Reconstruction of the Scalar-Tensor Lagrangian from a ΛCDM Background and Noether Symmetry

Savvas Nesseris
Department of Physics
University of Ioannina

arXiv:0705.3586v4 [astro-ph]

in collaboration with
L. Perivolaropoulos and S. Capozziello

Main points of the talk

- Derive the Dynamical System for Scalar-Tensor theories
- Input a cosmic history (ΛCDM) and get the non-minimal coupling $F(\Phi)$ and the potential $U(\Phi)$
- Derive the Critical Points in radiation, matter and de-Sitter eras
- Compare the analytic forms of F and U with numeric results
- Show that the forms for F, U are also motivated by imposing a Noether symmetry

Dynamics of Scalar-Tensor cosmologies

- Sc-Ten theories are given by the Lagrangian

$$\mathcal{L} = \frac{F(\Phi)}{2}\, R - \frac{1}{2}\, \epsilon\, g^{\mu\nu}\partial_\mu\Phi\partial_\nu\Phi - U(\Phi) + \mathcal{L}_{\rm m}[\psi_{\rm m}; g_{\mu\nu}]$$

Potential

Kinetic term

Non-minimal coupling

$\epsilon = \pm 1$ **for standard scalar and phantom fields respectively**

- Assuming a FRW backround

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + a^2(t)\,\mathrm{d}\mathrm{x}^2$$

Dynamics of Scalar-Tensor cosmologies

- Get the EOM by varying the action with respect to the metric

$$3FH^2 = \rho_{\rm m} + \rho_r + \frac{1}{2}\epsilon\dot{\Phi}^2 - 3H\dot{F} + U \qquad (1)$$

$$-2F\dot{H} = \rho_{\rm m} + \frac{4}{3}\rho_r + \epsilon\dot{\Phi}^2 + \ddot{F} - H\dot{F} \qquad (2)$$

$$\epsilon(\ddot{\Phi} + 3H\dot{\Phi}) = 3F(\Phi)_{,\Phi}(\dot{H} + 2H^2) - U(\Phi)_{,\Phi} \qquad (3)$$

- The matter and radiation densities obey the usual eqs:

$$\dot{\rho}_{\rm m} + 3H\rho_{\rm m} = 0\,,$$
$$\dot{\rho}_{\rm r} + 4H\rho_{\rm r} = 0\,.$$

Dynamics of Scalar-Tensor cosmologies

- Rewrite (1) in the form:

$$3FH^2 = \rho_{\rm m} + \rho_r + \frac{1}{2}\epsilon\dot{\Phi}^2 - 3H\dot{F} + U$$

$$1 = \frac{\rho_{\rm m}}{3FH^2} + \frac{\rho_r}{3FH^2} + \epsilon\frac{\Phi'^2}{6F} + \frac{U}{3FH^2} - \frac{F'}{F} \qquad (4)$$

where $\quad ' = \frac{d}{d\ln a} \equiv \frac{d}{dN} = \frac{1}{H}\frac{d}{dt}$

Dynamics of Scalar-Tensor cosmologies

- Define the dimensionless variables

$$
\begin{aligned}
x_1 &= -\frac{F'}{F}\,, \\
x_2 &= \frac{U}{3FH^2}\,, \\
x_3^2 &= \frac{\Phi'^2}{6F}\,, \\
x_4 &= \frac{\rho_{\rm r}}{3FH^2} = \Omega_r
\end{aligned}
$$

- Set $\Omega_m \equiv \frac{\rho_{\rm m}}{3FH^2}$ and rewrite the previous eq.

$$\Omega_m = 1 - x_1 - x_2 - \epsilon x_3^2 - x_4$$

Dynamics of Scalar-Tensor cosmologies

- Then the dynamical system takes the following form:

$$x_1' = 3 - 2x_1 - 3x_2 + x_4 + 3\epsilon x_3^2 + x_1^2 + 2\frac{H'}{H} - x_1\frac{H'}{H}$$

$$x_2' = x_2\left[x_1(1-m) - 2\frac{H'}{H}\right]$$

$$\epsilon(x_3^2)' = \epsilon x_3^2 x_1 - 6\epsilon x_3^2 - 2x_1 + m x_2 x_1 - 2\epsilon x_3^2\frac{H'}{H} - x_1\frac{H'}{H}$$

$$x_4' = -4x_4 + x_4 x_1 - 2x_4\frac{H'}{H}$$

where

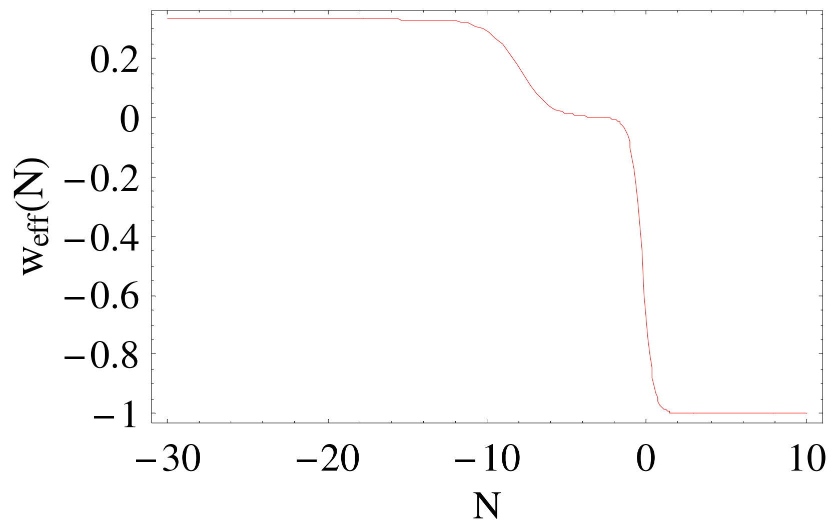
$$m \equiv \frac{U_{,\Phi}/U}{F_{,\Phi}/F}$$

has an explicit dependance on N

Dynamics of Scalar-Tensor cosmologies

- The total effective equation of state is:

$$w_{eff} = -1 - \frac{2}{3}\frac{H'(N)}{H(N)}$$



- and it has a value:

$$w_{eff} = \frac{1}{3} \quad \text{Radiation Era}$$
$$w_{eff} = 0 \quad \text{Matter Era}$$
$$w_{eff} = -1 \quad \text{deSitter Era}$$

- Input a cosmic history:

$$H(N)^2 = H_0^2\left[\Omega_{0\mathrm{m}}e^{-3N} + \Omega_{0r}e^{-4N} + \Omega_\Lambda\right]$$

where $N \equiv \ln a$

and $\Omega_\Lambda = 1 - \Omega_{0\mathrm{m}} - \Omega_{0r}$.

Dynamics of Scalar-Tensor cosmologies

$$\frac{H'(N)}{H(N)} = -2 \quad N < N_{rm} \quad \simeq -\ln\frac{\Omega_{0m}}{\Omega_{0r}}$$

$$\frac{H'(N)}{H(N)} = -\frac{3}{2} \quad N_{rm} < N < N_{m\Lambda} \quad \simeq -\frac{1}{3}\ln\frac{\Omega_{\Lambda}}{\Omega_{0m}}$$

$$\frac{H'(N)}{H(N)} = 0 \quad N > N_{m\Lambda}$$

Transition points from

radiation to matter

matter to deSitter

Not autonomous at all times!

- Now we can study the dynamics of the system by finding the critical points and their stability in each one of the three eras.

The "attractor" critical points will serve as a prediction for the numerical evolution of our dynamical system.

The Critical Points

A brief introduction

see Copeland, astro-ph/0603057

Consider a system of ODEs

$$\dot{x} = f(x,y,t)$$
$$\dot{y} = g(x,y,t)$$

A point (xc, yc) is called a critical point when

$$(f,g)|_{(x_c,y_c)} = 0$$

Check the stability of a trajectory close to the CP by perturbing the system

$$x = x_c + \delta x$$
$$y = y_c + \delta y$$

→

$$\frac{\mathrm{d}}{\mathrm{d}N}\begin{pmatrix}\delta x\\ \delta y\end{pmatrix} = \mathcal{M}\begin{pmatrix}\delta x\\ \delta y\end{pmatrix}$$

where

$$\mathcal{M} = \begin{pmatrix}\frac{\partial f}{\partial x} & \frac{\partial f}{\partial y}\\ \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y}\end{pmatrix}_{(x=x_c,y=y_c)}$$

The solution to the system can be found to be

$$\delta x = C_1 e^{\mu_1 N} + C_2 e^{\mu_2 N}$$
$$\delta y = C_3 e^{\mu_1 N} + C_4 e^{\mu_2 N}$$

where μ_1, μ_2 are the eigenvalues of M and will characterize our point

The Critical Points

- If $\mu_1<0$ and $\mu_2<0$ then the CP is an attractor
- If $\mu_1>0$ and $\mu_2>0$ then the CP is a source
- If $\mu_1<0$ and $\mu_2>0$ then the CP is a saddle
- If $|M|<0$ and $Re(\mu_1)$,$Re(\mu_2)<0$ then the CP is a stable spiral

The Critical Points

Era	CP	x_1	x_2	x_3^2	x_4	Ω_m	Ω_{DE}	Eigenvalues
	R_1	2	0	-1	0	0	1	(2,3,1,6-2m)
	R_2	1	0	0	0	0	1	(1,2,-1,5-m)
Radiation	R_3	-1	0	0	0	2	-1	(-1,-2,-3,3+m)
$w_{eff}=\frac{1}{3}$	R_4	$\frac{4}{-1+m}$	$\frac{15-8m+m^2}{3(m-1)^2}$	$\frac{2(m-5)m}{3(m-1)^2}$	0	0	1	$(\frac{4}{m-1}, \frac{m+3}{m-1},$ see [a])
	R_5	0	0	0	1	0	0	(1,-1,-2,4)
	M_1	2	0	-1	0	0	1	$(1,2,1/2,5-2m)$
	M_2	3/2	0	-1/2	0	0	1	$(1/2,3/2,-1/2,-3/2(m-3))$
Matter	M_3	0	0	0	0	0	0	(-1,-3/2,-2,3)
$w_{eff}=0$	M_4	$\frac{3}{m-1}$	$\frac{15-11m+2m^2}{4(m-1)^2}$	$\frac{1-9m+2m^2}{4(m-1)^2}$	0	0	1	$(\frac{4-m}{m-1}, \frac{3}{m-1},$ see [b])
	M_5	1	0	-1/4	1/4	0	3/4	$(1,-1/2,-1,4-m)$
	Λ_1	2	0	-1	0	0	1	$(-2,-1,-1,2-2m)$
	Λ_2	3	0	-2	0	0	1	$(-1,0,1,3-3m)$
deSitter	Λ_3[c]	3	0	-2	0	0	1	$(-1,0,1,3-3m)$
$w_{eff}=-1$	Λ_4	0	1	0	0	0	1	$-4, -3, -\frac{\sqrt{24m+1}+5}{2}, \frac{\sqrt{24m+1}-5}{2}$
	Λ_5	4	0	-4	1	0	0	$(1,1,2,4-4m)$

[a] the other two eigenvalues are: $-\frac{3m+\sqrt{8m^3-63m^2+118m+1}-11}{2(m-1)}, \frac{-3m+\sqrt{8m^3-63m^2+118m+1}+11}{2m-2}$

[b] the other two eigenvalues are: $-\frac{7m+\sqrt{48m^3-263m^2+358m+1}-19}{4(m-1)}, \frac{-7m+\sqrt{48m^3-263m^2+358m+1}+19}{4m-4}$

[c] Notice that Λ_2 and Λ_3 are degenerate

"Attractors"

R3 for m<-3

R4 for -3<m<1

M4 for m<1

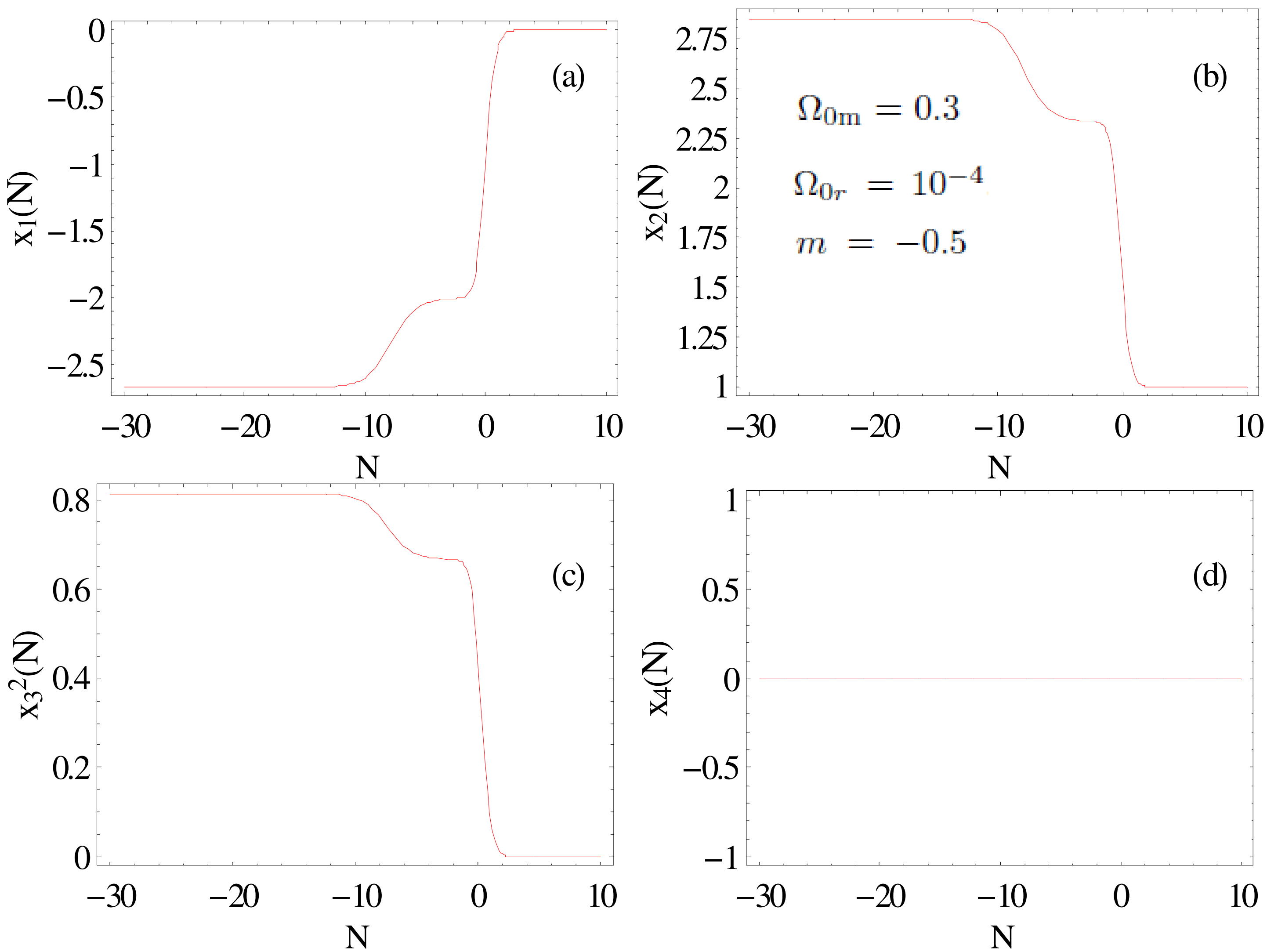
Λ1 for m>1

Λ4 for m<1

The system should follow the "attractors" through each era, but what do the numerics say?

Numeric Results

- The evolution of the system:



"Attractors"

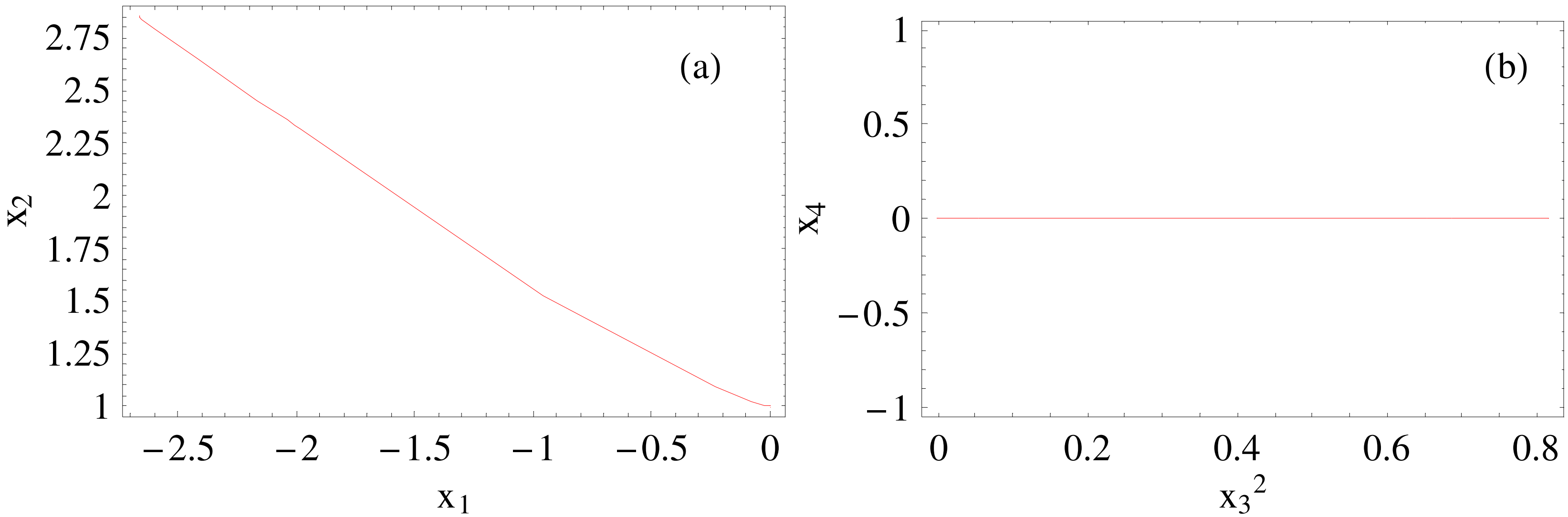
R4=(-2.67, 2.85, 0.81, 0)

M4=(-2, 2.33, 0.67, 0)

Λ4=(0, 1, 0, 0)

Numeric Results

- The phase space trajectories:



The system indeed passes through the "attractors" in each era!

But, can we do anything analytically?

Yes! As we can reconstruct F and U analytically in each era.

Analytic Results

Consider a critical point of the form $(\bar{x}_1, \bar{x}_2, \bar{x}_3^2, \bar{x}_4)$ =const

$$x_1 = -\frac{F'}{F} \quad \Longrightarrow \quad F = F_0 e^{-\bar{x}_1 N}$$

$$x_3^2 = \frac{\Phi'^2}{6F} \quad \Longrightarrow \quad \Phi(N) = 2\sqrt{6}\,\frac{\bar{x}_3}{\bar{x}_1}\,F_0^{1/2}\left(1 - e^{-\bar{x}_1 N/2}\right) + \Phi_0$$

or

$$F(\Phi) = \frac{1}{24}\frac{\bar{x}_1^2}{\bar{x}_3^2}(\Phi - C)^2 \equiv \xi(\Phi - C)^2$$

$$U = cF^m$$

Doing the same for the potential

$$U(\Phi) = \lambda(\Phi - C)^{2m}$$

$$\lambda = \begin{cases} 3\bar{x}_2 \frac{\Omega_{0r}}{F_0^{4/\bar{x}_1}} \xi^{1+4/\bar{x}_1}, & \text{Rad. Era} \\ 3\bar{x}_2 \frac{\Omega_{0m}}{F_0^{3/\bar{x}_1}} \xi^{1+3/\bar{x}_1}, & \text{Mat. Era} \\ 3\bar{x}_2 \xi(1 - \Omega_{0r} - \Omega_{0m}), & \text{dS Era} \end{cases}$$

Another way to get these results is to use a Noether symmetry approach...

If $\mathcal{L}(q^i, \dot{q}^i)$ is a point-like Lagrangian in the space of q^i

then the Noether theorem states that L is invariant under a transformation **X** generated by $\mathbf{X} = \alpha^i(\mathbf{q})\frac{\partial}{\partial q^i} + \left(\frac{d}{d\lambda}\alpha^i(\mathbf{q})\right)\frac{\partial}{\partial \dot{q}^i}$

if $L_{\mathbf{X}}\mathcal{L} = 0$

- In scalar-tensor gravity the point-like Lagrangian is

$$\mathcal{L} = -3a\dot{a}^2F - 3F_{,\Phi}\dot{\Phi}a^2\dot{a} + a^3\left(\frac{1}{2}\dot{\Phi}^2 - U(\Phi)\right) - Da^{-3(\gamma-1)}$$

the generator of the Noether symmetry is

$$\mathbf{X} = \alpha\frac{\partial}{\partial a} + \beta\frac{\partial}{\partial \Phi} + \dot{\alpha}\frac{\partial}{\partial \dot{a}} + \dot{\beta}\frac{\partial}{\partial \dot{\Phi}}$$

Noether symmetry approach

and we can calculate the Lie derivative of L

$L_{\mathbf{X}}\mathcal{L} = 0$ ⇨ mess…

- Separating the linearly independent terms gives rise to a coupled system of ODEs

$$\alpha + 2a\frac{\partial\alpha}{\partial a} + a^2\frac{\partial\beta}{\partial a}\frac{F_{,\Phi}}{F} + a\beta\frac{F_{,\Phi}}{F} = 0$$

$$\left(2\alpha + a\frac{\partial\alpha}{\partial a} + a\frac{\partial\beta}{\partial\Phi}\right)F_{,\Phi} + aF_{,\Phi\Phi}\beta + 2F\frac{\partial\alpha}{\partial\Phi} - \frac{a^2}{3}\frac{\partial\beta}{\partial a} = 0$$

$$3\alpha - 6F_{,\Phi}\frac{\partial\alpha}{\partial\Phi} + 2a\frac{\partial\beta}{\partial\Phi} = 0$$

$$\frac{U_{,\Phi}}{U} = -\frac{3\alpha}{a\beta}$$

Noether symmetry approach

- It can be solved for α, β by separation of variables and as by-products we get U and F

$$F'' = \frac{3s(s+1)(s+2)F'^4}{(2s+3)F^2} + \frac{(s+1)\left(8s^2+16s+3\right)F'^2}{2(2s+3)F} + \frac{s(2s+3)}{3}$$

$$U = cF^m$$

A particular solution is $F = \xi(\Phi - \Phi_0)^2$

where $\xi = -\frac{(2s+3)^2}{24(s+1)(s+2)}$ or $\xi = -\frac{1}{6}$

For the potential we get $U(\Phi) = U_0(\Phi - \Phi_0)^{\frac{6(s+1)}{2s+3}}$

These solutions are the same as the ones that followed from the reconstruction.

Is this a coincidence? Work in progress...

More numeric results...

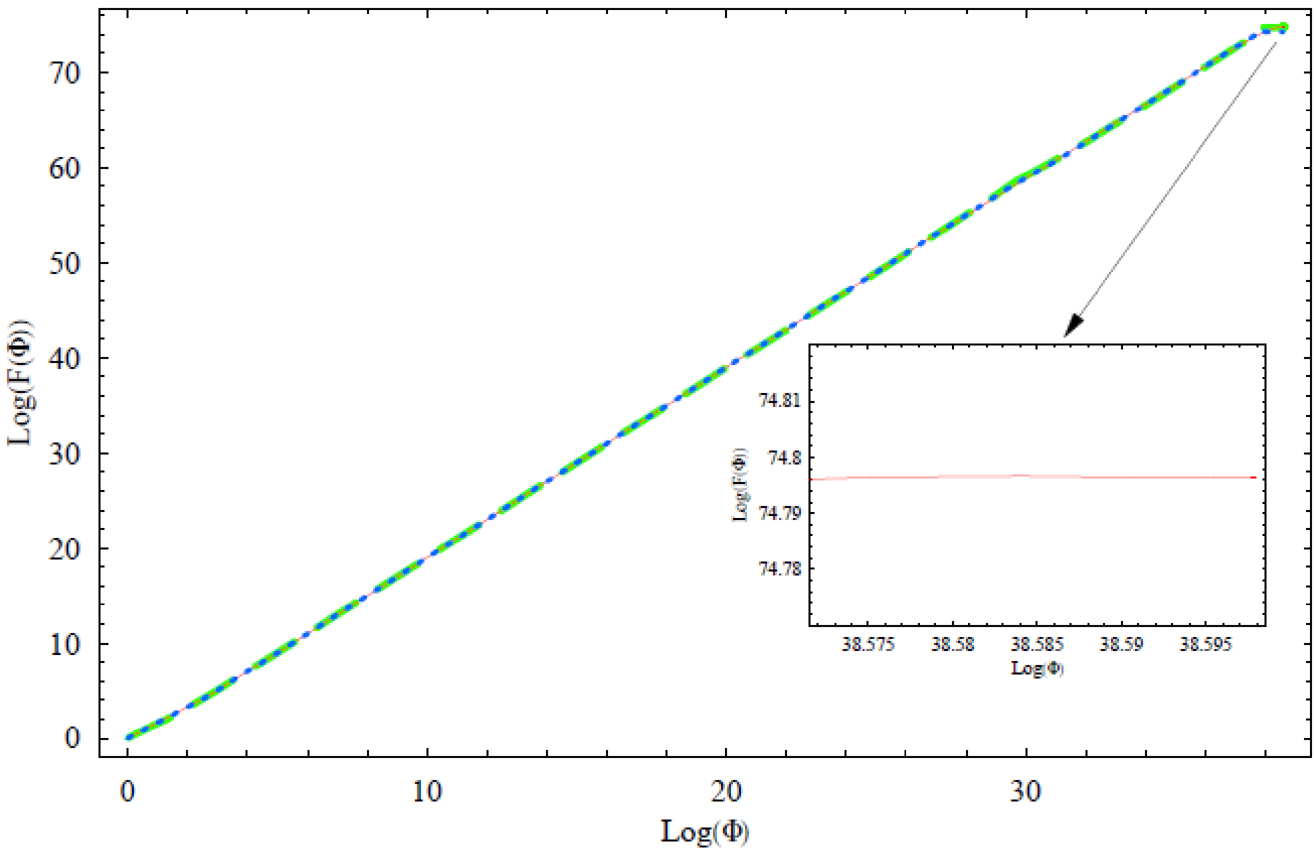


FIG. 4: The form of $log(F(\Phi))$ in the numerical reconstruction (red continuous line), its analytical approximation (blue dotted line) and a fit of the numerical reconstruction using Eq.(3.56) (green long-dashed line). The agreement between the three approaches is very good. The reason for the existence of the small plateau, see the zoomed region, is that as the system evolves towards the deSitter era the potential $F(\Phi)$ "freezes" much faster than the field Φ.

Conclusions

- We derived the Dynamical System for Scalar-Tensor theories

- For a particular cosmic history (ΛCDM) we found analytic forms of $F(\Phi)$ and $U(\Phi)$

- We compared the analytic forms of F and U with numeric results

- We showed that the forms for F, U may also be motivated by a Noether Symmetry Approach