Kerr-Schild Method and Geodesic Structure in Codimension-2 Brane Black Holes

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Abstract

We consider the black hole solutions of five-dimensional gravity with a Gauss-Bonnet term in the bulk and an induced gravity term on a 2-brane of codimension-2. Applying the Kerr-Schild method we derive additional solutions which include charge and angular momentum. Moreover, we study the geodesic structure of such spacetimes.

* In collaboration with S. Aguilar and N. Zamorano (Universidad de Chile).

BTZ on codimension-2

B.Cuadros-Melgar, E.Papantonopoulos, M.Tsoukalas, V.Zamarias, *Phys.Rev.Lett.* **100**, 221601 (2008)

Action

$$
\begin{aligned}
S_{\mathrm{grav}} &= \frac{M^3_{(5)}}{2}\left\{\int d^5x\sqrt{-g^{(5)}}\left[R^{(5)}\right.\right. \\
&+ \alpha\left(R^{(5)2} - 4R^{(5)}_{MN}R^{(5)MN} + R^{(5)}_{MNKL}R^{(5)MNKL}\right)\Big] \\
&+ \left. r_c^2\int d^3x\sqrt{-g^{(3)}}\,R^{(3)}\frac{\delta(\rho)}{2\pi b}\right\} + \int d^5x\mathcal{L}_{bulk} + \int d^3x\mathcal{L}_{brane}\frac{\delta(\rho)}{2\pi b}\,.
\end{aligned}
$$

Metric

$$ds_5^2 = f^2(\rho)\left(-n(r)^2dt^2 + n(r)^{-2}dr^2 + \frac{r^2}{l^2}d\phi^2\right) + d\rho^2 + b^2(\rho)d\theta^2\ . \qquad (1)$$

Metric

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Einstein Equations

$$G_M^{(5)N} + r_c^2 G_\mu^{(3)\nu} g_M^\mu g_\nu^N \frac{\delta(\rho)}{2\pi b} - \alpha H_M^N = \frac{1}{M_{(5)}^3}\left[T_M^{(B)N} + T_\mu^{(br)\nu} g_M^\mu g_\nu^N \frac{\delta(\rho)}{2\pi b}\right] \ ,$$

Junction Conditions: Einstein Equations on the brane

$$G^{(3)}_{\mu\nu} = \frac{1}{M^3_{(5)}(r_c^2 + 8\pi(1-\beta)\alpha)} T^{(br)}_{\mu\nu} + \frac{2\pi(1-\beta)}{r_c^2 + 8\pi(1-\beta)\alpha} g_{\mu\nu} \,. \qquad (2)$$

Solutions

Table 1: Bulk Solutions

$n(r)$	$f(\rho)$	$b(\rho)$	$-\Lambda_5$	Notes
BTZ	$\cosh\left(\frac{\rho}{2\sqrt{\alpha}}\right)$	$\forall b(\rho)$	$\frac{3}{4\alpha}$	$l^2 = 4\alpha$
	$\cosh\left(\frac{\rho}{2\sqrt{\alpha}}\right)$	$2\,\beta\,\sqrt{\alpha}\,\sinh\left(\frac{\rho}{2\sqrt{\alpha}}\right)$	$\frac{3}{4\alpha}$	$l^2 = 4\alpha$
	± 1	$\gamma^{-1/2}\,\sinh\left(\gamma^{1/2}\,\rho\right)$	$\frac{3}{l^2}$	$\gamma = -\frac{2\Lambda_5}{3+4\alpha\Lambda_5}$
$\forall n(r)$	$\cosh\left(\frac{\rho}{2\sqrt{\alpha}}\right)$	$2\,\beta\,\sqrt{a}\,\sinh\left(\frac{\rho}{2\sqrt{\alpha}}\right)$	$\frac{3}{4\alpha}$	T^{br}
corrected	$\cosh\left(\frac{\rho}{2\sqrt{\alpha}}\right)$	$2\,\beta\,\sqrt{\alpha}\,\sinh\left(\frac{\rho}{2\sqrt{\alpha}}\right)$	$\frac{3}{4\alpha}$	$l^2 = 4\alpha$
BTZ	± 1	$2\,\beta\,\sqrt{\alpha}\,\sinh\left(\frac{\rho}{2\sqrt{\alpha}}\right)$	$\frac{1}{4\alpha}$	$l^2 = 12\alpha$

BTZ: $$n^2(r) = -M + \frac{r^2}{l^2}\,,$$

BTZ-corrected: $$n^2(r) = -M + \frac{r^2}{l^2} - \frac{\zeta}{r}\,.$$

Brane Equations

$n(r)$	T^{br}	Brane
BTZ	$\Lambda_3 = -1/l^2$	Vacuum
BTZ-corrected	$\left(\frac{\zeta}{2r^3}, \frac{\zeta}{2r^3}, -\frac{\zeta}{r^3}\right)$	Scalar field

Table 2: Brane Solutions

Applying the Kerr-Schild Method

Background Metric $\Rightarrow$ New Solution

$$\tilde{g}_{MN} = g_{MN} + 2H(r,\rho)\ell_M \ell_N \,. \tag{3}$$

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Null geodesic vector

$$\ell_{M;N}\ell^N = 0\,, \tag{4}$$

$$\ell_M \ell^M = 0\,. \tag{5}$$

$$\ell_M = \left(C_1\,,\ \frac{1}{n^2}\sqrt{C_1^2 - \left(\frac{C_2^2}{r^2} + \xi^2\right)n^2}\,,\ C_2, \sqrt{\frac{\xi^2}{f^2} - \frac{C_3^2}{b^2}}\,,\ C_3 \right)\,. \tag{6}$$

Main solutions

♣ **Charged BTZ string**

$$H(r,\rho) = f^2(\rho)\frac{Q^2}{2}\ln r$$

$$ds^2 = f^2\left(-\hat{n}^2 d\hat{t}^2 + \frac{dr^2}{\hat{n}^2} + r^2 d\phi^2\right) + d\rho^2 + b^2 d\theta^2\,, \qquad (7)$$

with $\hat{n}^2 = -M + r^2/l^2 - Q^2 \ln r$.

Brane energy-momentum tensor,

$$T_\mu^\nu = diag\left(-\frac{Q^2}{2r^2}, -\frac{Q^2}{2r^2}, \frac{Q^2}{2r^2}\right)\,. \qquad (8)$$

♣ BTZ string + brane scalar field

$$H(r,\rho) = f^2(\rho)\frac{\zeta}{2r}$$

$$\hat{n}^2 = -M + r^2/l^2 - \zeta/r$$

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$$H(r,\rho) = f^2(\rho)\frac{\zeta}{2r}$$

$$\hat{n}^2 = -M + r^2/l^2 - \zeta/r$$

♣ Charged BTZ string + brane scalar field

Background metric: Charged BTZ string

Brane energy-momentum tensor,

$$T_\mu^\nu = diag\left(-\frac{Q^2}{2r^2} + \frac{\zeta}{2r^3}, -\frac{Q^2}{2r^2} + \frac{\zeta}{2r^3}, \frac{Q^2}{2r^2} - \frac{\zeta}{r^3}\right), \tag{9}$$

♣ BTZ string with angular momentum

$$H(r,\rho) = f^2(\rho)c$$

$$ds^2 = f^2\left[-\hat{n}^2 d\hat{t}^2 + \frac{dR^2}{\hat{n}^2} + R^2\left(\frac{-J}{2R^2}d\hat{t} + d\hat{\phi}\right)^2\right] + d\rho^2 + b^2 d\theta^2\,, \qquad (10)$$

where $\hat{n}^2(r) = -\tilde{M} + R^2/l^2 + J^2/4R^2$.

Analogous solutions when we use $f(\rho) = 1$ and $b(\rho) = \gamma\sinh(\rho/\gamma)$.

Geodesic Structure

Lagrangian

$$\mathcal{L} = f^2(\rho)\left(- n(r)^2 \dot{t}^2 + \frac{\dot{r}^2}{n(r)^2} + r^2 \dot{\phi}^2 \right) + \dot{\rho}^2 + b^2(\rho)\dot{\theta}^2 \,. \tag{11}$$

$$\frac{\partial \mathcal{L}}{\partial \dot{t}} = -2E \,, \tag{12}$$

$$\frac{\partial \mathcal{L}}{\partial \dot{\phi}} = 2L \,, \tag{13}$$

$$\frac{\partial \mathcal{L}}{\partial \dot{\theta}} = 2K \,, \tag{14}$$

Then,

$$
\begin{aligned}
2\mathcal{L} \;&=\; \cosh^2\left(\frac{\rho}{2\sqrt{\alpha}}\right)\left[\frac{-E^2}{\cosh^4\left(\frac{\rho}{2\sqrt{\alpha}}\right)n(r)^2}+\frac{\dot{r}^2}{n(r)^2}+\frac{L^2}{r^2\cosh^4\left(\frac{\rho}{2\sqrt{\alpha}}\right)}\right] \\
&\quad +\dot{\rho}^2+\frac{K^2}{4\beta^2\alpha\sinh^2\left(\frac{\rho}{2\sqrt{\alpha}}\right)}=h\,, \qquad (15)
\end{aligned}
$$

$h = 0,\ 1 \quad \Rightarrow$ lightlike and timelike geodesics.

Geodesics on the Brane

Effective Potential

$$V_{eff}^2 = n^2(r)\left(\frac{L^2}{r^2} - h\right). \tag{16}$$

Orbits

$$\frac{dr}{d\lambda} = \sqrt{E^2 - n^2(r)\left(\frac{L^2}{r^2} - h\right)}. \tag{17}$$

BTZ Case Radial geodesics $(L = 0)$

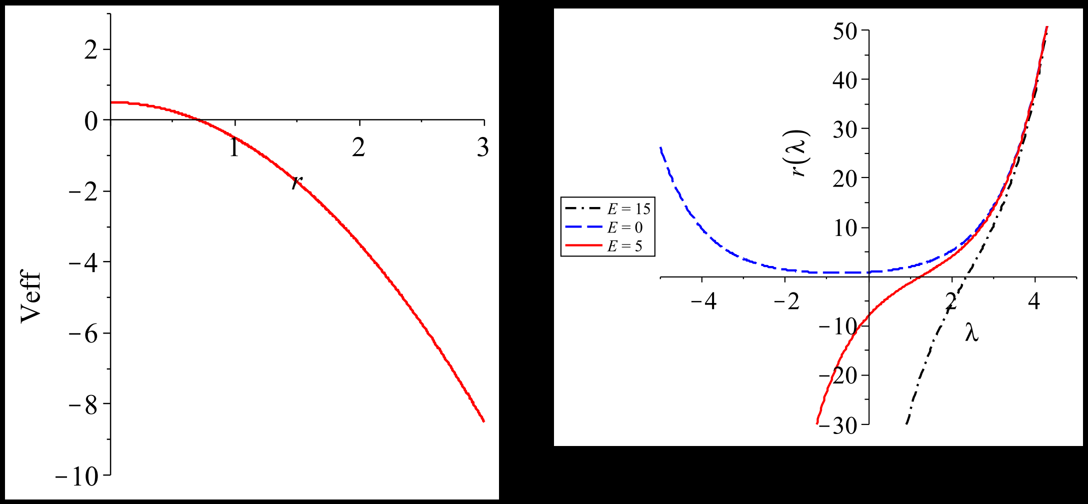
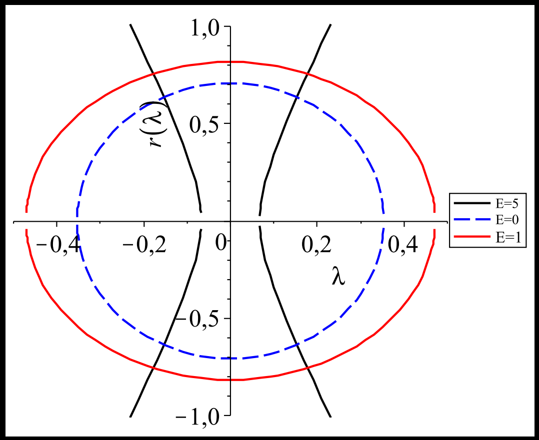


Figure 1: Effective potential and orbits for timelike radial particles on the brane.

$L \neq 0$



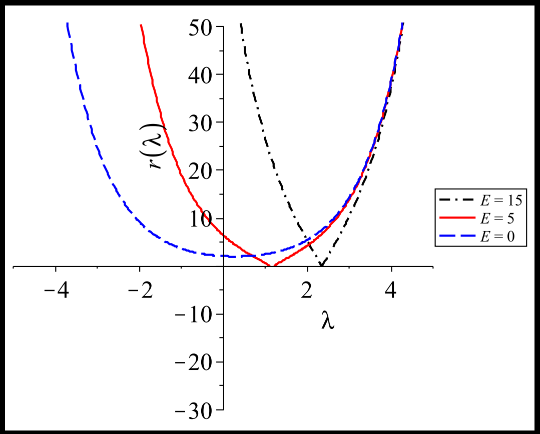
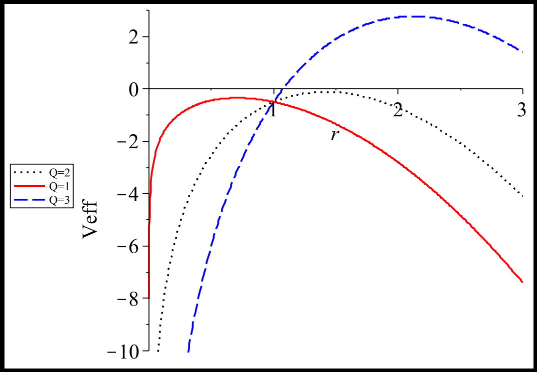


Figure 2: Orbits for lightlike (left) and timelike (right) geodesics with L.

Charged BTZ and BTZ + Scalar field Case



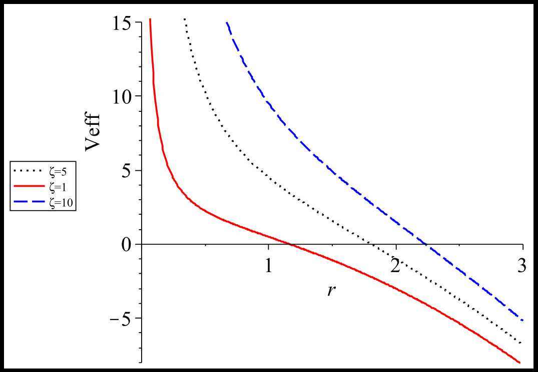


Figure 3: Effective potential for timelike geodesic on the brane.

Geodesics in the Bulk: $f(\rho) = \cosh\left(\frac{\rho}{2\sqrt{\alpha}}\right)\,,\; b(\rho) = 2\beta\sqrt{\alpha}\sinh\left(\frac{\rho}{2\sqrt{\alpha}}\right)$

$L = K = 0$

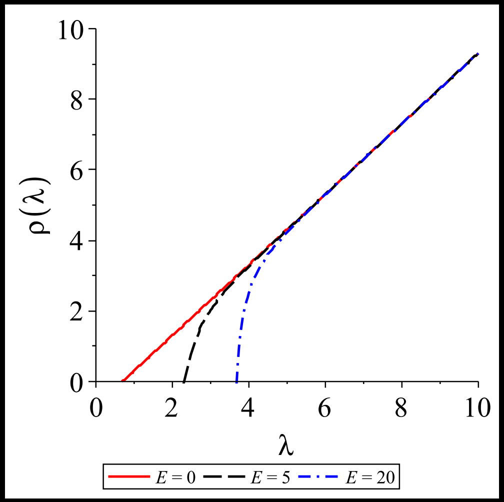


Figure 4: Timelike orbits for geodesics in the bulk.

$L \neq 0\,,\; K \neq 0$

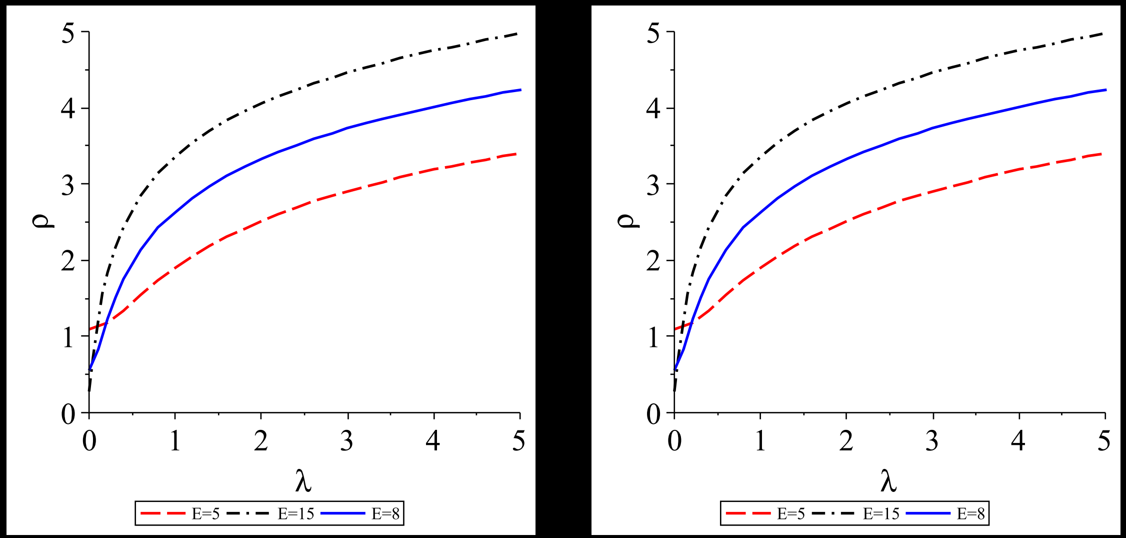


Figure 5: Orbits for lightlike (left) and timelike (right) particles with L.

$$f(\rho) = 1\,,\; b(\rho) = \gamma \sinh(\gamma^{-1}\rho)$$

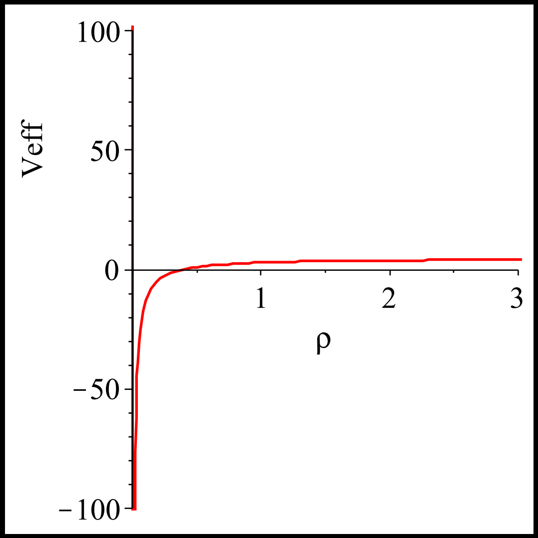


Figure 6: Effective potential for $f = 1$.

BTZ with Angular Momentum

$$
\begin{aligned}
h \;&=\; f^2\Big[-\Big(-4M+\frac{4r^2}{l^2}+\frac{J^2}{r^2}\Big)\frac{l^4(-2r^2E+LJ)^2}{f^4(4Ml^2r^2-4r^4-J^2l^2)^2} \\
&\quad +\frac{4\dot r^2}{-4M+4r^2/l^2+\frac{J^2}{r^2}}+\frac{r^2(-Jl^2(-2r^2E+LJ))}{r^2f^2(4Ml^2r^2-4r^4-J^2l^2)} \\
&\quad +\frac{2(-JEl^2+2Ml^2L-2Lr^2)}{f^2(4Ml^2r^2-4r^4-J^2l^2)^2}\Big]+\dot\rho^2+\frac{K^2}{b^2}\,,
\end{aligned} \tag{18}
$$

Geodesics on the Brane

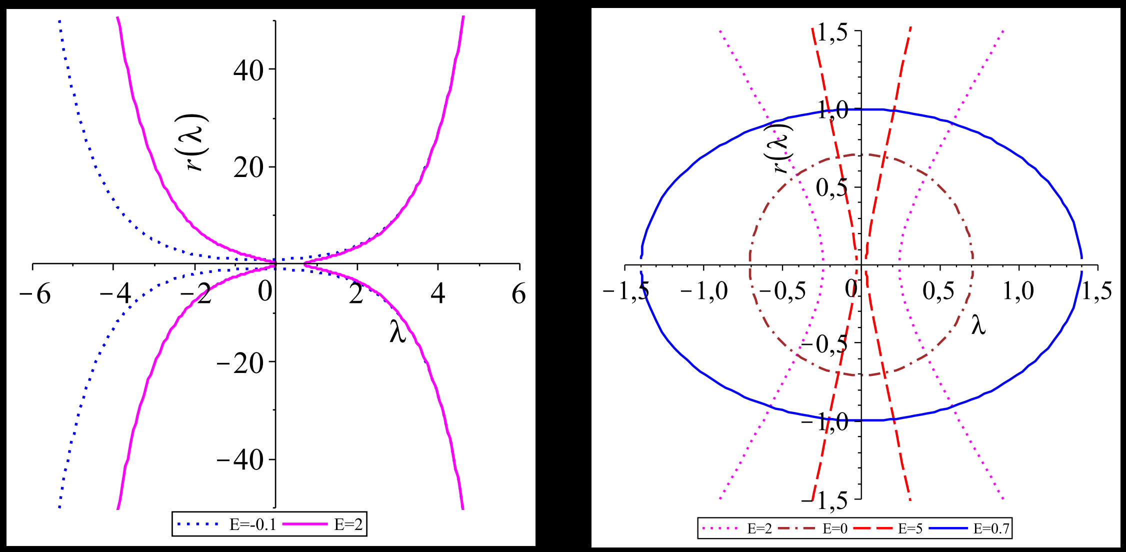


Figure 7: Timelike (left) and likelike (right) brane geodesics for BTZ with J.

Geodesics in the Bulk

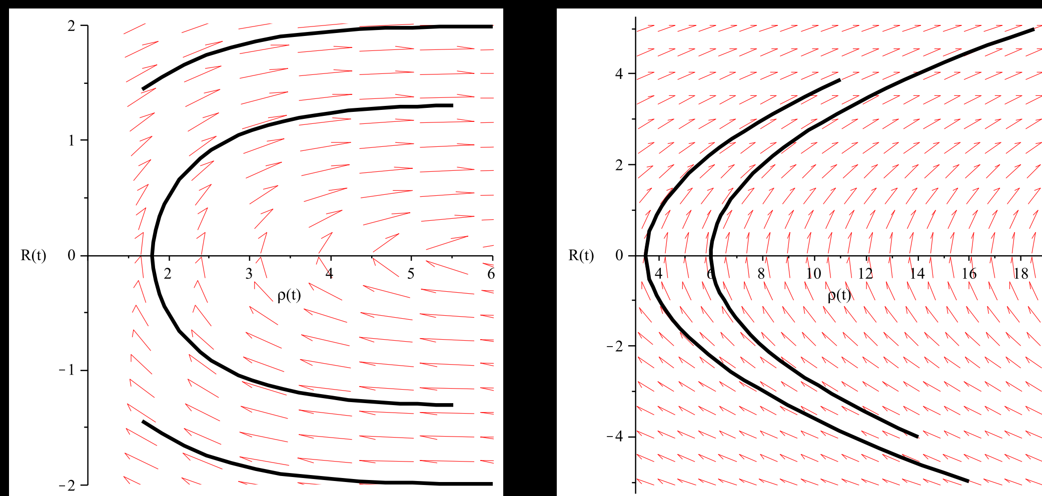


Figure 8: Lightlike (left) and timelike (right) phaseportrait for geodesics in the bulk for BTZ with J.

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♣ The orbits present several features, some of them can be inferred from an effective potential, others can be integrated directly, and a phaseportrait graph was needed in some cases.