adS. The bigger black hole has on the contrary positive specific heat and is thus thermodynamically stable. Furthermore for $T_{min} < T < T_{HP}$ the free energy of the plus branch is less than pure thermal adS which is energetically favoured. However, for $T > T_{HP}$ the situation is inversed and the large black hole state has less free energy than thermal adS space. AdS space can therefore tunnel towards this large black hole which becomes the energetically preferred state.

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