INFLATION : GENERATING THE COSMOLOGICAL PERTURBATIONS THROUGH SCALAR FIELDS

Lecture 1 :

- Short introduction to Inflation
- Generation of cosm. perturb^{n} (1st part)

Lecture 2 :

- Generation of cosm. perturb^{n} (2nd part)

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Some basics of the Big-Bang model:

Einstein eqs: $$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 8\pi G_N T_{\mu\nu}$$

$R_{\mu\nu}$ Riemann tensor

$T_{\mu\nu}$ Energy momentum tensor

$G_N \equiv \frac{1}{M_p^2}$

Homogeneity and isotropy $\Rightarrow$ $T_{\mu\nu} = \mathrm{Diag}(\rho, P, P, P)$

ρ: energy density

P: Pressure

$$ds^2 = dt^2 - a^2(t)\, d\vec{x}^2 \quad ; \quad d\vec{x}^2 = \frac{dr^2}{1-Kr^2} + r^2\left(d\theta^2 + \sin^2\theta\, d\varphi^2\right)$$

$a(t)$: scale factor

$\Downarrow$

$$\begin{cases} H^2 \equiv \left(\dfrac{\dot a}{a}\right)^2 = \dfrac{8\pi G_N}{3}\rho - \dfrac{K}{a^2} \\[2ex] \dfrac{\ddot a}{a} = -\dfrac{4\pi G_N}{3}\left(\rho + 3P\right) \end{cases}$$

Useful equations $$\dot H = -4\pi G_N (\rho + P)$$
$$\dot\rho = -3H(P+\rho)$$

Some def:

- $\Omega = \rho / \rho_c$

$$\rho_c = 3H^2 / 8\pi G_N \quad ; \quad \rho_c(\text{today}) \approx 10^4 \, eV \, cm^{-3}$$

$$H^2 = \frac{8\pi G_N}{3} \rho - \frac{k}{a^2} \Rightarrow \Omega - 1 = \frac{k}{a^2 H^2}$$

- $R_{CUR} = \dfrac{H^{-1}}{|\Omega - 1|^{1/2}}$ sets the scale at which curvature becomes relevant

$$R_{CUR}(t = t_0) \gg H_0^{-1} \sim \text{present horizon}$$

- Particle horizon: $R_H(t) = a(t) \displaystyle\int_0^t \frac{dt'}{a(t')}$

the physical distance photons travel up to time t

$$a(t) \sim t^n \Rightarrow R_H(t) \sim \frac{n H^{-1}}{(1-n)} \sim H^{-1}$$

Take a scale $\lambda = \dfrac{2\pi a}{k}$

$\dfrac{k}{aH} \ll 1$ scale λ outside the horizon

$\dfrac{k}{aH} \gg 1$ scale λ inside the horizon

SHORTCOMINGS OF THE STANDARD BIG BANG THEORY

1) <u>The flatness problem :</u>

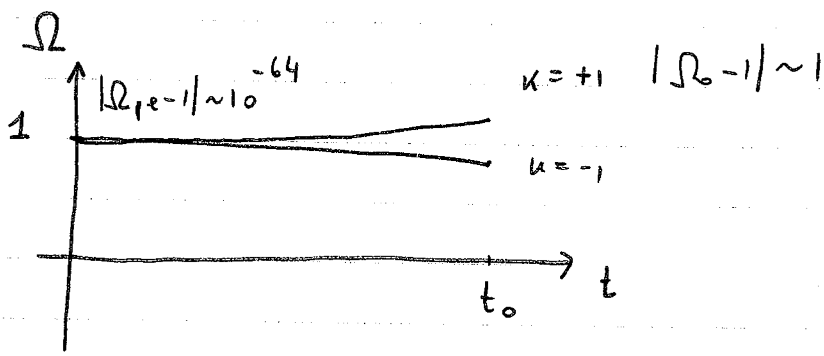
during RD epoch $\Omega - 1 \sim \frac{K}{a^2 a^4} \propto a^2$

because $H^2 \sim \rho \sim a^4 \sim T^4$

Neglect the MD epoch :

$$\frac{|\Omega - 1|_{T \approx T_{Pl}}}{|\Omega - 1|_{T = T_0}} \simeq \left(\frac{a_{Pl}}{a_0}\right)^2 \approx \left(\frac{T_0}{T_{Pl}}\right)^2$$

$T_0 \sim 10^{-4}\,eV$, $T_{Pl} \sim 10^{19}\,GeV \Rightarrow \mathcal{O}(10^{-64})$!!



FINE-TUNING PROBLEM, FATAL ?

2) The entropy problem:

→ It is related to the flatness problem

Assume expansion is adiabatic $\Rightarrow S \sim a^3 T^3 \sim$ const.

$$\Rightarrow S(t=t_{pl}) = S_0 \sim (H_0^{-1} T_0)^3 \sim 10^{90}$$

$H_0 \sim 10^{-33} eV$, $T_0 \sim 10^{-4} eV$

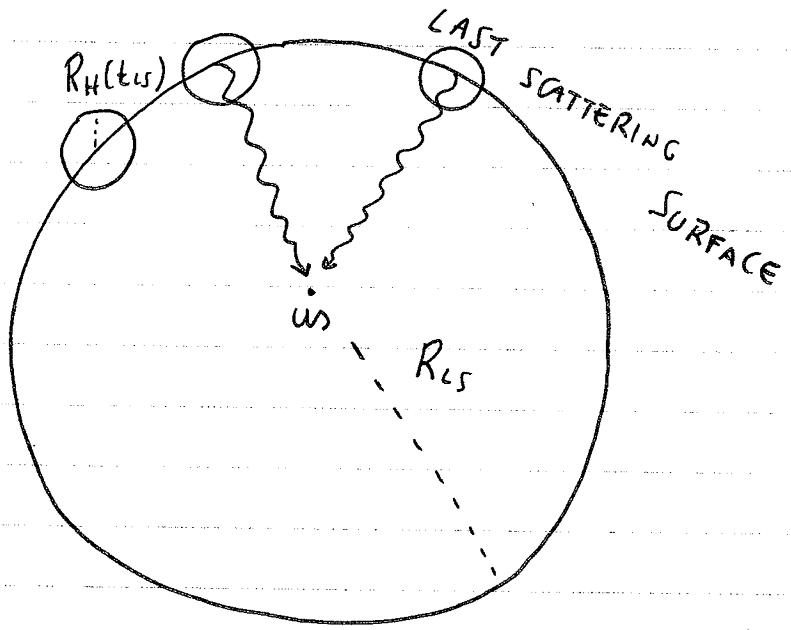
$$\Omega - 1 = \frac{k}{a^2 H^2} \sim \frac{k M_p^2}{a^2 T^4} = \frac{k M_p^2}{(a^2 T^2) T^2}$$

$$= \frac{k M_p^2}{S^{2/3} T^2} = \frac{k M_p^2}{S_0^{2/3} T^2}$$

$$\Omega - 1 (t=t_{pl}) \sim k S_0^{-2/3} \sim 10^{-60}$$

$\Rightarrow$ Suggestion: Solve the entropy / flatness problem releasing adiabaticity assumption

3) Horizon problem:



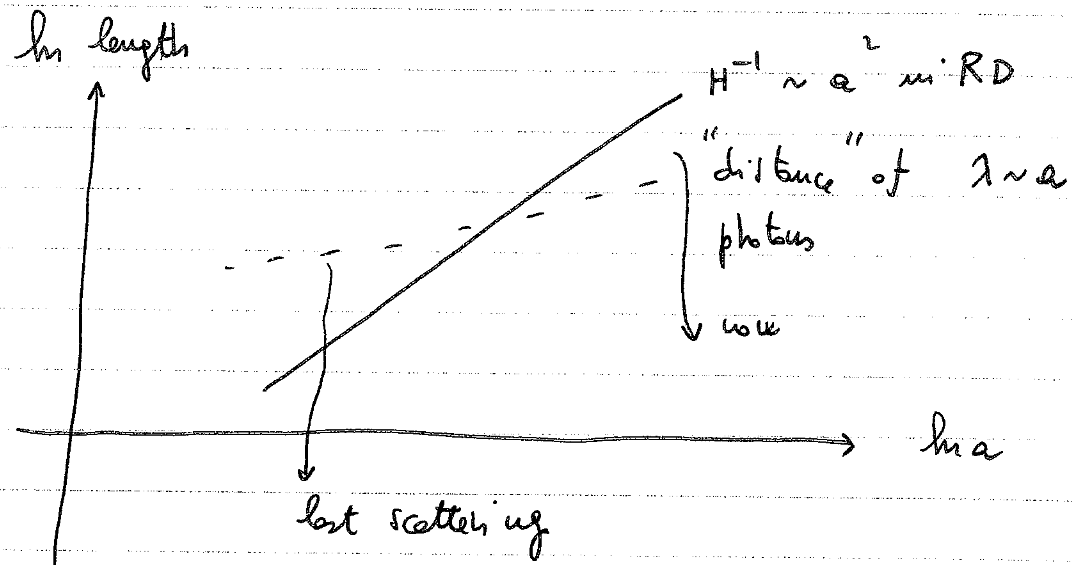
$$R_{LS} = R_H(t_0)\left(\frac{a_{LS}}{a_0}\right) \sim R_H(t_0)\left(\frac{T_0}{T_{LS}}\right) \sim H_0^{-1}\left(\frac{10^{-4}\,eV}{0.3\,eV}\right)$$

$$R_H(t_{LS}) = H^{-1}(t=t_{LS}) = R_H(t_0)\left(\frac{T_{LS}}{T_0}\right)^{-3/2}$$

$$\left(\frac{R_{LS}}{R_H(t_{LS})}\right)^3 \sim \left(\frac{T_{LS}}{T_0}\right)^{3/2} \sim 10^6$$

The last scattering surface contains $\sim 10^6$ horizon volumes!

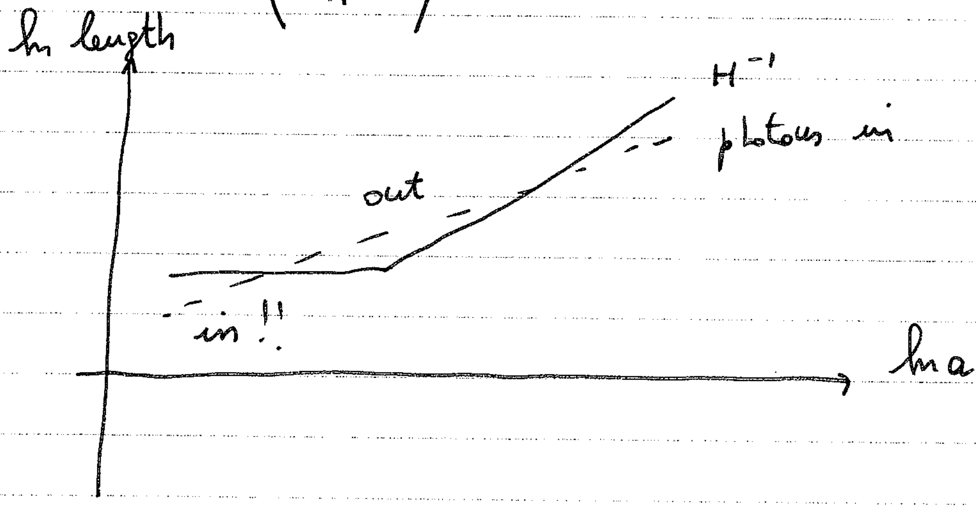
Photon temperature uniform over many horizon regions!



Suggestion to solve the horizon problem:

CHANGE SLOPE OF H^{-1} IN SUCH A WAY THAT THE LINES H^{-1} & λ MEET AGAIN!

MATH: $\left(\frac{\lambda}{H^{-1}}\right)^{\dot{}} > 0 \Rightarrow \ddot{a} > 0$



$$\boxed{\text{INFLATION} \Leftrightarrow \ddot{a} > 0}$$

HOW TO GET INFLATION ?

$$\ddot{a} > 0 \Rightarrow \frac{\ddot{a}}{a} = -\frac{4}{3}\pi G_N \left(\rho + 3P \right) < 0$$

$(\rho + 3P) < 0$ RD, MD No!

$$\frac{P}{\rho} = \frac{1}{3}, 0$$

Forget about this problem and assume that inflation is taking place and take the extreme case:

$$P = -\rho \quad \text{(VACUUM)}$$

$$\Rightarrow H = \text{const} = H_*, \quad a = a_* e^{H_*(t-t_*)}$$

exponential grow

Let's solve all problems one by one

Horizon problem:

Need to impose that the largest scale we observed today $\sim H_0^{-1}$ is reduced to a scale smaller than H_*^{-1} during inflation

$$H_0^{-1}\left(\frac{a_f}{a_0}\right)\left(\frac{a_i}{a_f}\right) \sim H_0^{-1}\left(\frac{T_0}{T_f}\right) e^{-N} \lesssim H_*$$

a_i = a at the beginning of inflation
a_f = " " " end " "

$N = H_* (t_f - t_i)$

$$\Rightarrow N \gtrsim 70 + \ln\left(\frac{T_f}{H_*}\right)$$

We have "exponentially" decrease the problem

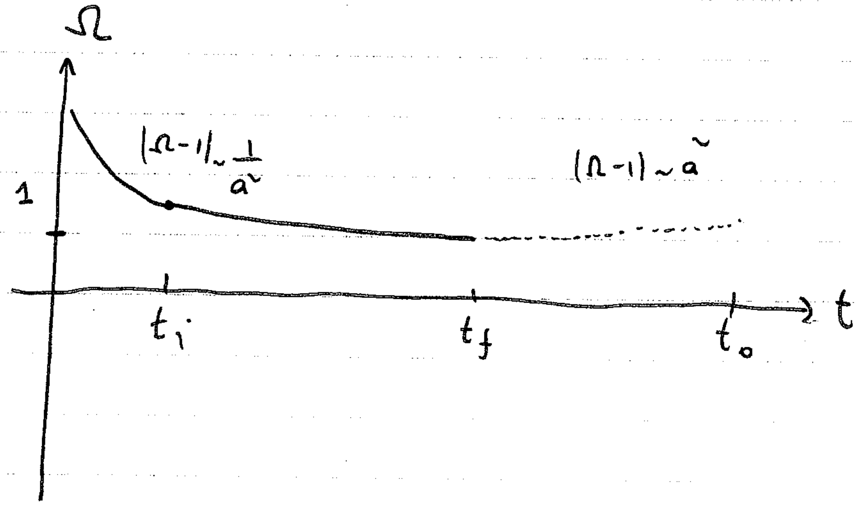
Flatness problem:

During inflation $\Omega - 1 \sim \frac{k}{a^2 H_*^2} \sim \frac{1}{a^2}$

Remember that $|\Omega - 1|$ at the beginning of the RD epoch was $\mathcal{O}(10^{-60})$

Have to impose $\frac{|\Omega-1|_{t_f}}{|\Omega-1|_{t_i}} = \left(\frac{a_f}{a_i}\right)^{-2} = e^{-2N} \lesssim 10^{-60}$

$N \gtrsim 70$



- Prediction of INFLATION: $\Omega_0 = 1$ if $N \gg 70$

Entropy problem:

Remember: have to get rid of adiabaticity

Inflation does it: at the end of inflation the energy in the VACUUM is realeted in the form of Radiation through a transition

$$S \sim a^3 \Rightarrow S_i \sim (a_i T_i)^3 \sim 1$$

$$S_f \sim (a_f T_f)^3 \sim 10^{90} \text{ (IMPOSED!)}$$

$$\Rightarrow \frac{a_f}{a_i} \gtrsim e^N \sim 10^{30} \left(\frac{T_i}{T_f}\right) \Rightarrow N \gtrsim 70$$

All problems solved by $N \gtrsim 70$

HOW TO GET INFLATION:

Take a scalar field ϕ :

$$S = \int d^4x \sqrt{-g}\, \mathcal{L} = \int d^4x \sqrt{-g} \left[\frac{1}{2} g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi - V(\phi) \right]$$

$$\partial^\mu \frac{\delta(\sqrt{-g}\,\mathcal{L})}{\delta \partial_\mu \phi} - \frac{\delta(\sqrt{-g}\,\mathcal{L})}{\delta \phi} = 0 \qquad \phi = \phi(t)$$

$$\Rightarrow \quad \ddot{\phi} + 3H\dot{\phi} - \frac{\nabla^2 \phi}{a^2} + V'(\phi) = 0$$

$$T_{\mu\nu} = \partial_\mu \phi \partial_\nu \phi - g_{\mu\nu} \mathcal{L}$$

$$\rho_\phi = \frac{1}{2}\dot{\phi}^2 + \cancel{\frac{(\nabla\phi)^2}{2a^2}} + V(\phi)$$

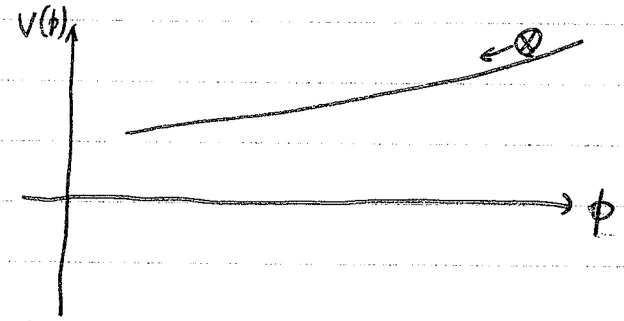
$$P_\phi = \frac{1}{2}\dot{\phi}^2 - \cancel{\frac{(\nabla\phi)^2}{6a^2}} - V(\phi)$$

$$\frac{P_\phi}{\rho_\phi} \approx \frac{\frac{1}{2}\dot{\phi}^2 - V}{\frac{1}{2}\dot{\phi}^2 + V}$$

INFLATION $\Rightarrow \quad P_\phi < -\frac{1}{3}\rho_\phi \Rightarrow V(\phi) \gg \dot{\phi}^2$

VACUUM ENERGY DOMINATES OVER KINETIC ENERGY

Very flat potential



How flat?

$$\ddot{\phi} + 3H\dot{\phi} + V' = 0 \quad , \quad \text{remove } \ddot{\phi}$$

$$\Rightarrow \quad \dot{\phi} = -\frac{V'}{3H} \quad ; \quad \dot{\phi}^2 \ll V(\phi) \Rightarrow \left(\frac{V'}{V}\right)^2 \ll \frac{1}{M_P^2}$$

$$|\ddot{\phi}| \ll |3H\dot{\phi}| \quad \Rightarrow \quad V'' \ll H^2$$

Slow roll parameters:

$$\begin{cases} \epsilon = -\dfrac{\dot{H}}{H^2} = \dfrac{4\pi G_N \dot{\phi}^2}{H^2} \approx \dfrac{1}{16\pi G_N}\left(\dfrac{V'}{V}\right)^2 \\[2ex] \eta = \dfrac{1}{8\pi G_N}\left(\dfrac{V''}{V}\right) = \dfrac{1}{3}\dfrac{V''}{H^2} \\[2ex] \delta = -\dfrac{\ddot{\phi}}{H\dot{\phi}} = \eta - \epsilon \end{cases}$$

$$\frac{\ddot{a}}{a} = \dot{H} + H^2 = (1-\epsilon)H^2 > 0 \quad \Leftrightarrow \quad \epsilon < 1$$

Notice : 1) the slow-roll parameters are not constant in time, but run little:

$$\dot{\epsilon} \sim \left(\frac{\dot{\phi}\ddot{\phi}}{H^2} - \frac{\dot{\phi}^2}{H^3}\dot{H} \right)\frac{1}{M_p^2}$$

$$= \frac{\dot{\phi}^2}{H^2}\frac{1}{M_p^2}\frac{\ddot{\phi}H}{\dot{\phi}H} + \frac{\dot{\phi}^2}{H^2 M_p^2}\frac{\dot{H}}{H^2}H$$

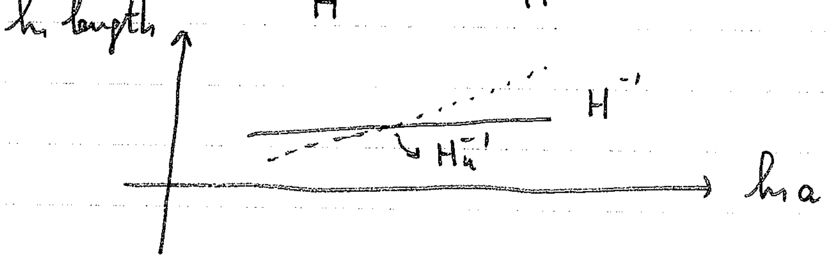
$$\sim H\left(\delta\epsilon - \epsilon^2\right)$$

2) H is not exactly constant (P_ϕ is not exactly equal to $-\rho_\phi$)

Define H_k the value at which a given scale $\lambda = \frac{a}{k}$ leaves the horizon : $k = a H_k$

$$\frac{d \ln H_k^2}{d \ln k} = \left(\frac{d \ln H_k^2}{dt}\right)\left(\frac{dt}{d \ln a}\right)\left(\frac{d \ln a}{d \ln k}\right)$$

$$= 2\frac{\dot{H}}{H} \times \frac{1}{H} \times 1 = 2\frac{\dot{H}}{H^2} = -2\epsilon$$



Total number of efolds:

$$N = \int_{t_i}^{t_f} dt \; H(t)$$

$$= \int_{\phi_i}^{\phi_f} \frac{dt}{d\phi} H(\phi) \, d\phi$$

$$= \int_{\phi_i}^{\phi_f} \frac{H}{\dot{\phi}} d\phi = -3 \int_{\phi_i}^{\phi_f} d\phi \; \frac{H^2}{V'}$$

$$\approx 8\pi G \int_{\phi_f}^{\phi_i} \frac{V(\phi)}{V'(\phi)} d\phi$$

ΔN = # of e-folds to go till the end of inflation

$$\Delta N = 8\pi G \int_{\phi_f}^{\phi_{\Delta N}} \frac{V(\phi)}{V'(\phi)} d\phi$$

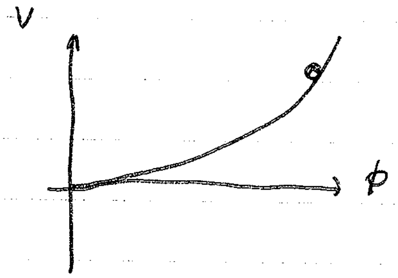
Take $V(\phi) = \frac{m^2}{2}\phi^2 \Rightarrow N \sim 8\pi G \int_{\phi_f}^{\phi_i} \phi \, d\phi = 4\pi G \phi_i^2$

$V(\phi_i) \sim M_P^4 \Rightarrow \phi_i^2 \sim M_P^4/m^2 \Rightarrow N \sim \left(M_P/m\right)^4 \ggg 1$

($m \sim 10^{13}$ GeV, see later)

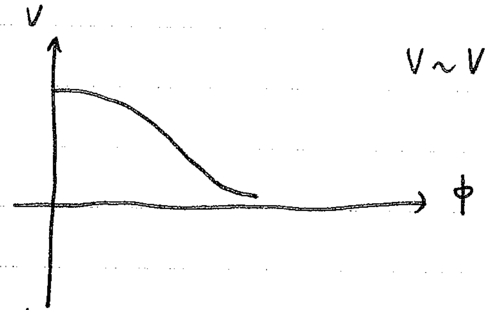
Rough classification:

Large-field models



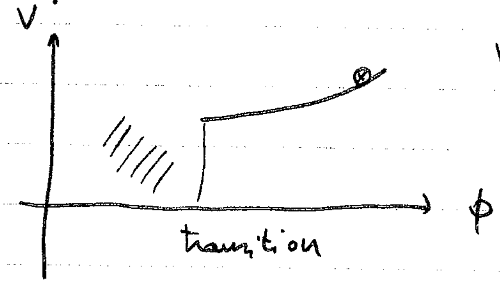
$V \sim \phi^p$

Small-field models



$V \sim V_0 - \mu^{4-p} \phi^p$

Hybrid models

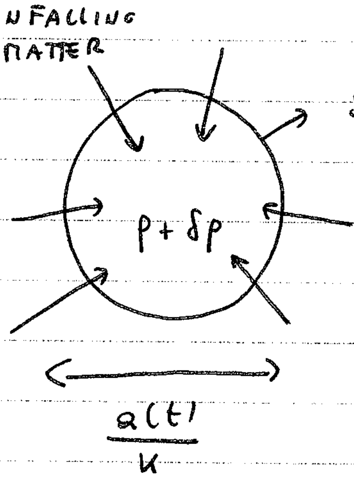


$V \sim V_0 + \mu^{4-p} \phi^p$

THE ORIGIN OF STRUCTURE & INFLATION

Q: How did the first galaxies form?

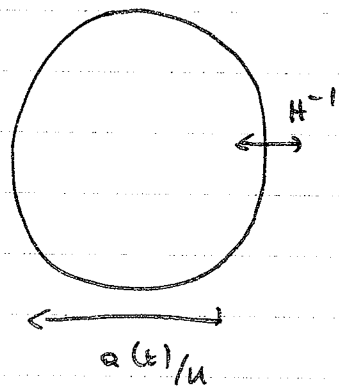
A: By gravitational collapse of slightly overdense regions



Q: When does it start?

A: Horizon entry: $\frac{a(t)}{k} = H^{-1}(t)$

$$\dot{a}(t)/k = 1$$



INFLATION PROVIDES THE SEEDS OF THIS INSTABILITY

Study a scalar field in De-Sitter and start with the simplest case: massless;

$$\chi = \chi(\vec{x}, t)$$

$$\delta\chi(\vec{x},t) = \int \frac{d^3k}{(2\pi)^{3/2}} e^{i\vec{k}\cdot\vec{x}} \delta\chi_k(t)\, a_k + h.c.$$

$$\delta\ddot{\chi} + 3H\delta\dot{\chi} - \frac{\nabla^2\delta\chi}{a^2} = 0$$

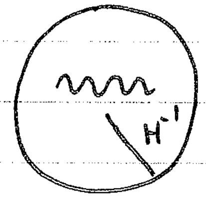
$$\Rightarrow \quad \delta\ddot{\chi}_k + 3H\,\delta\dot{\chi}_k + \frac{k^2}{a^2}\delta\chi_k = 0$$

When written in conformal time:

$$d\tau = \frac{dt}{a} \Rightarrow a(t) \sim e^{Ht} \rightarrow a(\tau) = -\frac{1}{H\tau} \quad (\tau<0)$$

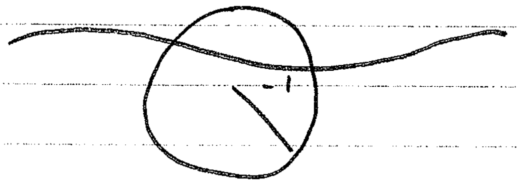
$$\frac{k}{aH} = (-k\tau)$$ defines if a scale is larger of smaller than horizon

- $\lambda \ll H^{-1}$, $k \gg aH$



$$\delta\ddot{\chi}_k + \frac{k^2}{a^2}\delta\chi_k = 0 \quad \Rightarrow \underline{\text{oscillations}}$$

- $\lambda \gg H^{-1}$, $k \ll aH$



$$\delta\ddot{\chi}_k + 3H\delta\dot{\chi}_k = 0$$

$$\underline{\delta\chi_k = \text{const.}}$$

NOT A SURPRISE : $H^{-1} = a(t)\int_t^{\infty}\frac{dt'}{a(t')}$ = event horizon

(NO QUANTUM HAIR De Sitter) : evolution is frozen for $\lambda > H^{-1}$

Define :

$$\delta\sigma_k = \delta\chi_k \, a$$

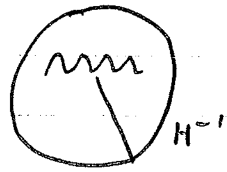
$$d\tau = \frac{dt}{a}$$

$$\delta\sigma_k'' + \left(k^2 - \frac{a''}{a}\right)\delta\sigma_k = 0$$

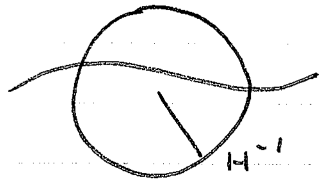
- $k^2 \gg a''/a = \frac{2}{\tau^2} \Rightarrow |k\tau| \gg 1 \quad (\lambda > H^{-1})$

$$\delta\sigma_k'' + k^2\,\delta\sigma_k = 0$$

$$\delta\sigma_k = \frac{e^{-ik\tau}}{\sqrt{2k}}$$



- $k^2 \ll \frac{a''}{a} = \frac{2}{\tau^2} \Rightarrow |k\tau| \ll 1 \quad (\lambda < H^{-1})$



$$\delta\sigma_k = B(k)\, a$$

Fix $B(k)$ imposing that the two solutions match at $k = aH$

$$|B(k)|\,a = \frac{1}{\sqrt{2k}} \Rightarrow B(k) = \frac{1}{a\sqrt{2k}} = \frac{H}{\sqrt{2k^3}}$$

$$\delta\chi_k = \frac{H}{\sqrt{2k^3}} \quad \text{on super-horizon scales}$$

Indeed the equation:

$$\delta\sigma_k'' + \left(k^2 - \frac{a''}{a}\right)\delta\sigma_k = 0$$

has the exact solution

$$\delta\sigma_k = \frac{A e^{-ik\tau}}{\sqrt{2k}}\left(1 - \frac{i}{k\tau}\right) + \frac{B e^{ik\tau}}{\sqrt{2k}}\left(1 + \frac{i}{k\tau}\right)$$

$B = 0$, $A = 1$ when imposing that for $-k\tau \gg 1$ get plane-waves with positive energy (MINKOWSKY VACUUM)

Notice: if $\frac{m^2}{2}\chi^2$ term is present:

$$\left(\delta\sigma_k'' + \left(k^2 - \frac{a''}{a} + \frac{m^2}{H^2}\frac{1}{\tau^2}\right)\delta\sigma_k = 0\right)$$

$$\delta\chi_k \simeq \frac{H}{\sqrt{2k^3}}\left(\frac{k}{aH}\right)^{\frac{3}{2}-\nu} \; ; \; \nu^2 = \frac{9}{4} - \frac{m^2}{H^2}$$

on superhorizon scales

Def :

$$\langle 0 | (\delta\chi(\vec{x},t))^2 | 0 \rangle$$

$$= \int \frac{d^3k}{(2\pi)^3} \, |\delta\chi_k|^2$$

$$\equiv \int \frac{dk}{k} \, \mathcal{P}_{\delta\chi}(k) \quad ; \quad \mathcal{P}_{\delta\chi}(k) = \frac{k^3}{2\pi^2} |\delta\chi_k|^2$$

power spectrum

For a massless field in de Sitter on superhorizon scales:

$$\mathcal{P}_{\delta\chi} = \left(\frac{H}{2\pi}\right)^2$$

We may define a spectral index n

$$\mathcal{P}_{\delta\chi}(k) = A^2 \left(\frac{k}{aH}\right)^{n-1}$$

$\Rightarrow$ Massless fields in the de Sitter background have $n = 1$

Take a massless scalar field in quasi de Sitter : $\epsilon \ll 1$, but not too

$$a(\tau) = -\frac{1}{H\tau(1-\epsilon)}$$

$$\frac{a''}{a} \simeq \frac{1}{\tau^2}(2+3\epsilon)$$

$$\delta\sigma_k'' + \left(k^2 - \frac{a''}{a}\right)\delta\sigma_k = 0$$

$$\Rightarrow \quad (\lambda \gg H^{-1}) \quad \delta\chi_k = \frac{\delta\sigma_k}{a} = \frac{H}{\sqrt{2k^3}}\left(-k\tau\right)^{-\epsilon}$$

$$P_{\delta\chi} = \left(\frac{H}{2\pi}\right)^2 \left(\frac{k}{aH}\right)^{-2\epsilon}$$

- Trick to obtain the same result: account for the change of H : $H \to H_k$

$$P_{\delta\chi} = \left(\frac{H_k}{2\pi}\right)^2 \Rightarrow n-1 = \frac{d \ln P_{\delta\chi}}{d \ln k} = -2\epsilon$$

Perturbations are GAUSSIAN! (linear)

We imagine an ensemble of Universes, ours is typical (each perturbation is a random field) _ Gaussian means that

$$\langle \delta \chi_k \rangle = 0$$

$$\langle \delta \chi^*_{\vec{k}} \, \delta \chi_{\vec{k}'} \rangle = \delta^{(3)}(\vec{k}-\vec{k}') \times \frac{2\pi}{k^3} P_{\delta \chi}(k)$$

is the only object needed

Remember: gravity is non linear $\Rightarrow$ Non-Gaussianity!

INCLUDE GRAVITY

We forgot about gravity (wrong)

Suppose we have a scalar field $\phi = \phi(\vec{x},t)$ driving INFLATION $\equiv$ INFLATON

$$\begin{tikzcd} \delta\phi(\vec{x},t) \arrow[r, Rightarrow] & \delta T^{\phi}_{\mu\nu} \arrow[dl, Rightarrow, "G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 8\pi G T_{\mu\nu}"] \\ \delta G_{\mu\nu},\ \delta g_{\mu\nu} \arrow[u, Rightarrow] & \end{tikzcd}$$

Heuristic explanation of why the inflaton is perturbed:

$$\phi(\vec{x},t) = \phi_0(t) + \delta\phi(\vec{x},t)$$

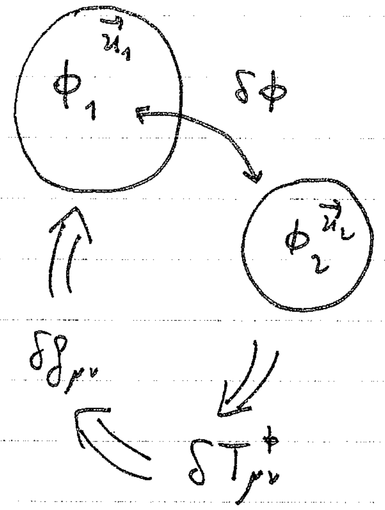
$$\delta\ddot{\phi} + 3H\delta\dot{\phi} - \frac{\nabla^2\delta\phi}{a^2} + V''\delta\phi = 0$$

$$\ddot{\phi}_0 + 3H\dot{\phi}_0 + V'(\phi_0) = 0 \Rightarrow \dddot{\phi}_0 + 3H\ddot{\phi}_0 + V''\dot{\phi}_0 \approx 0 \quad (\dot{H} \simeq 0)$$

$$\Rightarrow \quad \delta\phi = -\dot{\phi}_0 \delta t(\vec{x})$$

$$\phi(\vec{x},t) = \phi_0(t - \delta t(\vec{x}))$$

The inflaton does not acquire the same value at different points



Metric fluctuations:

$$g_{\mu\nu}(\vec{x},t) = \underset{\text{FRW}}{\underset{\parallel}{g^0_{\mu\nu}}} + \delta g_{\mu\nu}(\vec{x},t)$$

Metric perturbations may be decomposed according to the way they transform under rotations on const. time hypersurfaces:

- scalar
- vector
- tensor

$g_{\mu\nu}$ is symmetric $\Rightarrow$ 10 elements

6 = 10 − 4 are physical (can make a transformation $x^\mu \to x^\mu + \delta x^\mu$, $\mu = 0, \dots, 3$)

- Helmholtz's theorem : $u_i = \partial_i v + v_i$ (vorticity

$\nabla \cdot \vec{v} = 0$, $v_{[i;j]} = 0$

$\Downarrow$

2 d.o.f. for vectors

- h_{ij} has 6 entries but $\partial^i h_{ij} = 0$ transverse

$h^i_{\ i} = 0$ Traceless

$\Downarrow$

6 − 4 = 2 for tensors

$\Downarrow$

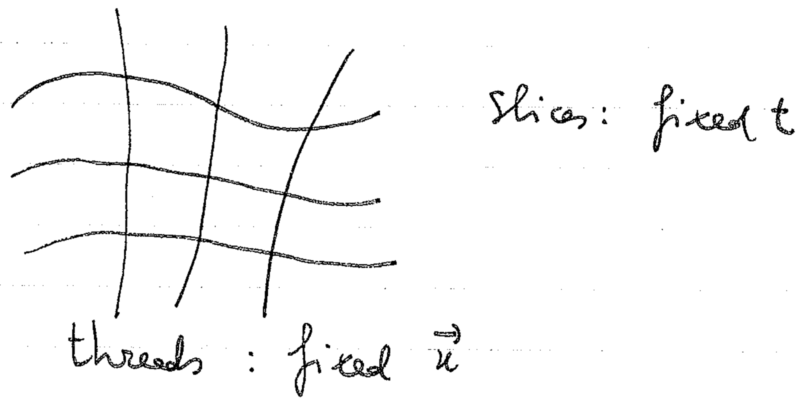
6 − 2 − 2 = 2 d.o.f. for scalars

Consider from now on only scalars:

$$ds^2 = a^2\left[(1+2A)\,d\tau^2 - (1-2\psi)\,d\vec{x}^2\right]$$

GR is a gauge theory whose gauge transformations are generic coordinate transformations from a local reference to another

To define the perturbations we need to specify a SLICING & a THREADING of spacetime, corresponding to some coordinate system



GAUGE CHOICE $\Longleftrightarrow$ SLICING & THREADING

We are interested only in slicing

Take a scalar :

$$\begin{cases} x^0 \to x^0 + \delta x^0 = \tilde{x}^0 \\ \\ \delta f(x) = f(x) - f_0(x) \end{cases}$$

$$\tilde{f}(\tilde{x}) = f(x)$$
$$\tilde{f}_0(\tilde{x}) = f_0(\tilde{x}_0)$$

$$\widetilde{\delta f}(\tilde{x}) = \tilde{f}(\tilde{x}) - \tilde{f}_0(\tilde{x})$$

$$= f(x) - f_0(\tilde{x})$$

$$= f(x) - \dot{f}_0(x)\,\delta x^0 - f_0(x)$$

$$= \delta f - \dot{f}_0\,\delta x^0$$

$$ds^2 = \widetilde{ds}^2 \Rightarrow a^2(\tilde{x}^0)(1+2\tilde{A})(d\tilde{x}^0)^2$$

$$= a^2(x^0)(1+2A)(dx^0)^2$$

$$a^2(\tilde{x}^0) = a^2(x^0) + 2aa'\,\delta x^0, \quad d\tilde{x}^0 = dx^0 + (\delta x^0)'\,dx^0$$

$$\Rightarrow \tilde{A} = A - (\delta x^0)' - \frac{a'}{a}\,\delta x^0$$

Analogously one can show $\tilde{\psi} = \psi + \frac{a'}{a}\delta x^0$

Instead of choosing the slicing we can work with gauge invariant quantities:

$$\boxed{\zeta = \psi + H\frac{\delta\rho}{\dot{\rho}}}$$

$^{(3)}R = \frac{4}{a^2}\nabla^2\psi$

a) curvature perturbation on slices of uniform energy density

$$\zeta = \psi\Big|_{\delta\rho=0}$$

b) $\delta\rho$ perturbation on flat ($\psi=0$) slices

$$\zeta = \frac{H\,\delta\rho}{\dot{\rho}}\Bigg|_{\psi=0} = \frac{-\delta\rho}{3(P+\rho)}\Bigg|_{\psi=0}$$

The curvature perturbation is constant on superhorizon scales if the adiabatic condition on the pressure holds:

$$\delta \nabla_\mu T^{\mu\nu} = 0 \Rightarrow \delta\dot{\rho} = -3H(\delta\rho + \delta P) - 3\dot{\psi}(\bar{P} + \bar{\rho})$$

$$\delta P = \delta P_{\text{nonad}} + \frac{\dot{P}}{\dot{\rho}}\delta\rho$$

Go to the uniform energy density slice

$$\Rightarrow \delta\rho = 0 \quad \& \quad \psi = \zeta$$

$$\Rightarrow \dot{\zeta} = -\frac{H}{\bar{P} + \bar{\rho}}\delta P_{\text{nonadiabatic}}$$

If $P = P(\rho)$ $\delta P_{\text{nonad}} = 0$

$$\boxed{\dot{\zeta} = 0}$$

Follows from energy conservation <u>not</u> gravity!

The curvature perturbation generated by the inflaton field:

On flat slices $\zeta = - \frac{\delta\rho}{3(P+\rho)}$

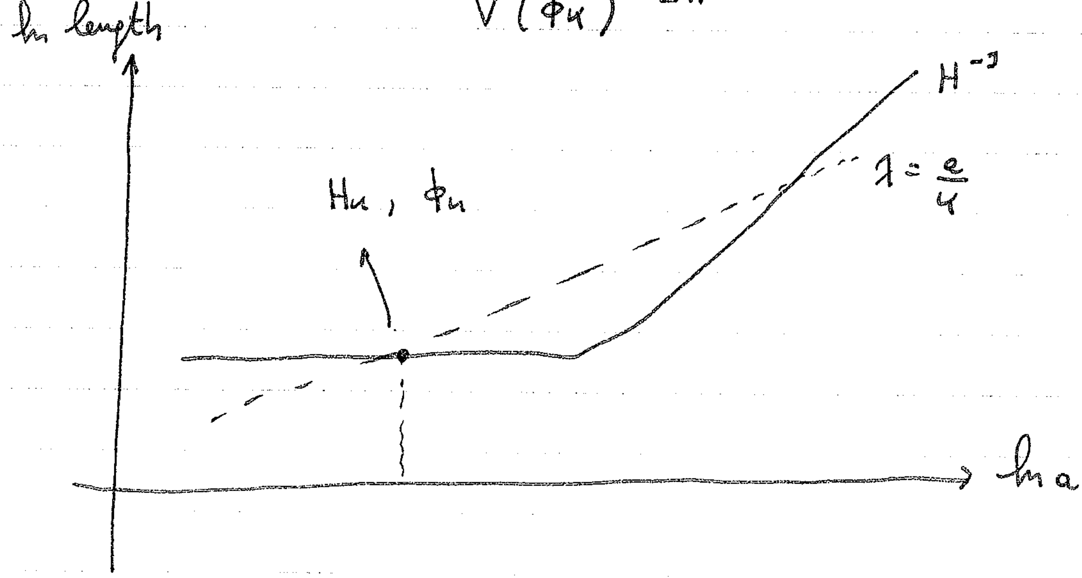
$$= - \frac{V'\,\delta\phi}{3\,\dot{\phi}^2}$$

$$= \frac{-3V'\,\delta\phi}{\frac{(V')^2}{H^2}}$$

$$\Downarrow$$

$$\zeta_k = -3\frac{H_k^2}{V'(\phi_k)}\,\delta\phi_k = H_k\frac{\delta\phi_k}{\dot{\phi}_k}$$

$$= -\frac{3\,H_k^3}{V'(\phi_k)}\,\frac{1}{2\pi}$$



Inflation provides a curvature perturbation on very large scales from initial quantum fluctuations

$$\mathcal{P}_\zeta = \frac{1}{2M_p^2\epsilon}\left(\frac{H}{2\pi}\right)^2\left(\frac{k}{aH}\right)^{n_\zeta - 1}$$

$$\zeta_k \sim \frac{H_k}{\dot\phi_k}\,\delta\phi_k \sim \frac{1}{2\pi}\left(\frac{H_k^2}{\dot\phi_k}\right)$$

$$\frac{d\ln\dot\phi_k}{d\ln k} = \left(\frac{d\ln\dot\phi_k}{dt}\right)\left(\frac{dt}{d\ln a}\right)\left(\frac{d\ln a}{d\ln k}\right) = \frac{\ddot\phi_k}{\dot\phi_k}\times\frac{1}{H}\times 1$$

$$= -\delta = \epsilon - \eta$$

$$\Rightarrow n_\zeta - 1 = \frac{d\ln\mathcal{P}_\zeta}{d\ln k} = \frac{d\ln H_k^4}{d\ln k} - \frac{d\ln\dot\phi_k^2}{d\ln k}$$

$$= -4\epsilon + 2(-\epsilon+\eta)$$

$$= 2\eta - 6\epsilon$$

The spectrum is tilted (slightly)

Example: $V(\phi) = \frac{m^2}{2}\phi^2$

$$3H\dot{\phi} = -V' = -m^2\phi$$

$$\zeta \sim \frac{H\,\delta\phi}{\dot{\phi}} = \frac{H^2}{2\pi}\left(-\frac{1}{m^2\phi}\right)H$$

$$\Delta N = \int_{t_{\Delta N}}^{t_f} dt'\, H(t')$$

$$= \int_{\phi_f}^{\phi_{\Delta N}} d\phi \quad 8\pi G\,\frac{V}{V'}$$

$$\simeq \frac{1}{2}\,\frac{\phi_{\Delta N}^2}{M_p^2}$$

$$\zeta_{\Delta N} = \frac{m^2\phi_{\Delta N}^2}{6M_p^2}\,\frac{1}{2\pi}\left(-\frac{1}{m^2\phi_{\Delta N}}\right)H$$

$$= \frac{-1}{12\pi M_p^2}\,\frac{m\,\phi_{\Delta N}^2}{\sqrt{6}\,M_p}$$

$$= -\frac{1}{12\pi\sqrt{6}}\left(\frac{m}{M_p}\right)\times 2 \quad \sim 5\times 10^{-5}$$

$$\Rightarrow \quad m \sim 10^{13}\ \text{GeV}$$

After inflation has ended and all vacuum energy has been released into radiation $\longrightarrow$ RD $\longrightarrow$ MD

$$\zeta_m = -\frac{1}{3}\frac{\delta\rho_m}{\rho_m} = \zeta_{\text{primordial}} = \frac{H\,\delta\phi}{\dot\phi}$$

$$\zeta_R \sim \frac{\delta\rho_R}{\rho_R} \sim \frac{\delta T}{T} \sim 10^{-5}$$

<u>GRAVITATIONAL WAVES:</u>

$$S = \frac{M_p^2}{2}\int d^4x\,\sqrt{-g}\;\frac{1}{2}\,\partial_\sigma h^{ij}\,\partial^\sigma h_{ij}$$

$$v_k = \frac{a M_p}{\sqrt{2}}\,h_k \;\Rightarrow\; \left(v_k'' + \left(k^2 - \frac{a''}{a}\right) v_k = 0 \right)$$

$$P_h = \frac{k^3}{2\pi^2}\sum_{\lambda=\pm} |h_k|^2 = \frac{8}{M_p^2}\left(\frac{H}{2\pi}\right)^2\left(\frac{k}{aH}\right)^{n_T}$$

$$n_T = -2\epsilon$$

Since $P_\zeta = \left(\frac{H}{2\pi M_p}\right)^2 \frac{1}{2\epsilon}$

$\Rightarrow$ in slow-roll inflation : $\frac{T}{S} \sim \epsilon \sim n_T$

Proof that the curvature perturbation is conserved at any order of perturbation theory on super-horizon scales for adiabatic fluids :

On scales larger than the horizon $\frac{k}{a} \ll H$ we can neglect all gradients and the Universe should look a collection of separate almost homogeneous universes

We choose a threading of spatial coordinates comoving with the fluid

$$u^\mu = \frac{dx^\mu}{dt}, \quad v^i = \frac{u^i}{u^0} = \frac{dx^i}{dt} = 0$$

rate of expansion $\theta = \nabla_\mu u^\mu = \frac{1}{N e^{3\alpha}} \partial_0 \left(e^{3\alpha} \right)$

$$= \frac{3\dot{\alpha}}{N}$$

where $g_{00} = N^2$, $g_{ij} = e^{2\alpha} \tilde{\gamma}_{ij}$, $\det \tilde{\gamma}_{ij} = 1$

The energy conservation equation $u_\nu \nabla_\mu T^{\mu\nu} = 0$

$$\Rightarrow \quad \frac{d}{d\tau} \rho + (P+\rho)\, \theta = 0 \quad \text{where } \frac{dt}{d\tau} = u^0 = \frac{1}{N}$$

$$\Rightarrow \quad \dot{\rho} + 3(P+\rho)\, \dot{\alpha} = 0$$

Define $a(t)\, e^{-\psi} = e^{\alpha}$

$$\Rightarrow \quad \theta = \frac{1}{\mathcal{N}}\left(\frac{\dot a}{a} - \dot\psi\right) = \frac{3\dot\alpha}{\mathcal{N}}$$

$$\dot\rho + 3(P+\rho)\,\dot\alpha = 0$$

$$\Rightarrow \quad \frac{\dot a}{a} - \dot\psi = 3\dot\alpha = -\frac{\dot\rho}{P+\rho}$$

$$N(t_2, t_1, x^i) \equiv \frac{1}{3}\int_{\tau_1}^{\tau_2} \theta\, d\tau = \frac{1}{3}\int_{t_1}^{t_2} \theta\, \mathcal{N}\, dt$$

is the # of e-folds of expansion along an integral curve of the 4-velocity (x^i comoving with fluid)

$$N(t_2, t_1, x^i) = -\frac{1}{3}\int_{t_1}^{t_2} dt\, \frac{\dot\rho}{\rho+P}\Big|_{x^i}$$

$$\Rightarrow \quad \psi(t_2, x^i) - \psi(t_1, x^i) = -N(t_2, t_1, x^i) + \ln\frac{a(t_2)}{a(t_1)} \qquad (*)$$

The change in ψ from one slice to another equals the difference the actual # of e-folds and the background one

In a flat slice $\quad N(t_2, t_1, x^i) = \ln\frac{a(t_2)}{a(t_1)}$

Consider now two different slices A & B which coincide at $t = t_1$

$$\Rightarrow \; -N_A(t_2, t_1, x^i) + N_B(t_2, t_1, x^i)$$

$$= \psi_A(t_2, x^i) - \psi_B(t_2, x^i)$$

Now, choose the slice A such that it is flat at $t = t_1$ and ends on a uniform energy slice at $t = t_2$ and B to be flat both at t_1 & t_2

$$\Rightarrow -\psi_A(t_2, x^i) = N_A(t_2, t_1, x^i) - N_0(t_2, t_1, x^i)$$

(since B is flat)

From (*)

$$-\psi(t_2, x^i) + \psi(t_1, x^i) = -\ln \frac{a(t_2)}{a(t_1)} - \frac{1}{3} \int_{\rho(t_1, x^i)}^{\rho(t_2, x^i)} \frac{d\rho}{\rho + P}$$

if $P = P(\rho)$

$$\Rightarrow \zeta = -\psi + \frac{1}{3} \int_{\rho(t)}^{\rho(t, x^i)} \frac{d\rho}{P + \rho} \quad \text{is constant}$$

$\Rightarrow \zeta$ can be computed using the ΔN-formalism: it is the difference in N between the uniform-density slicing and the flat slice from t_1 to t_2

$$\Rightarrow \quad \zeta = \delta N$$

$$\delta N = \delta N\big(\phi(\vec{x},t)\big) \qquad\qquad N = \int H\, dt$$

$$\zeta = \frac{\partial N}{\partial \phi}\,\delta\phi$$

$$= \frac{\partial N}{\partial t}\,\frac{1}{\dot\phi}\,\delta\phi$$

$$= H\,\frac{\delta\phi}{\dot\phi}$$

One can go higher in order

$$\zeta = \frac{\partial N}{\partial \phi}\,\delta\phi + \frac{1}{2}\,\frac{\partial^2 N}{\partial \phi^2}(\delta\phi)^2 + \dots$$

$$= H\,\frac{\delta\phi}{\dot\phi} + \frac{1}{2}\,\frac{\partial}{\partial\phi}\left(\frac{\partial N}{\partial t}\,\frac{1}{\dot\phi}\right)(\delta\phi)^2$$

$$= H\,\frac{\delta\phi}{\dot\phi} + \frac{1}{2}\,\frac{\partial}{\partial\phi}\left(\frac{H}{\dot\phi}\right)(\delta\phi)^2$$

$$= H\,\frac{\delta\phi}{\dot\phi} + \frac{1}{2}\,\frac{\partial}{\partial t}\left(\frac{H}{\dot\phi}\right)\frac{(\delta\phi)^2}{\dot\phi}$$

$$\frac{\partial}{\partial t}\left(H/\dot\phi\right) = \dot H/\dot\phi - \frac{H}{\dot\phi^2}\,\ddot\phi = -\epsilon\,\frac{H^2}{\dot\phi} - \frac{H}{\dot\phi^2}\,\ddot\phi$$

$$\Rightarrow \zeta = H\,\frac{\delta\phi}{\dot\phi} + \frac{1}{2}\Big(-\epsilon + \delta\Big)\left(\frac{H\,\delta\phi}{\dot\phi}\right)^2 =$$

$$= H\frac{\delta\phi}{\dot\phi} + \frac{1}{2}(\eta - 2\epsilon)\left(\frac{H\delta\phi}{\dot\phi}\right)^2$$

$$\Rightarrow \quad \zeta = \zeta_L + \mathcal{O}(\epsilon,\eta)\,(\zeta_L)^2$$

Non-Gaussianity (at the end of inflation) is TINY

One example which makes use of the fact that ζ is not conserved at superhorizon scales when $P \neq P(\rho)$

Consider a scalar field σ with VEV $\bar\sigma$ almost massless ($m^2 \ll H^2$) during inflation:

We know that $\quad \delta\sigma_k = \left(\frac{H_k}{2\pi}\right)$

During inflation $\bar\sigma$ remains frozen and starts oscillating when $m \gtrsim H$ (after inflation)

$$\ddot{\bar\sigma} + 3H\dot{\bar\sigma} + m^2\bar\sigma = 0$$

$$\frac{1}{2}\frac{d}{dt}\dot{\bar\sigma}^2 + 3H\dot{\bar\sigma}\dot{\bar\sigma} + m^2\bar\sigma\dot{\bar\sigma} = 0$$

If $m \gg H$ the field oscillates many times in one Hubble time $\Rightarrow$

$\langle \dot{\bar\sigma}^2 \rangle = \langle \bar\sigma^2 \rangle m^2$; $\langle \cdots \rangle$ over many oscillations

$\langle \rho_\sigma \rangle = m^2 \langle \bar\sigma^2 \rangle$

$\langle \rho_\sigma \rangle = \left\langle \frac{1}{2} \dot{\bar\sigma}^2 + \frac{m^2}{2} \bar\sigma^2 \right\rangle$

$\langle \dot\rho_\sigma \rangle = \left\langle \frac{1}{2}\, 2 \dot{\bar\sigma} \ddot{\bar\sigma} + m^2 \bar\sigma \dot{\bar\sigma} \right\rangle$

$= -3H \langle \dot{\bar\sigma} \dot{\bar\sigma} \rangle = -3H \langle \dot{\bar\sigma}^2 \rangle$

$= -3H \langle \rho_\sigma \rangle$

$\langle \rho_\sigma \rangle \propto a^{-3} \Rightarrow \bar\sigma \sim a^{-3/2}$

$\ddot{\delta\sigma}_k + 3H \dot{\delta\sigma}_k + m^2 \delta\sigma_k = 0 \Rightarrow$ scaling of $\delta\sigma$ is $a^{-3/2}$

$\Rightarrow$ for $m \gtrsim H$ $\quad \frac{\delta\sigma}{\sigma}$ remains constant

$\Rightarrow \frac{\delta\rho_\sigma}{\rho_\sigma} \sim \frac{m^2 \bar\sigma \delta\sigma}{m^2 \bar\sigma^2} \sim \frac{\delta\sigma}{\bar\sigma}$ remains constant

Suppose there are no inflaton fluctuations during inflation — After inflaton decay:

$\zeta = \psi + H \frac{\delta\rho}{\dot\rho} = \psi + H \sum_i \frac{\delta\rho_i}{\dot\rho}$ $\quad i =$ fluids

$= \sum_i \frac{\dot\rho_i}{\dot\rho} \zeta_i$, $\quad \zeta_i = \psi + H \frac{\delta\rho}{\dot\rho_i}$

During RD:

$$\zeta = \frac{\dot\rho_R}{\dot\rho}\zeta_R + \frac{\dot\rho_\sigma}{\dot\rho}\zeta_\sigma \quad , \text{ but } \zeta_R = 0$$

($\dot\zeta_R = \dot\zeta_\sigma = 0$ because of energy conservation)

$\dot\zeta \neq 0 : (\dot\rho_\sigma / \dot\rho)^{\cdot} \neq 0$

The field σ decays when $\frac{\dot\rho_\sigma}{\dot\rho} = r$

and all the perturb^2 in σ are transfered to radiation

$$\zeta_{\text{after decay}} \equiv r\,\zeta_\sigma = r\,\zeta_\sigma^{\text{primordial}}$$

$$= r\,\frac{\delta\rho_\sigma}{\rho_\sigma} = r\,\left(\frac{\delta\sigma}{\sigma}\right)_{\text{at inflation}}$$

This is the <u>CURVATON</u>

$$\zeta = r\,\frac{\delta\rho_\sigma}{\rho_\sigma} = r\,\frac{(\bar\sigma\,\delta\sigma + (\delta\sigma)^2)}{\bar\sigma^2}$$

$$= r\,\frac{\delta\sigma}{\bar\sigma} + r\left(\frac{\delta\sigma}{\bar\sigma}\right)^2$$

$$= \zeta_L + \frac{1}{r}(\zeta_L)^2$$

If $r \ll 1$ large Non-Gaussian components