Introduction

Black holes in String Theory very broad subject

string theory gives rise to (more general) black hole solutions:

- charged black branes + extremal (supersymmetric) versions of these

two main themes:
  - use black branes in connection with gauge/gravity correspondence
ead to learn about thermal phases of dual non-gravitational theories
  (e.g. super Yang-Mills theory) living on brane (holography)
- microscopic description of black hole entropy via string theory

Focus in these lectures:
  - black holes, strings and rings in higher dimensions
    (intimately connected to black objects in string theory)
  - black branes in string theory (U-duality, thermodynamics, stability, .. )

Many other interesting recent developments in string theory:
  - black hole entropy and attractors, relation to topological strings
  - fuzzball conjecture (see lecture Mathur)
Outline

- Introduction to black branes and their thermodynamics + relevance for the gauge/gravity correspondence

Part 1:
- Black objects in higher dimensional spaces
  - Kaluza-Klein black holes (spaces with extra compact directions)
  - rotating (stationary) black objects (in asymptotically flat space)
  - newly found connection between the phase structure of these two

Part 2:
- non-and near- extremal branes in string theory via boost/U-duality map
- applications of KK black holes to string theory:
  - correlated stability conjecture
    (relation between thermodynamic and classical stability)

See e.g. Review articles by: Kol (PhysRept)/Harmark,Niarchos,NO (CQG) also Tasi lecture of Peet (+ various other reviews)
Motivation (higher D gravity + String Theory)

- Study **black objects in higher dimensional gravity**
  - richer phase structure
  - *(non)-uniqueness theorems* in higher dimensional gravity
  - new **topologies** of event horizons possible
  - gravitational **phase transitions** between different solutions with event horizons (topology change)
  - **Gregory-Laflamme instability** (new phases)
  - possible objects in **universe/accelerators**
  - + many **ST applications** (black branes, BH entropy, AdS/CFT)

▶ **Two cases** studied — most progress in recent years — less explored

- **asymptotically flat** spaces: five dimensions six and beyond
  (stationary solutions)
  - MP black holes, black rings, black Saturns, black di-rings,

- **Kaluza-Klein** spaces: \( d \)-dim Minkowski x circle (tori) other Ricci flat...
  (static solutions)
  - non-uniform strings, localized black holes
  - bubble-black hole sequences, merger point evolution of GL instability
  e.g. CY
String/Gauge Theory Motivations

- phase structure of Kaluza-Klein black holes related to objects and phenomena in string theory/gauge theory
  
  Bostock, Ross/Aharony, Marsano, Minwalla, Wiseman/Harmark, NO

  phase structure of non- + near-extremal branes (with circle in transverse space)
  
  - new insights into phase structure of strongly coupled large $\mathcal{N}$ theories
    - qualitative/quantitative tests of gauge/gravity correspondence
  - correlated stability conjecture
  - new stable phase of LST
  - entropy of 3-charge BHs on circle

- Black rings + supersymmetric cousins play important role in string theory
  
  - supersymmetric black rings, supertubes
  - microscopic counting of entropy, 4D-5D connection
  - foaming black rings and fuzzball proposal
  - plasma rings…..
Black hole thermodynamics is intimately related to gauge theory dynamics in very precise way, via brane solutions in string theory.

- Solutions of pure gravity with event horizon, in $D$-dimensions
- Boost/U-duality
- Black holes

- Solutions of String/M-theory
- Non-extremal p-branes
- Near-extremal p-branes
- Gauge/gravity correspondence

- (p+1)-dimensional (supersymmetric) gauge theories at strong coupling
black holes in higher dimensions

What do we know about black objects (i.e. with event horizon) in higher dimensional gravity

- some, but still lot to discover

- stationary black holes

- $\mathcal{M}^D$

- $\mathcal{M}^{D-1} \times S^1$

- KK black holes
Black hole non-uniqueness

In 4 dimensions: given mass, angular momentum and charge: unique black hole solution

For $D$-dimensional asymptotically flat space times: only static and neutral black hole in pure gravity is Schwarzschild-Tangherlini black hole

$$\frac{\Omega_{d-2}(d-2)}{16\pi G} r_0^{d-3}$$

Q: Does this uniqueness extend to higher dimensional GR? Recent years of research gives answer: No!

two examples of such non-uniqueness known:
Rotating ring in five dimensions

5D asymptotically flat rotating BH solutions

- Myers-Perry black hole (generalizes 4D Kerr solution)
- Rotating black ring (Emparan-Reall)

+ Black saturn, black di-rings etc. (exact solutions)

horizon topology

$S^3$

$S^2 \times S^1$

Infinite non-uniqueness for configurations not in thermal equilibrium

Non-uniqueness even persists for supersymmetric generalization in ST

EmparanReall/Elvang,Figueras /Elvang,Emparan,Figueras

Elvang,Emparan,Mateos,Reall
Kaluza-Klein black holes

- black holes asymptoting to d-dimensional Minkowvski space times
  a circle (Kaluza-Klein spaces) = \( \mathcal{M}^d \times S^1 \)
  = time x \( d \)-dimensional cylinder \( \mathbb{R}^{d-1} \times S^1 \)

circle direction breaks symmetry \( \rightarrow \) gives rise to new possibilities of BH solutions

- what are the static & neutral BH solutions on the cylinder ?
- why richer phase structure ?
- how can we parameterize extra freedom ?

- consider case with spherical symmetry for \( \mathbb{R}^{d-1} \) part of cylinder:

at \( \infty \) can think of any BH solution as coming from Newtonian source located at origin of \( \mathbb{R}^{d-1} \) but with mass distribution in circle direction: source \( \rho(z) \)

measure at horizon proper radius
of \( S^{d-2} \) around cylinder \( \rightarrow \) profile \( r(z) \)
- heuristically we can connect \( r(z) \) to mass distribution \( \rho(z) \) by imagining d-dim static Sch BH for each value of z
Possible BH solutions

- do all profile/mass distributions correspond to BH solutions?

- clearly No – BH solution in GR automatically takes into account self-gravitation of the mass distribution, so not even for Newtonian matter would we expect that what are possible BH solutions?

![Diagrams of black hole localized on the cylinder, uniform black string, and non-uniform black string](image)

plus copies: repeat same profile number of times (e.g. 4 times)

![Repeated profiles](image)

these are presently known solutions on cylinder (assuming spherical symmetry)
Mass and tension for KK BHs

consider static solutions of vacuum Einstein equations

- coordinates for $\mathcal{M}^d \times S^1$ ($d \geq 4$)

$$ds^2 = -dt^2 + dr^2 + r^2 d\Omega_{d-2}^2 + dz^2 \quad z \sim z + L$$

asymptotics

$$g_{tt} \sim -1 + \frac{c_t}{r^{d-3}}, \quad g_{zz} \sim 1 + \frac{c_z}{r^{d-3}}$$

2 (gauge-invariant) asymptotic quantities

$$M = \frac{\Omega_{d-2} L}{16 \pi G_N} [(d-2)c_t - c_z], \quad T = \frac{\Omega_{d-2}}{16 \pi G_N} [c_t - (d-2)c_z]$$

mass \quad \text{tension}

1st law of thermo

$$\delta M = T \delta S + T \delta L$$

Smarr formula

$$TS = \frac{d-2}{d-1} M - \frac{T L}{d-1}$$

using Komar integral

(time-translation sym.)

dimensionless quantities:

$$\mu = \frac{16 \pi G_N}{L^{d-2}} M, \quad n = \frac{T L}{M} \quad \mu \geq 0, \quad 0 \leq n \leq d - 2$$

Harmark, NO/Kol, Sorkin, Piran/Traschen, Fox/Towsend, Zamaklar
Uniform black string (UBS) + Gregory-Laflamme instability

\[ ds^2 = -f dt^2 + f^{-1} dr^2 + r^2 d\Omega_{d-2}^2 + dz^2 \]

\[ f = 1 - \frac{r_0^{d-3}}{r^{d-3}} \]

d-dim Schw-Tang. BH x flat compactified direction

GL found:
classically unstable \( \mu < \mu_{GL} \)
classically stable

\[ \delta g_{\mu\nu} \sim e^{ikz} e^{\Omega t} \]

long wave-length instability
– present when critical GL wavelength can fit in the compact direction

non-uniform static solution emerges

threshold mode \( \Omega = 0 \)

global thermodynamic argument:

\[ \mu \ll \mu_{GL}, \quad S_{BH}(M) > S_{BS}(M) \]

suggests that UBS decays to LBH

(subject of discussion)
Non-Uniform black string (NUBS)

- UBS has marginal mode at $\mu = \mu_{GL}$
- new branch of solutions emerges from this point
  - not translational invariant along circle direction

$d < 13$ : branch numerically known

$$n(\mu) = \frac{1}{2} - 0.14(\mu - \mu_{GL}) + O((\mu - \mu_{GL})^2), \quad 0 \leq \mu - \mu_{GL} \ll 1$$

- critical dimension $D=14$:
  - slope of non-uniform branch reverses
    (2\textsuperscript{nd} order transition instead of 1\textsuperscript{st} order)

- analytical result: ansatz for metric of NUBS
Localized black hole on cylinder (LBH)

- for small mass (large circle radius) LBH becomes more and more like $d+1$ dimensional Schwarzschild-Tangherlini BH
  
  tension vanishes in limit of vanishing mass $n \to 0$ as $\mu \to 0$

- analytical results
  - ansatz for metric
  - first-order metric for LBH for $\mu \ll 1$
    (found using method of matched asymptotic expansion)
  - higher order corrections to metric and thermodynamics

  \[ n(\mu) = \frac{(d-2)\zeta(d-2)}{2(d-1)\Omega_{d-1}} \mu + \mathcal{O}(\mu^2) \]

- numerical results
  
  $d=4,5$: LBH solution found with impressive accuracy
Phase Diagram in six dimensions

- Uniform black string (US) $S^{d-2} \times S^1$
- Non-uniform black string (NUS) $S^{d-2} \times S^1$
- Localized black hole branch (LBH) $S^{d-1}$

- Schwarzschild $(d) \times S^1$
  emanates from uniform at Gregory-Laflamme point
  - Gregory,Laflamme/Gubser/Wiseman/Sorkin motived in part by: Horowitz,Maeda

- Schwarzschild $(d+1) + \mathcal{O} (\mu)$
  - Harmark,NO/Harmar/Kol,Gorbonos/Sorkin,Kol,Piran/Koduh,Wiseman
Copies of LBH and NUBS phases

- for solutions that vary in circle direction
  Horowitz/Harmark, NO
  can generate copies by copying $k$ times on circle and
  scaling appropriately in $\mathbb{R}^{d-1}$ part

  [Diagram of LBH and NUBS phases]

  ▲ from this we can argue existence
  of localized multi-black configurations
  of different mass/size

Dias.Harmark, Myers NO
Multi-black hole configurations in phase diagram

- range of \( n \) is such that it can take all values between small LBH solution and the small LBH\(_k\) solution
  
  \( \text{continuous non-uniqueness} \) for given mass \( \mu \)
  
  - takes infinite amount of continuous parameters to point to specific solution!

  (non-thermal equilibrium solutions)

- multi BH solutions are in \textit{unstable mechanical equilibrium}

  in accordance with this: entropy of single BH is greater than entropy of multi BH with \( k \) black holes (at same total mass)

  \[ S_k(\mu) < S(\mu) \]

- speculation: existence of (static) lumpy black holes
  (one big BH + two small BHs: small ones can merge into lumpy object before all horizons merge)
Consequences for non-uniform black strings

- what happens when we crank up mass for multi BH configuration
  - could point to existence of new non-uniform black strings (bumpy black strings)

Open Q: if exist, how connected to GL critical masses?

Possibility:
GL instability first decays to non-uniform black string, then bifurcation into new non-uniform black strings

→ could change our understanding of GL instability as happening between uniform black string and single black hole without any intermediate steps
Black rings and holes in higher dimensions

The non-uniqueness + phase structure of rotating rings in higher dimensions turns out to be intimately related to that of black hole solutions of vacuum Einstein equations in six or more dimensions.

I construct thin rotating black rings in higher dimensions \((D \geq 6)\) with horizon topology \(S^{D-3} \times S^1\).

- matched asymptotic expansion
  - metric of thin black ring in linearized gravity
  - near-horizon metric (dipole perturbations corresponding to bending a boosted black string)
  - match in overlap zone + require regular event horizon
    - zero pressure condition is required (balancing of ring)

First steps to qualitatively complete phase diagram of asymptotically flat, neutral and rotating blackfolds

- exploit connection between black holes/strings/branes in KK spacetimes and higher-dim rotating black holes
  - use the analogy and known phase structure of KK BHs to uncover the phase diagram of stationary BHs
  - conjecture existence of infinite number of pinched black holes
  - connection to black Saturns configs thru merger transitions
Thin black rings from circular boosted black strings

In limit of very large radius \( R \) of the \( S^1 \) of 5D black ring metric becomes that of a (critically) boosted 5D black string

aim: use perturbative approach to construct thin black rings in \( \mathcal{M}^{n+4} \)

starting from boosted black string in \( \mathcal{M}^{n+4} \)

horizon topology \( S^1 \times S^{n+1} \)
thin means: \( R \gg r_0 \)

Zeroth order solution in \( 1/R \) : straight boosted black string

\[
ds^2 = -\left(1 - \cosh^2 \alpha \frac{r_0^n}{r^n}\right) dt^2 - 2 \frac{r_0^n}{r^n} \cosh \alpha \sinh \alpha \, dt \, dz + \left(1 + \sinh^2 \alpha \frac{r_0^n}{r^n}\right) dz^2 + \left(1 - \frac{r_0^n}{r^n}\right)^{-1} \, dr^2 + r^2 d\Omega^2_{n+1}
\]

Distributional source of energy/momentum for thin black ring

\[
T_{tt} = \frac{r_0^n}{16\pi G} (n \cosh^2 \alpha + 1) \delta^{(n+2)}(r)
\]
\[
T_{t\ell} = \frac{r_0^n}{16\pi G} n \cosh \alpha \sinh \alpha \delta^{(n+2)}(r)
\]
\[
T_{zz} = \frac{r_0^n}{16\pi G} (n \sinh^2 \alpha - 1) \delta^{(n+2)}(r)
\]
Equilibrium condition

- boosted black string limit of black ring is described by three parameters $r_0, R, \alpha$

  expect physically: two parameters (e.g. given radius and mass, spin is fixed)

  dynamical balance condition relates the three parameters

  \[
  K_{\mu\nu} T^{\mu\nu} = 0 \quad \rightarrow \quad \frac{T_{zz}}{R} = 0 \quad \rightarrow \quad \sinh^2 \alpha = \frac{1}{n}
  \]

  critical boost:

  enables computation of all leading order thermodynamic quantities!

  \[
  M = \frac{\Omega_{n+1}}{8G} R r_0^n (n + 2)
  \]

  \[
  J = \frac{\Omega_{n+1}}{8G} R^2 r_0^n \sqrt{n + 1}
  \]

  \[
  A = \Omega_{n+1} 2\pi R r_0^{n+1} \sqrt{\frac{n+1}{n}}
  \]

  valid in large $J$ limit of black ring

  crucial assumption: horizon remains regular when boosted black string is curved

  important check: rederive equilibrium condition from regularity condition

  shows how GR encodes EOM of BHs as regularity conditions on geometry
Matched asymptotic expansion

- MAE = systematic approach to iteratively construct solution given known solution in some limit + then correcting it in perturbative expansion
  (applied e.g. to construct metric of small black holes on circle)
  thin black rings:
  two scales \( r_0 , R \)
  asymptotic zone: \( r \gg r_0 \)
  near-horizon zone: \( r \ll R \)
  \( r \) is distance from ring

  - step 0: solution in near-horizon zone to 0\(^{th}\) order in \( 1/R \)
    = boosted black string of infinite length

  - step 1: solve Einstein eqs. in linearized approximation around flat space for appropriate source (circular distribution of given mass/angular momentum density)
    - valid to first order in \( r_0^n \propto GM/R , r_0 \ll r \)

  - step 2: find linear corrections to boosted black string for perturbation that is small in \( 1/R \)
    - BCs fixed by matching to step 2 in overlap region: \( r_0 \ll r \ll R \)
    - require regularity at horizon

  - step 3: solve next-to-linearized solution in asymptotic zone + use BCs from step 2
    - not necessary for phys. quantities, since can use corrections near horizon + Smarr
Solution method

Step 1a: find metric in the linearized approximation around flat space sourced by a thin black ring localized on circle of radius $R$

$$\Box \bar{h}_{\mu\nu} = -16\pi T_{\mu\nu} \quad T_{tt} \neq 0 \ , \ \ T_{t\psi} \neq 0 \quad \psi \sim \frac{\bar{z}}{R}$$

$$r_0 \gg r \quad T_{\mu\nu} \sim \delta(r)$$

Step 1b: Consider overlap region $r_0 \ll r \ll R$
- effect of curving thin black string into locally arc of constant curvature radius $R$

$$g_{\mu\nu} = \eta_{\mu\nu} + \frac{r_0^n}{r^n} \left( h_{\mu\nu}^{(0)} + \frac{r \cos \theta}{R} h_{\mu\nu}^{(1)} \right)$$

$\eta_{\mu\nu}$ = flat space metric in ring-adapted coords.

Note: can explicitly check that regular solution requires $T_{\psi\psi} = 0$
expected: additional stresses lead to singularities

Step 2: Find general solution near black string (hardest part) $r_0 \leq r \ll R$
+ match to metric in overlap region

$$g_{\mu\nu} = g_{\mu\nu}^{(bs)} + \frac{\cos \theta}{R} h_{\mu\nu}(r)$$

• can be solved exactly in terms of hypergeometric functions
• involves only dipole perturbations: thermodynamics not corrected thru this order
(exception: five dimensions)
Dimensionless quantities

- meaningful comparison in terms of dimensionless quantities
  classical GR does not possess intrinsic scale: use mass

\[ j^{n+1} \propto \frac{J^{n+1}}{GM^{n+2}}, \quad a_H^{n+1} \propto \frac{A^{n+1}}{(GM)^{n+2}}, \quad \omega_H \propto (GM)^{\frac{1}{n+1}} \Omega_H, \quad t_H \propto (GM)^{\frac{1}{n+1}} T_H \]

- thin black ring
  \[ j^{n+1} \sim \left( \frac{R}{r_0} \right)^n \]
  \( j \) very large

- ultraspinning MP BH
  \[ j^{n+1} \sim \left( \frac{a}{r_0} \right)^{n-1} \]
  ( \( a \) is rotation parameter)

- compare the “phase structure” for large \( j \)

\[ a_H(j), \quad \omega_H(j), \quad t_H(j) \]

same as studying \( A(M, J) \) at fixed \( M \)
Higher-dimensional black rings vs. MP black holes

black ring

\[ a_H \sim \frac{1}{j^{1/n}} \]

black hole

\[ a_H \sim \frac{1}{j^{2/(n-1)}} \]

\[ \Rightarrow \text{black rings dominate entropically in ultraspinning regime (} n > 1 \) \]

e.g. in 7D

\[ a_H^{\text{ring}} \sim \frac{1}{j^{1/3}} \]

\[ a_H^{\text{hole}} \sim \frac{1}{j} \]

onset of membrane-like behavior of MP BH
Higher-dimensional black rings vs. MP black holes (cont’d)

<table>
<thead>
<tr>
<th></th>
<th>black ring</th>
<th>black hole</th>
</tr>
</thead>
<tbody>
<tr>
<td>angular velocity</td>
<td>$\omega_H \sim \frac{1}{2j}$</td>
<td>$\omega_H \sim \frac{1}{j}$</td>
</tr>
<tr>
<td>temperature</td>
<td>$t_H \sim j^{1/n}$</td>
<td>$t_H \sim j^{2/(n-1)}$</td>
</tr>
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- black ring has hole in middle (wheel), more efficient to carry spin than packed MP BH (disk) → ring rotates more slowly
- i.e. black ring cooler then MP BH at fixed mass:
  - same mass in a wheel makes it thicker than in a pancake
  - + temperature inversely proportional to thickness
GL Instability of ultraspinning MP BH

MP BH approaches black membrane geometry $\mathbb{R}^2 \times S^n$ for large $J$

$a \gtrsim r_0$ $a \sim J/M$

rotational axial symmetry so one translational symmetry along $T^2$

Emparan, Myers

uniform black membrane

pinched membrane (GL instability)
non-uniform phase

pinched-off membrane

localized black string

ultraspinning MP black hole

lumpy (pinched) black holes

pinched-off black hole

black ring
Phase Diagram for Kaluza-Klein BHs on two-torus

- import knowledge of KK black holes on circle + add one uniform direction
  see e.g. review: Harmark, Niarchos, NO

- square two-torus sides $L$

  \[ \ell^{n+1} \propto \frac{L^{n+1}}{GM} \]

- measure linear size of horizon along rotation plane/torus for fixed mass

- identify: $j \leftrightarrow \ell$

- example:
  KK BHs in 7 dimensions on $M^5 \times \mathbb{T}^2$

- uniform black membrane

- localized black string

- non-uniform black membrane

- $a_{H}^{\text{lbs}}(\ell) \sim \ell^{-\frac{1}{n}}$ (cf. thin BR)

- $a_{H}^{\text{ubm}}(\ell) \sim \ell^{-\frac{2}{n-1}}$ (cf ultraspinning MP BH)

- + copies by copying the solutions $k$ times on the circle
Towards completing the phase diagram

- based on analogy with phase diagram for KK BHs on torus: extrapolate to \( j = \mathcal{O}(1) \) regime

- proposal for phase diagram of stationary BHs (one angular momentum) in asymptotically flat space: main sequence = MP BH, pinched MP BH, black ring (uniform, non-uniform, localized)
Black satrns and multi-pinches

most likely features

• **main sequence**: BH with pinch at rotation axis meets black ring phase
• **infinite sequence of pinched BHs** emanating from BH curve (from copies of the GL zero mode)
• upper **black Saturn** curve + merger to circular pinch

less compelling arguments for: **pancaked + pinched black Satrns**
(but admit a simple and natural way for completing the phase diagram consistent with available info)
Comparison to five dimensions

5D : MP BHs have upper bound on $j$ (black membranes do not exist in 5D)
  - fat black ring instead of pinched MP (behave like drilled-through MP BH)
  - only one type of black Saturn at large $j$
Further properties

- so far: thermal equilibrium phases (equal temperatures/angular velocities when more than one black object present)

non-thermal equilibrium phases perfectly valid as stationary multi-BH configurations of GR \( \rightarrow \) continuous families of solutions, e.g.

- more general black Saturns, di-rings etc.

(analogue of localized multi-BH solutions in KK space with unequal mass BHs)

Dias,Harmark,Myers,NO

- stability properties

(like in 5D) black rings at large \( \ell, \) GL-type instability creating ripples on \( S^1 \) fragments the ring into black holes flying apart

- MP BH + pinched BH could suffer from same type of instability

- black ring solutions below the cusp could be unstable towards radial perturbations causing collaps into MP BH
Future directions + outlook

► KK black holes
  • further examine possibility of new non-uniform strings and lumpy black holes
  • analytical second order corrections, higher D compact spaces + adding J
    - apply effective FT technique of Chu, Goldberger, Rothstein
  • numerical studies
    - extend into non-perturbative domain, temp. converge for multi BH?
  • applications to ST + dual gauge theories (see next time)
    - multiply gapped eigenvalue distributions as saddle points in finite T SYM?
    - 3-charge multi BHs on circle + microscopic entropy
  • examine fluid analogy

► higher D black rings
  • black satsums, rings in (A)dS spaces, dipoles, etc.
  • other topologies $S^2 \times S^2$, $S^2 \times T^2$, $S^2 \times \Sigma_{g>1}$ (in six dimensions)
    - boosted black p-branes: $S^{D-2-p} \times T^p$ (in progress)
  • balance conditions, global properties (embedding of the different topologies)
  • exact solutions? - numerics, effective FT techniques for thermodynamics
  • SUSY black rings + ST
  • duality to plasma balls + rings in AdS (cf. Lahiri, Minwalla) – many similar features

► further explore relation between phase structure of KK BHs and rotating black objects

Harmark, Kristjansen, NO, Roenne/Chowdhury, Giusto, Mathur