Gravitational waves from brane black holes

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Randall-Sundrum scenarios

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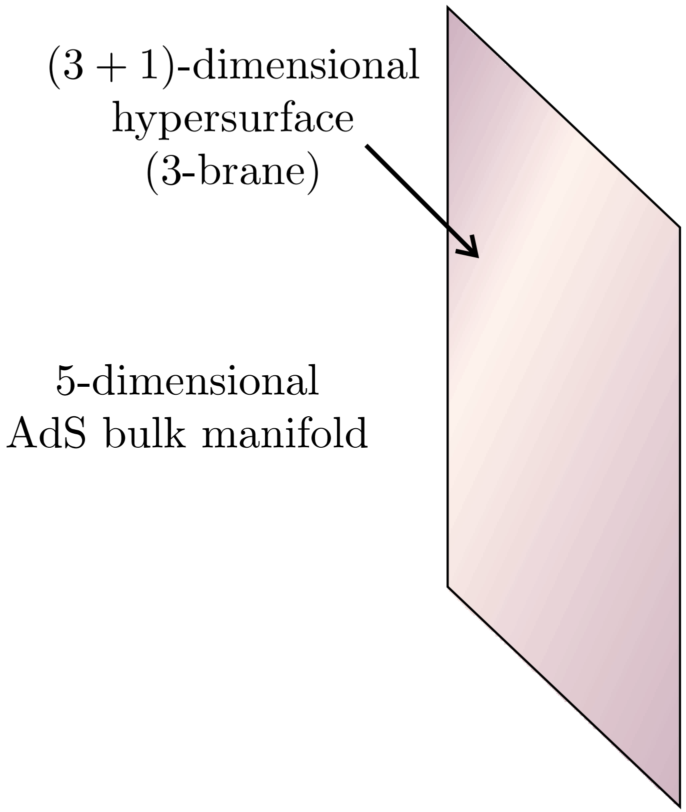

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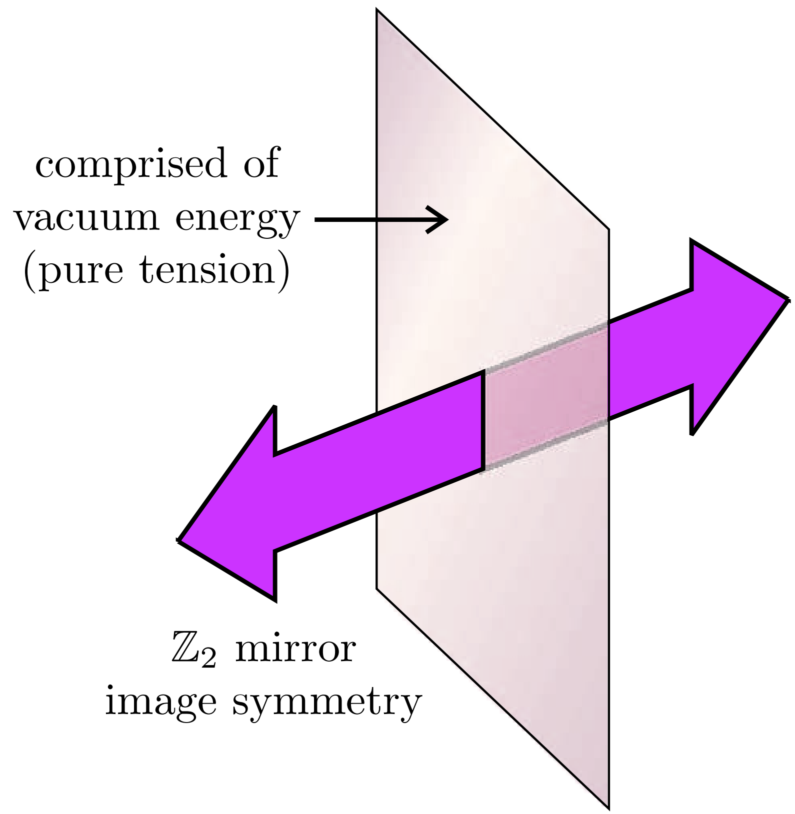

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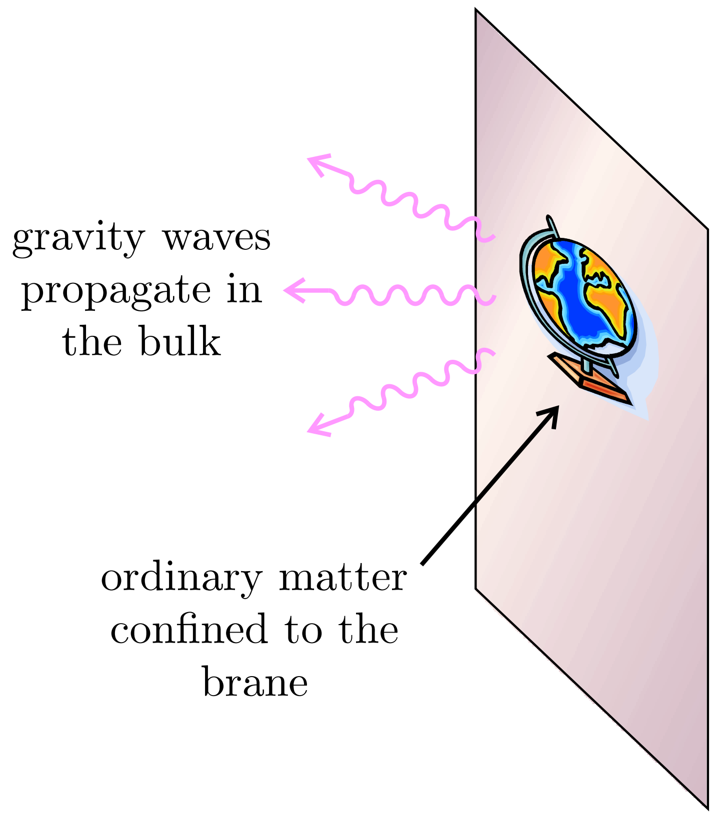

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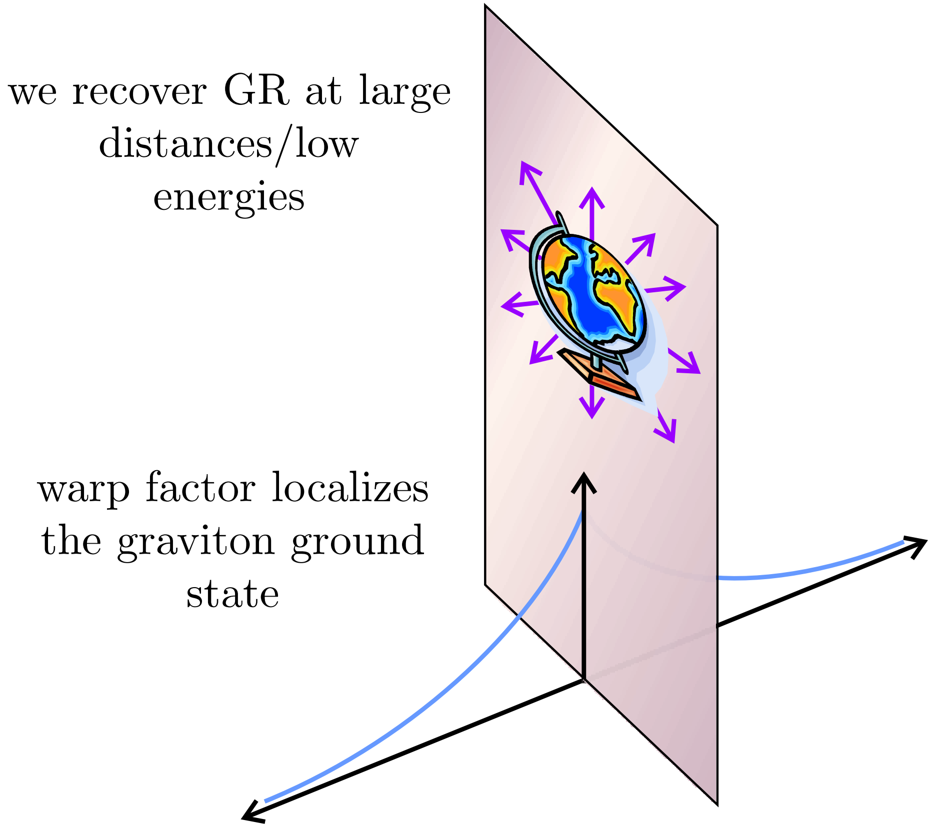

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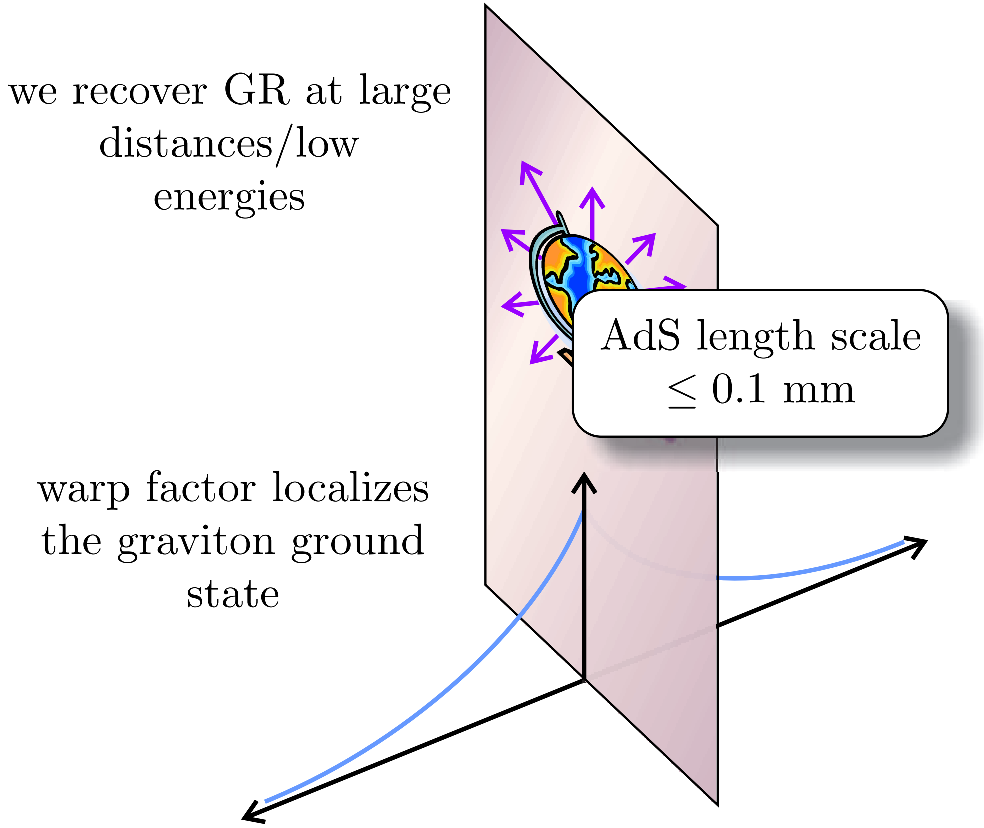

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    - behaviour of scalar and tensor perturbations altered from GR, but no observational "smoking gun"

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  - **this talk:** what are the gravitational wave signatures from braneworld black holes?

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A braneworld black hole model

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  - this is possibly due to fundamental results from the AdS/CFT correspondence
- the only calculable model I know of involves 2 branes: the black string braneworld

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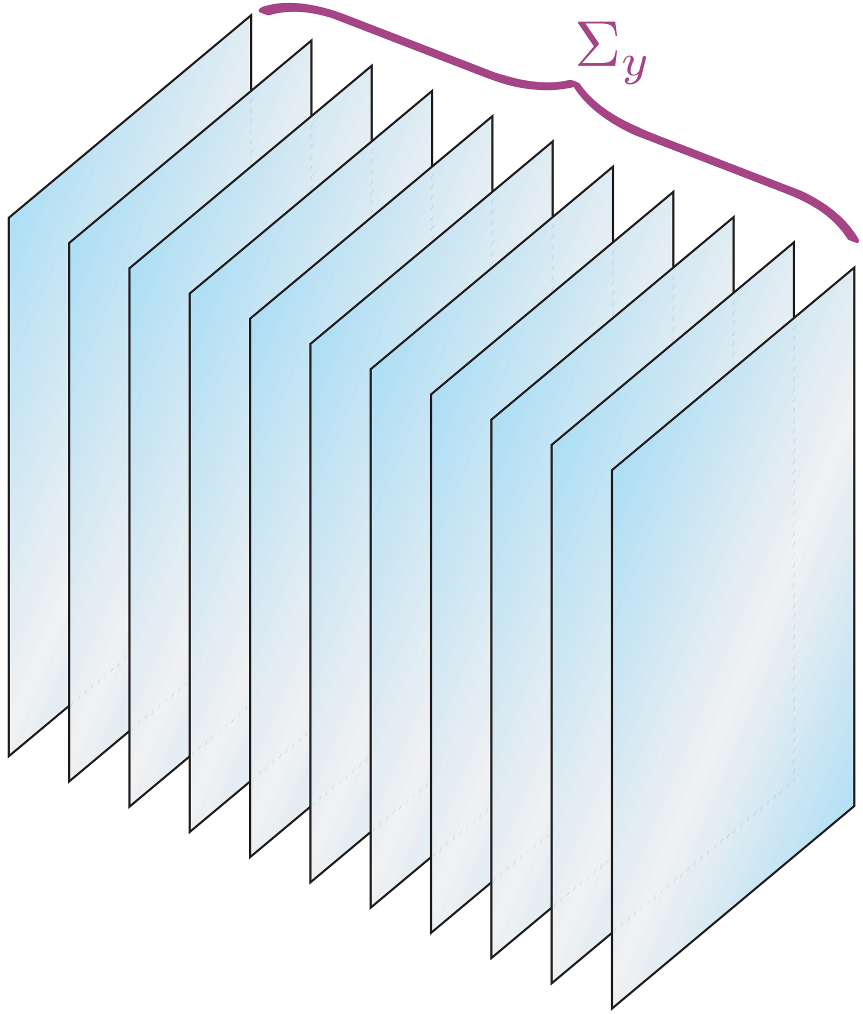

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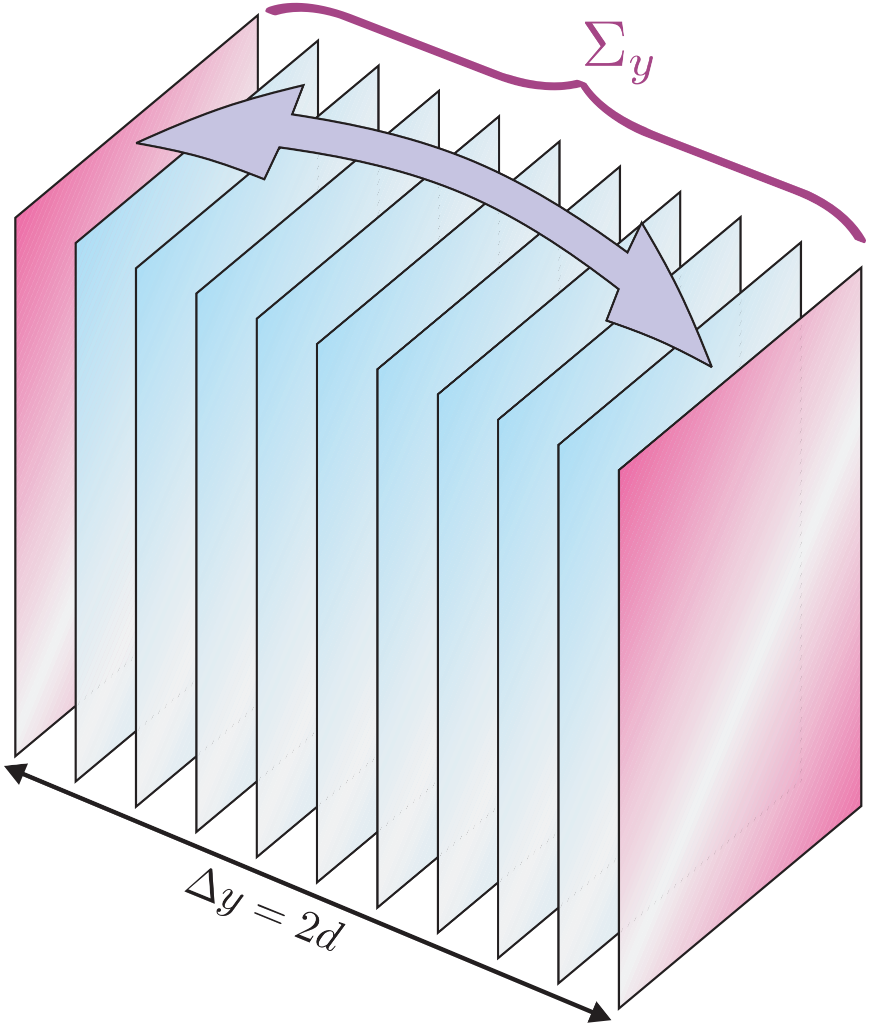

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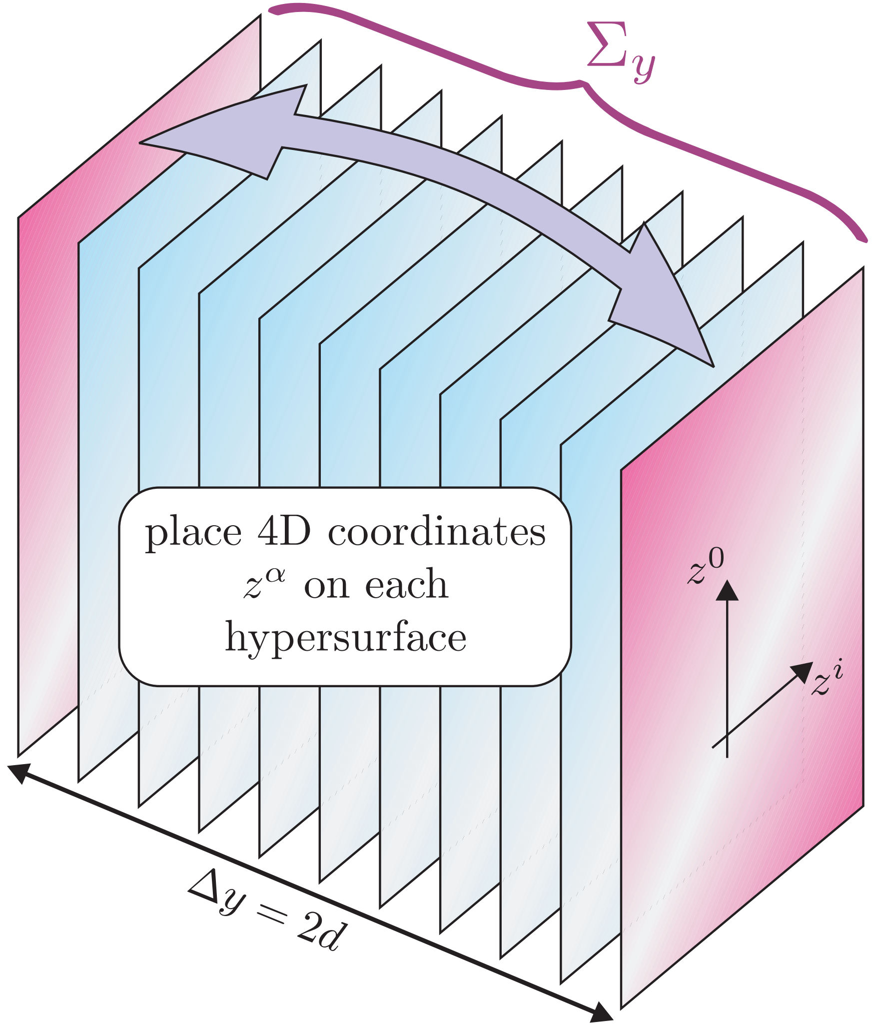

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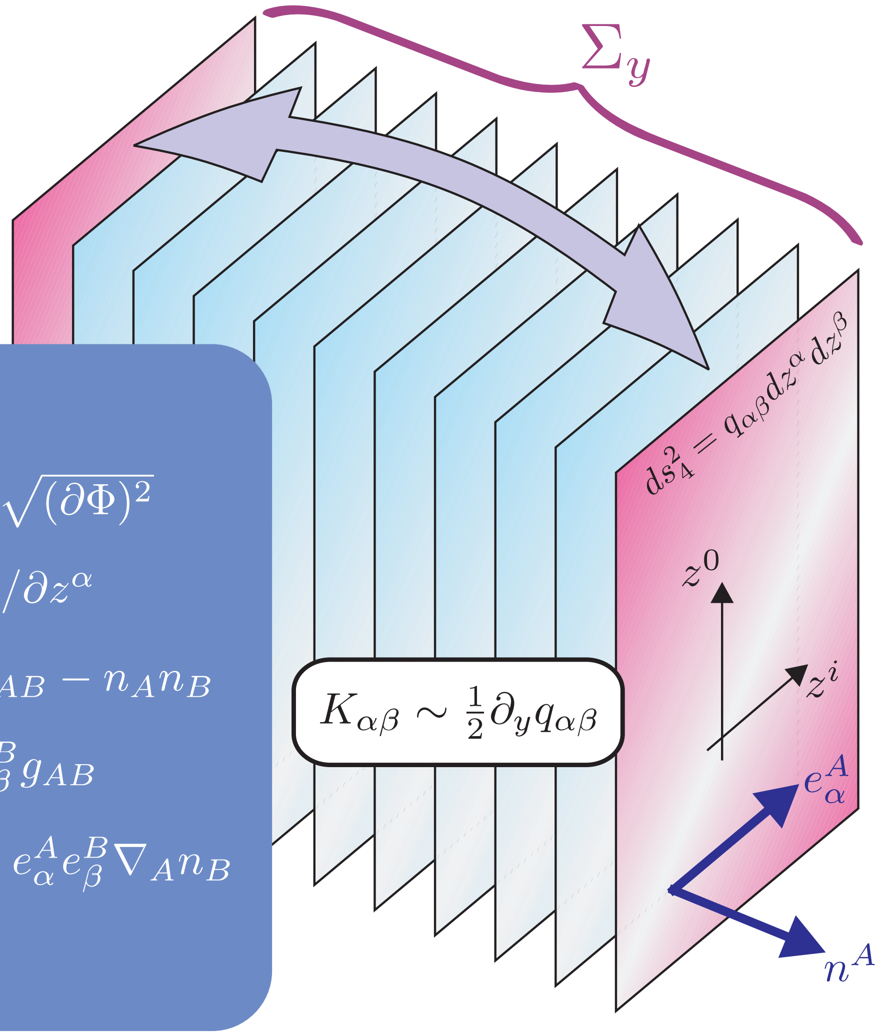
Some geometrical definitions:

- normal vector $n^A = \partial^A\Phi/\sqrt{(\partial\Phi)^2}$
- holonomic basis $e^A_\alpha = \partial x^A/\partial z^\alpha$
- projection tensor $q_{AB} = g_{AB} - n_A n_B$
- induced metric $q_{\alpha\beta} = e^A_\alpha e^B_\beta g_{AB}$
- extrinsic curvature $K_{\alpha\beta} = e^A_\alpha e^B_\beta \nabla_A n_B$
- $q_{AB} n^A = 0 = e^A_\alpha n_A$



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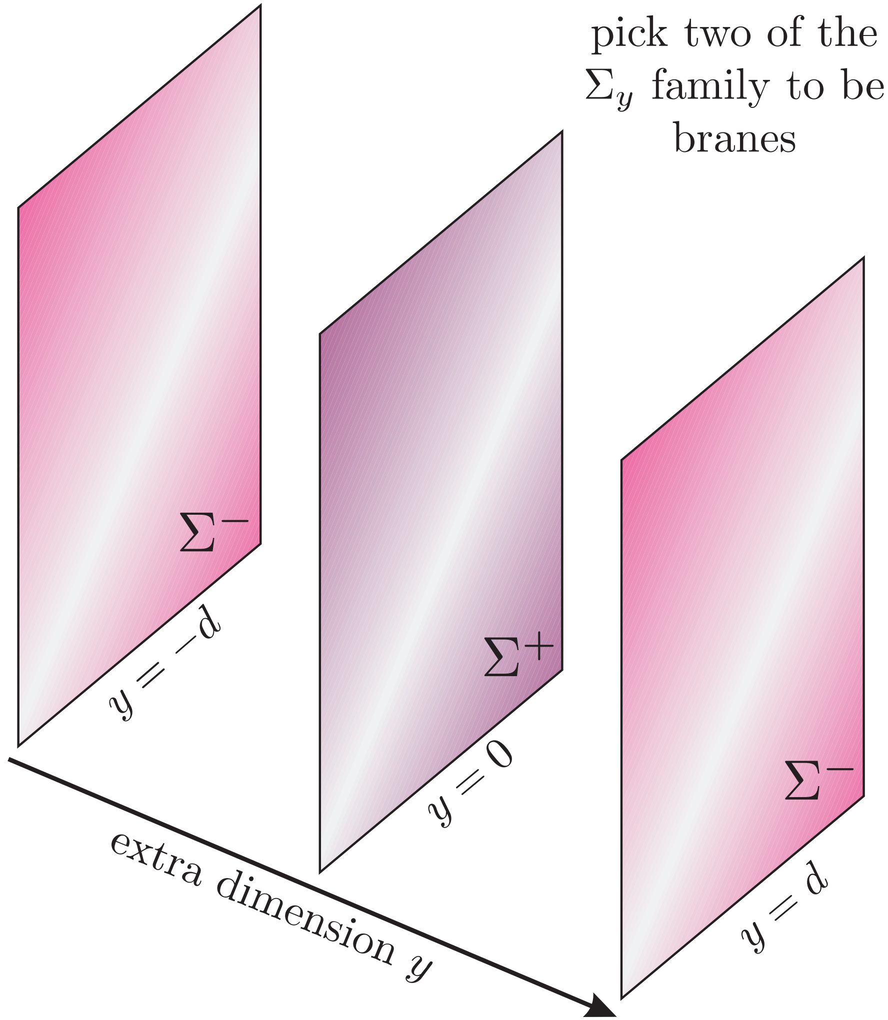

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$$S = \frac{1}{2\kappa_5^2}\int_{\mathcal{M}_{\rm L}} \left[{}^{(5)}R - 2\Lambda_5\right] + \frac{1}{2}\int_{\mathcal{M}_{\rm L}} \mathcal{L}_{\rm L}$$
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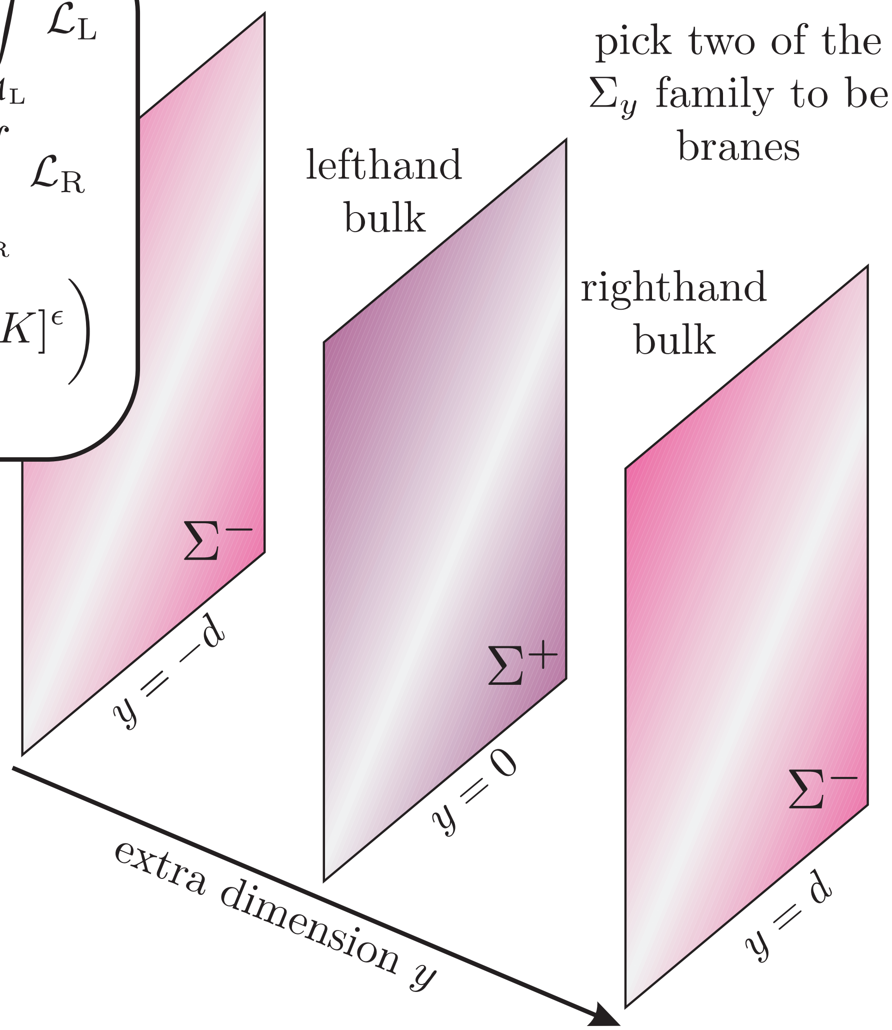

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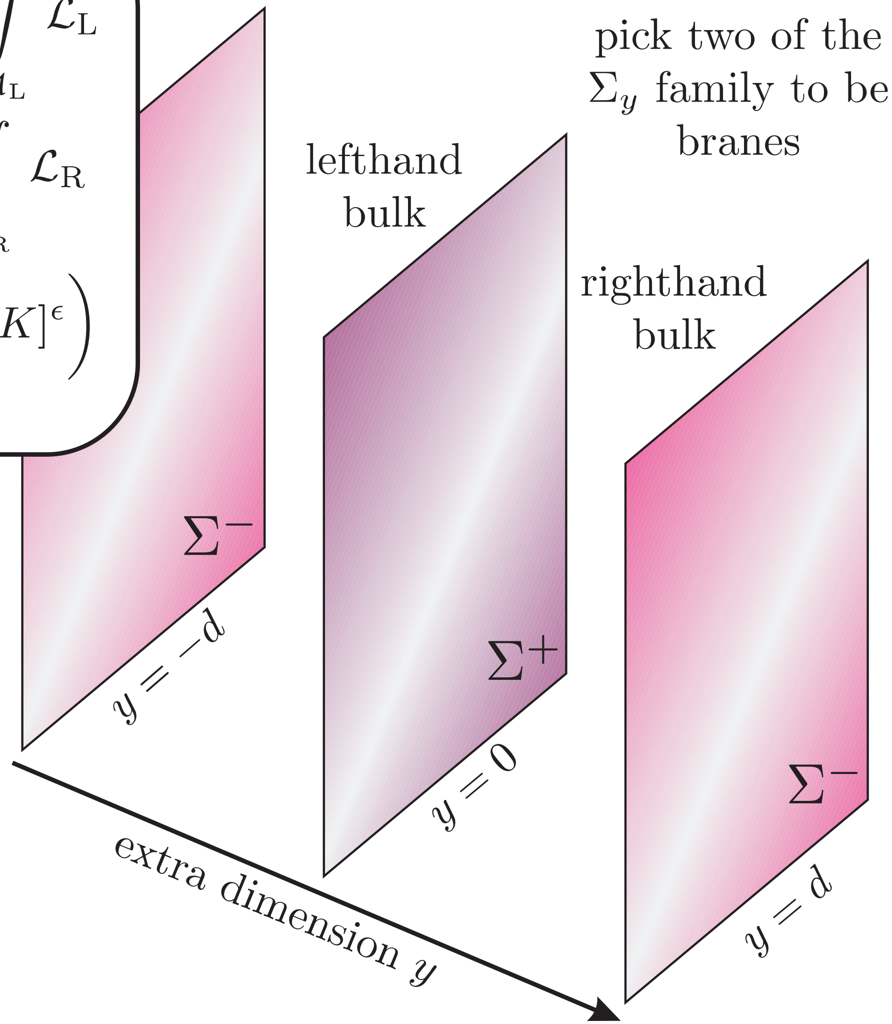
- brane tensions $\lambda^\pm$
- trace of extrinsic curvature K
- jump $[\cdots] = (\cdots)_\mathrm{R} - (\cdots)_\mathrm{L}$



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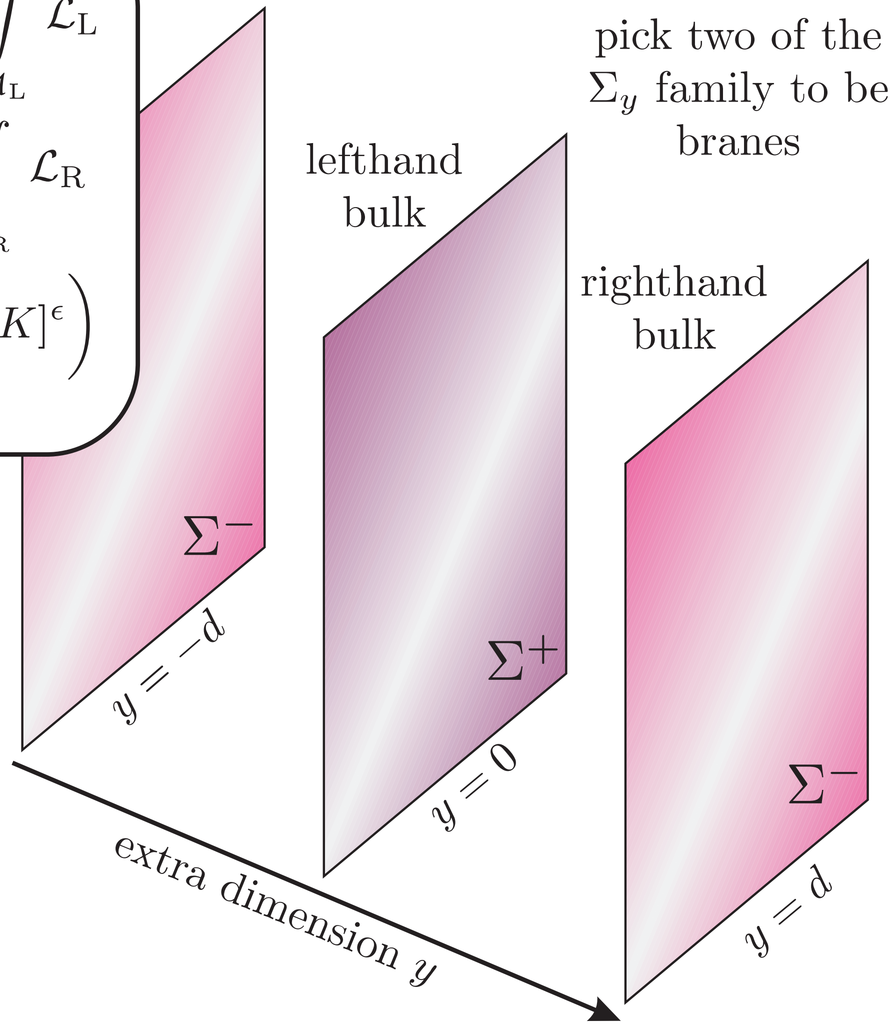
RS fine-tuning:
$\Lambda_5 = -6k^2 = -6/\ell^2$
$\lambda^\pm = \pm 6k/\kappa_5^2$



Action and field equations

bulk field equation

$$\frac{\delta S}{\delta g_{AB}} = 0 : \quad G_{AB} - 6k^2 g_{AB} = \kappa_5^2 \left[\theta(+y) T^{\mathrm{R}}_{AB} + \theta(-y) T^{\mathrm{L}}_{AB} \right]$$

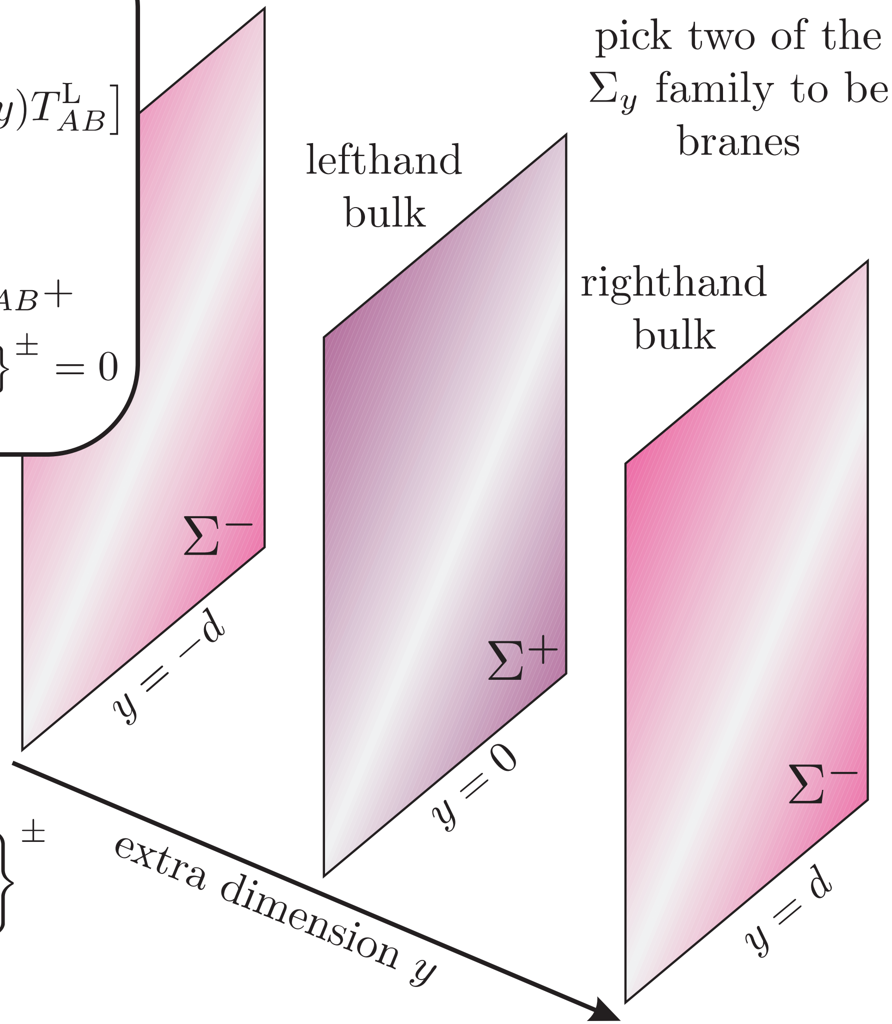
junction conditions

$$\frac{\delta S}{\delta q^{\pm}_{\alpha\beta}} = 0 : \quad Q^{\pm}_{AB} = \{ [K_{AB}] \pm 2k q_{AB} + \kappa_5^2 (T_{AB} - \tfrac{1}{3} T q_{AB}) \}^{\pm} = 0$$

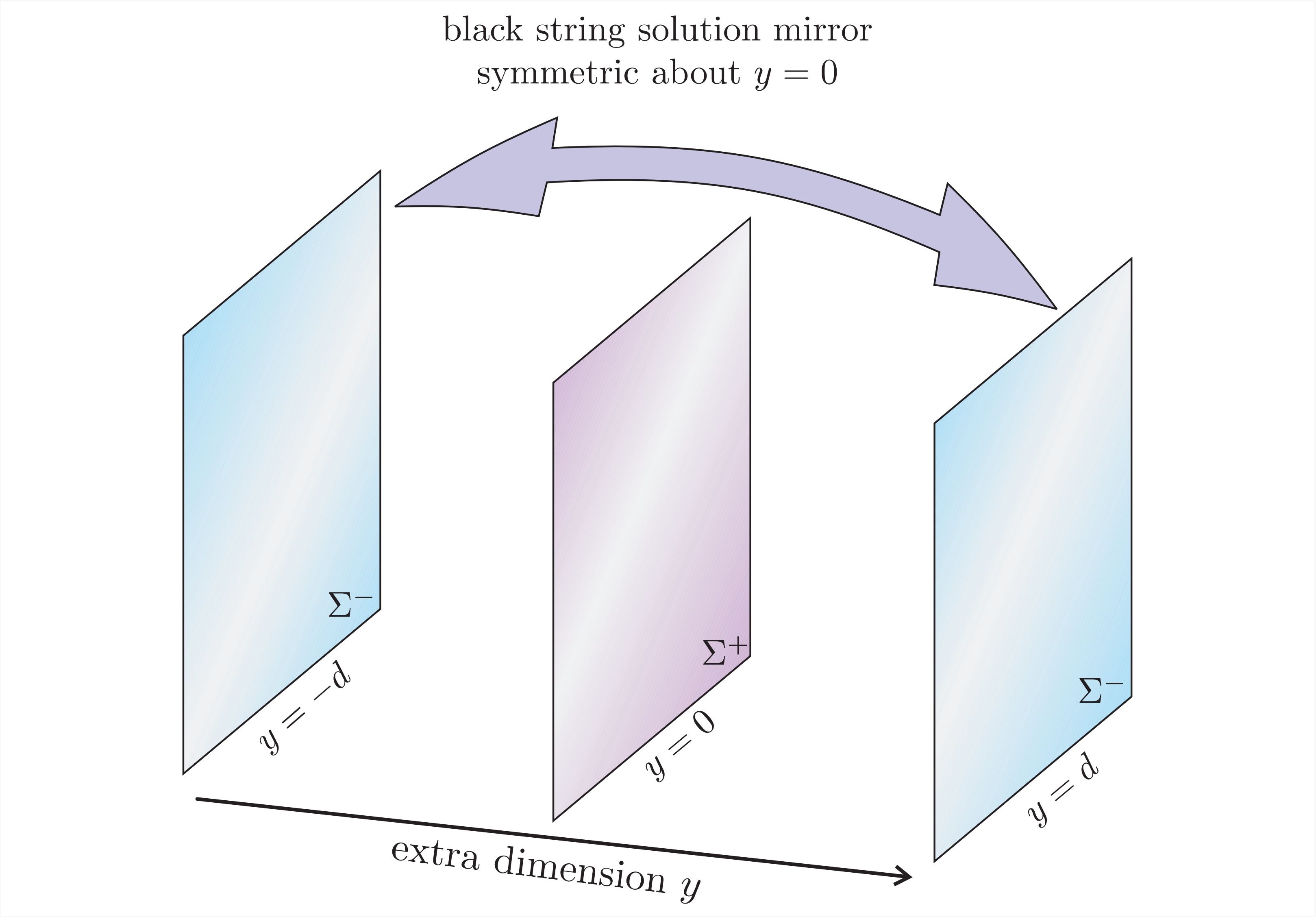
$$T^{\mathrm{L,R}}_{AB} = -\frac{2}{\sqrt{-g}} \frac{\delta \left(\sqrt{-g} \mathcal{L}_{\mathrm{L,R}} \right)}{\delta g^{AB}}$$

$$T^{\pm}_{AB} = e^{\alpha}_{A} e^{\beta}_{B} \left\{ -\frac{2}{\sqrt{-q}} \frac{\delta \left(\sqrt{-q} \mathcal{L} \right)}{\delta q^{\alpha\beta}} \right\}^{\pm}$$

$$K_{AB} = q^{C}{}_{A} \nabla_C n_B$$



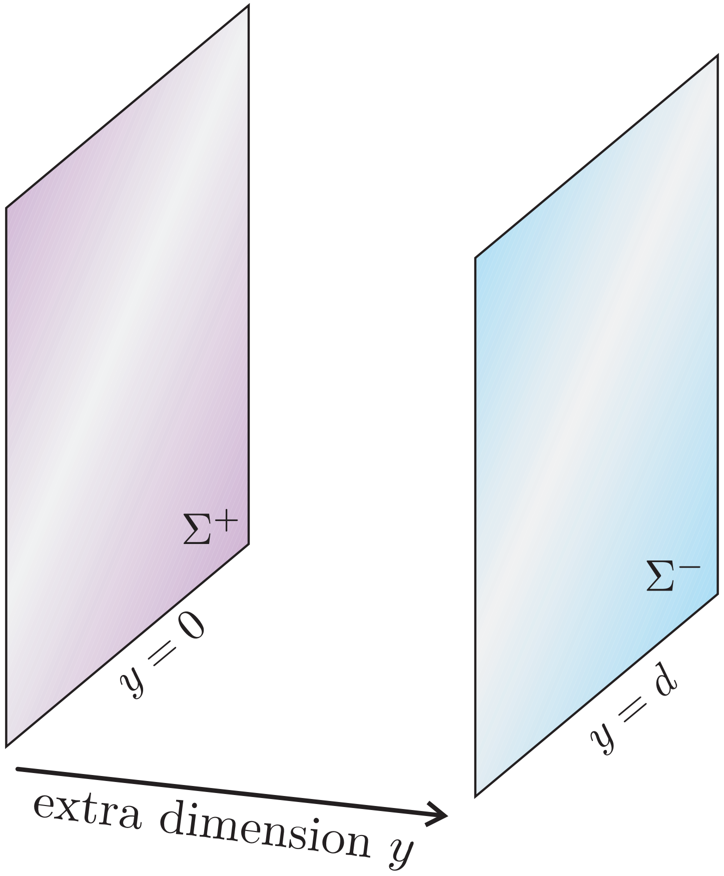
The black string background



The black string background

black string solution mirror
symmetric about $y = 0$

$\Rightarrow$ sufficient to just concentrate on one half of the bulk



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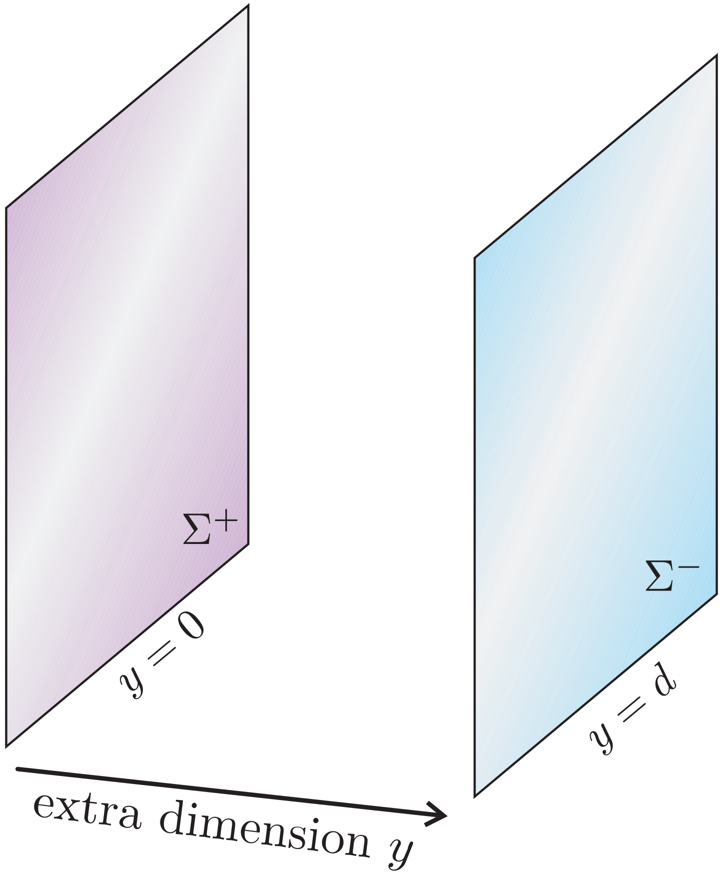
metric ansatz: $ds_5^2 = a^2(y) g_{\alpha\beta}(z) dz^\alpha dz^\beta + dy^2$



switch off all matter
$\mathcal{L}_{\mathrm{L,R}} = \mathcal{L}^\pm = 0$

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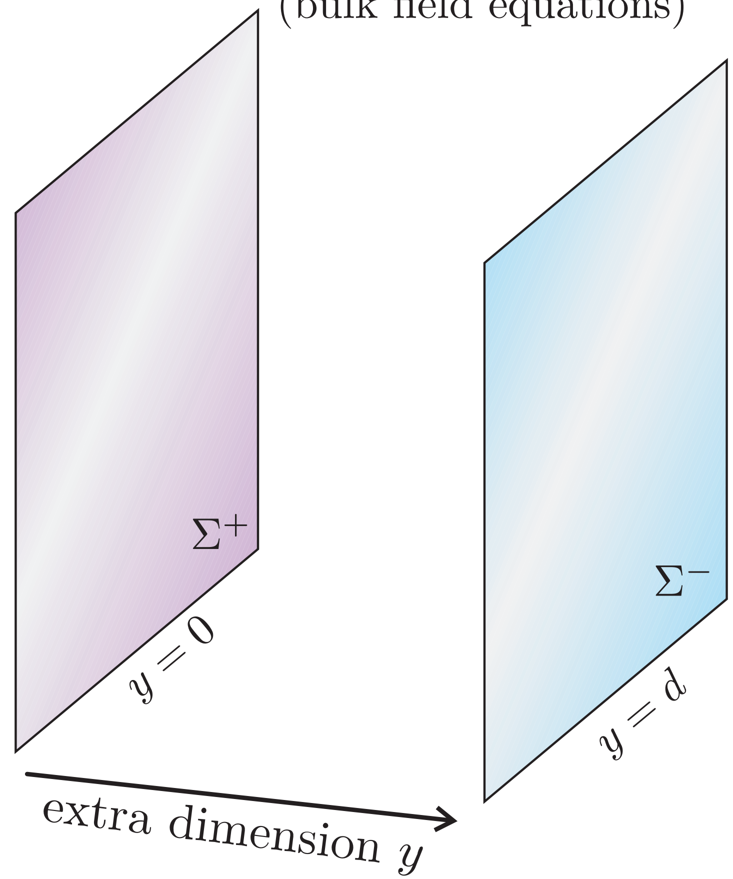

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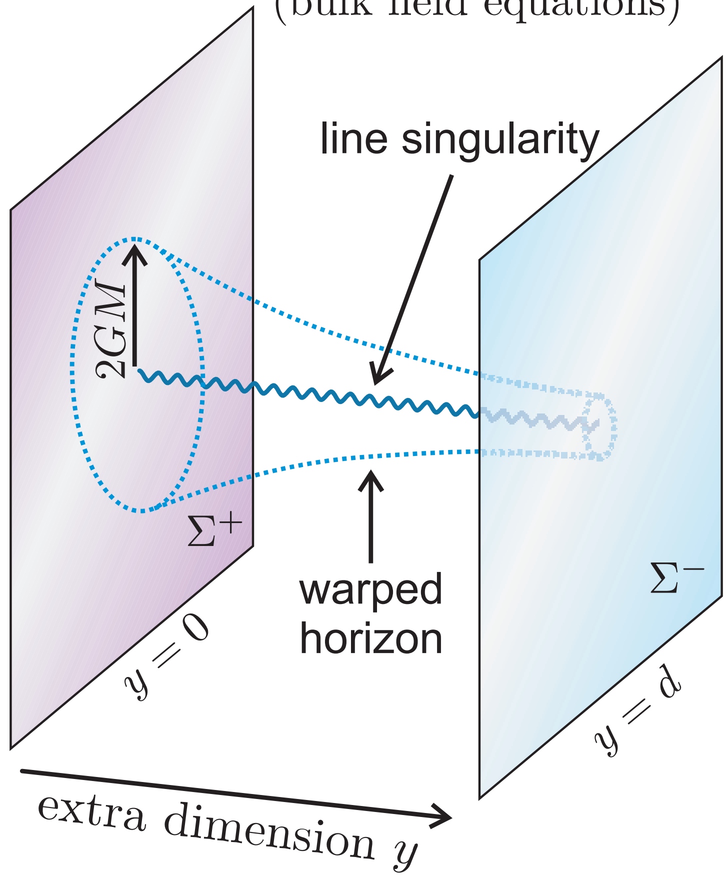

take $g_{\alpha\beta}$ to be Schwarzschild metric with mass M

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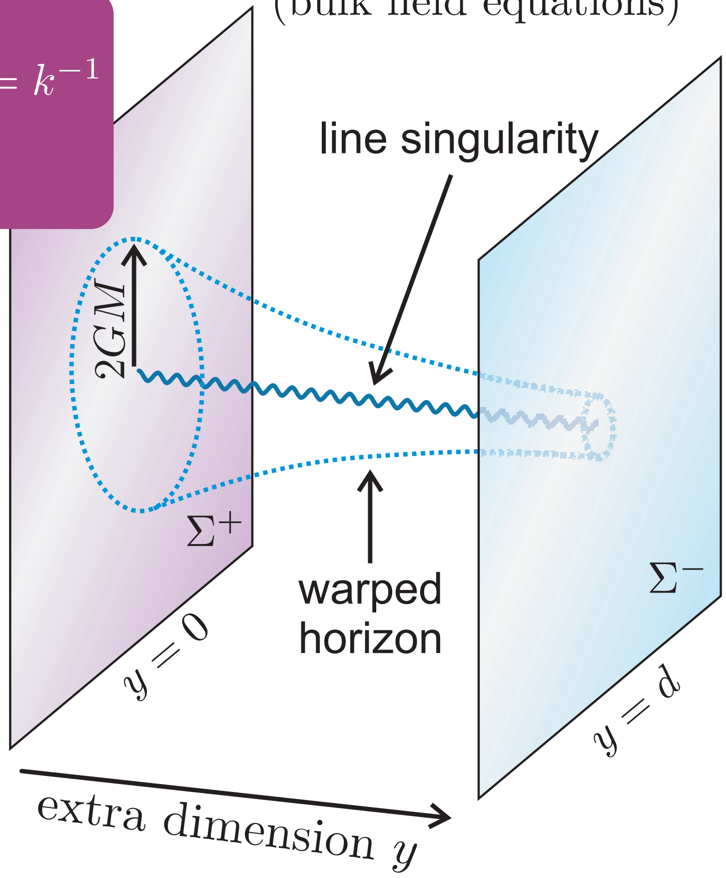
$g_{\alpha\beta}$ is any 4D solution of $R_{\alpha\beta} = 0$ (bulk field equations)

3 parameters:

- bulk curvature $\ell = k^{-1}$
- mass M
- separation d



take $g_{\alpha\beta}$ to be Schwarzschild metric with mass M

switch off all matter $\mathcal{L}_{\mathrm{L,R}} = \mathcal{L}^\pm = 0$

The black string background

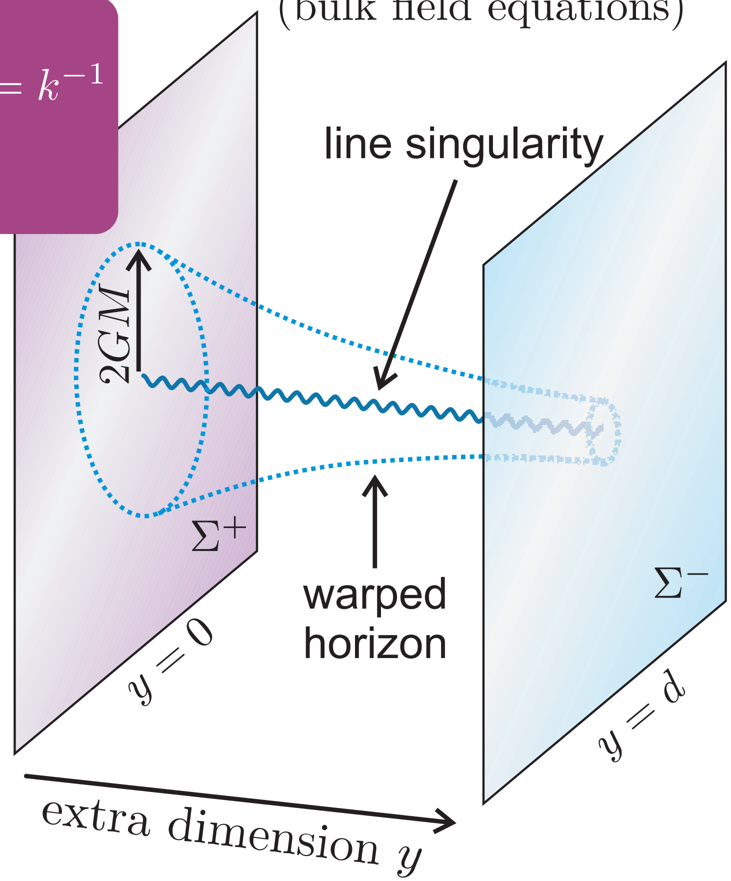
metric ansatz: $ds_5^2 = a^2(y) g_{\alpha\beta}(z) dz^\alpha dz^\beta + dy^2$

$a(y) = \exp(-ky)$

$g_{\alpha\beta}$ is any 4D solution of $R_{\alpha\beta} = 0$

(bulk field equations)

3 parameters:
- bulk curvature $\ell = k^{-1}$
- mass M
- separation d



take $g_{\alpha\beta}$ to be Schwarzschild metric with mass M

induced geometry on each Σ_y precisely Schwarzschild

switch off all matter $\mathcal{L}_{\mathrm{L,R}} = \mathcal{L}^{\pm} = 0$

Remarks

$$ds^2 = \exp(-2k|y|)(-f\,dt^2 + f^{-1}dr^2 + r^2 d\Omega^2) + dy^2$$

$$f = 1 - 2GM/r$$

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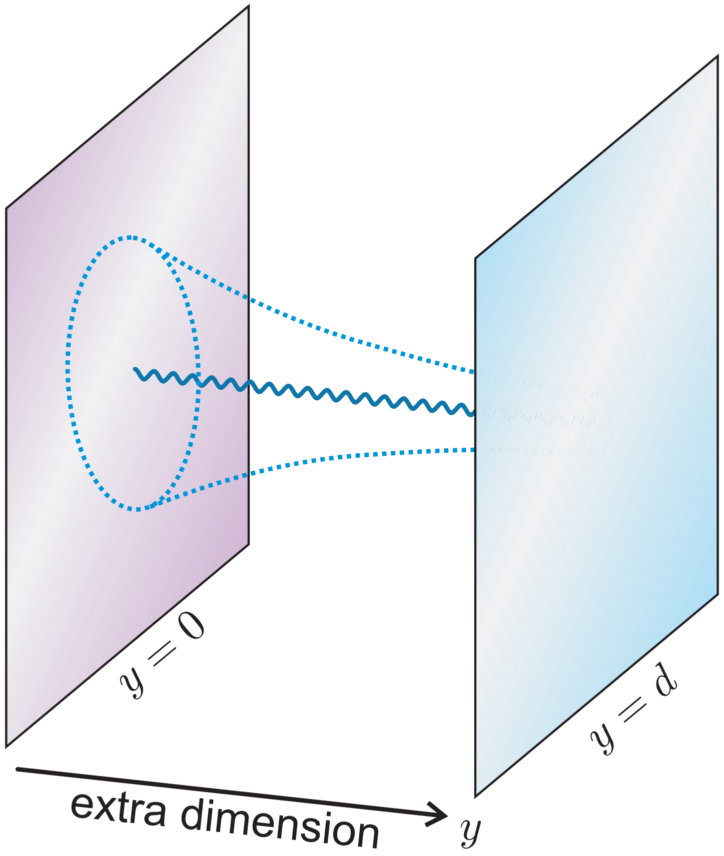
- we reside on positive tension brane at $y = 0$
  - different from RS1 — we do not try to solve hierarchy problem
- call the negative tension brane at $y = d$ the “shadow” brane

Linear perturbations

What do we perturb?

3 types of perturbations to consider:



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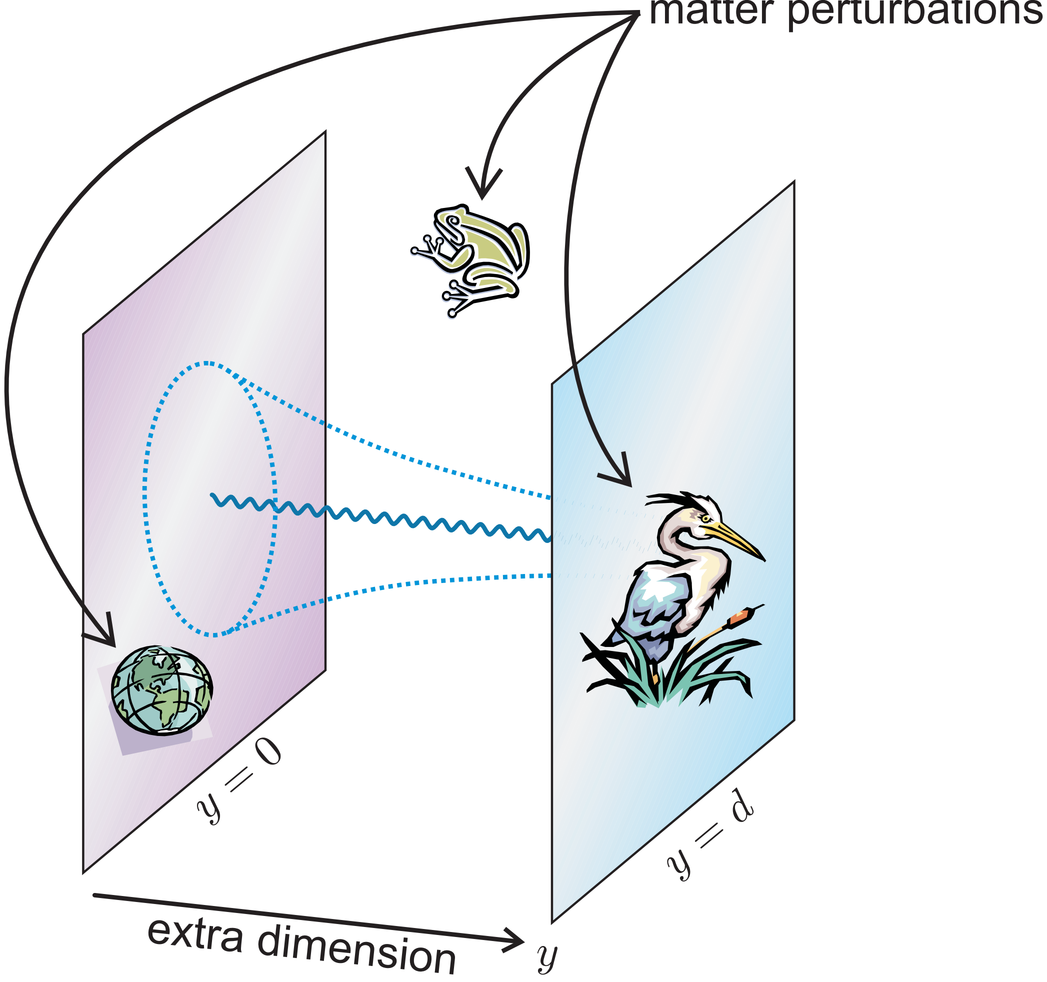

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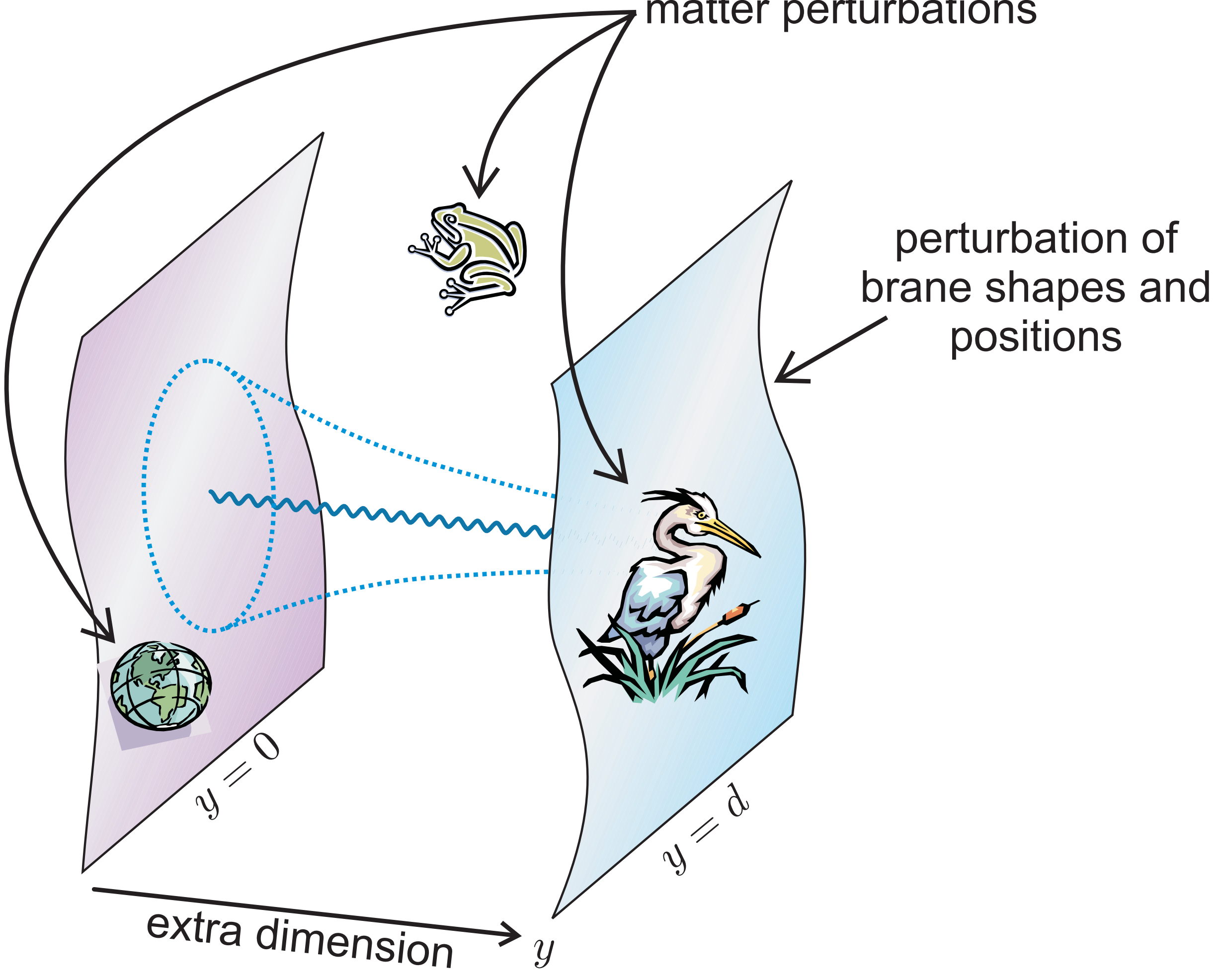

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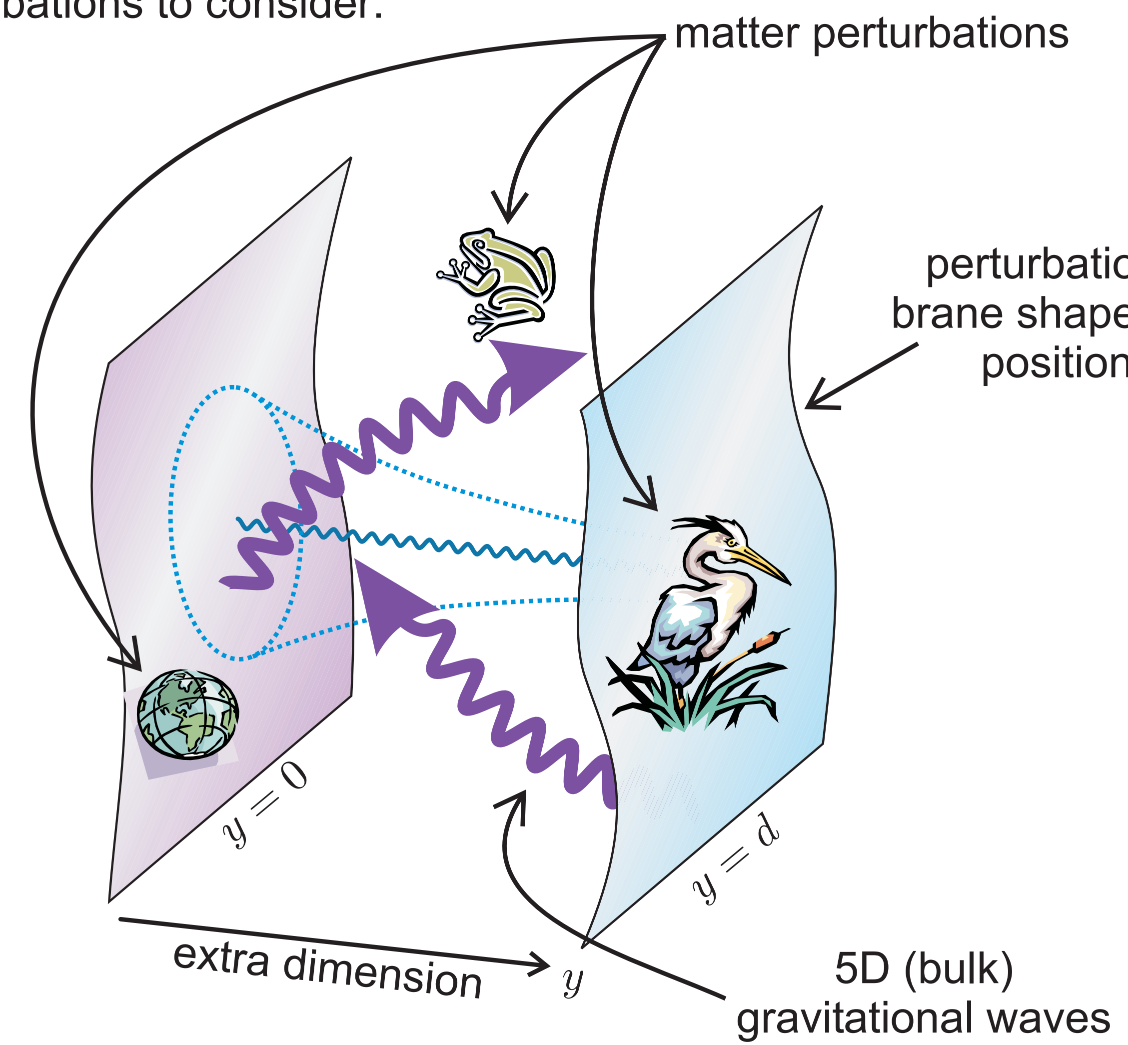

Perturbing the bulk

Bulk perturbations are governed by

$$\delta G_{AB} - 6k^2 \delta g_{AB} = \kappa_5^2 \left[\theta(+y) T^{\mathrm{R}}_{AB} + \theta(-y) T^{\mathrm{L}}_{AB} \right]$$

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writing $\delta g_{AB} = h_{AB}$ this is (c.f. Wald)

$$\nabla^C \nabla_C h_{AB} - \nabla^C \nabla_A h_{BC} - \nabla^C \nabla_B h_{AC} + \nabla_A \nabla_B h^C{}_C - 8k^2 h_{AB} = -2\kappa_5^2 \Sigma^{\mathrm{bulk}}_{AB},$$

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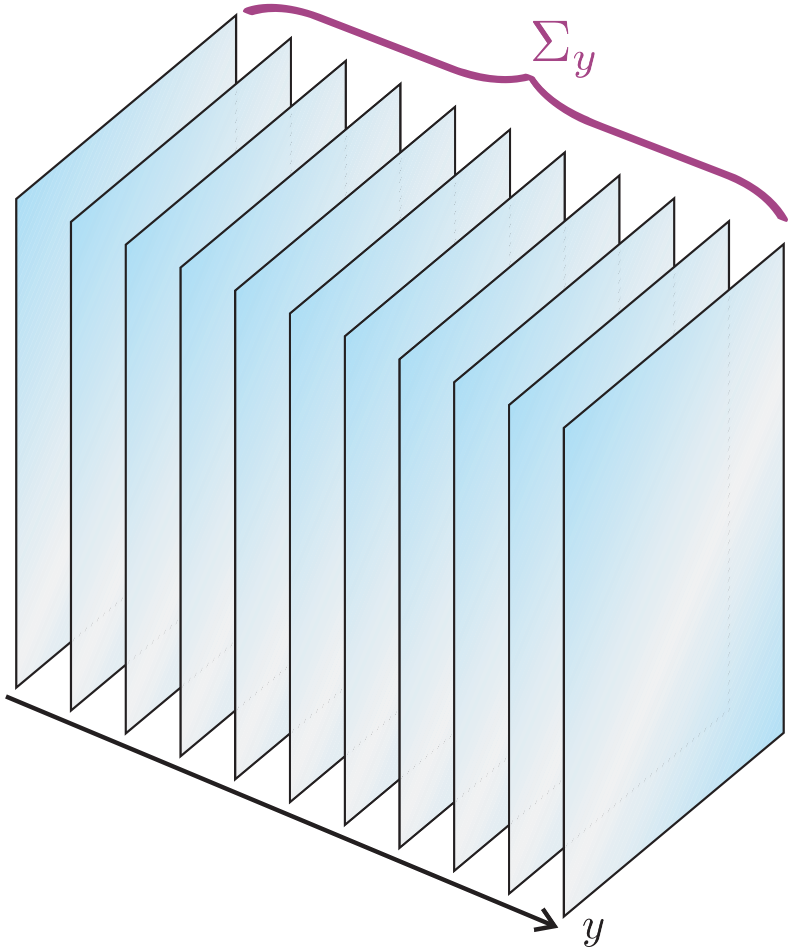
where

$$\Sigma^{\mathrm{bulk}}_{AB} = \Theta(+y)(T^{\mathrm{R}}_{AB} - \tfrac{1}{3} T^{\mathrm{R}} g_{AB}) + \Theta(-y)(T^{\mathrm{L}}_{AB} - \tfrac{1}{3} T^{\mathrm{L}} g_{AB})$$

Brane bending

Recall that our Σ_y foliation was defined by the level surfaces of $\Phi(x^A) = y$ in the background



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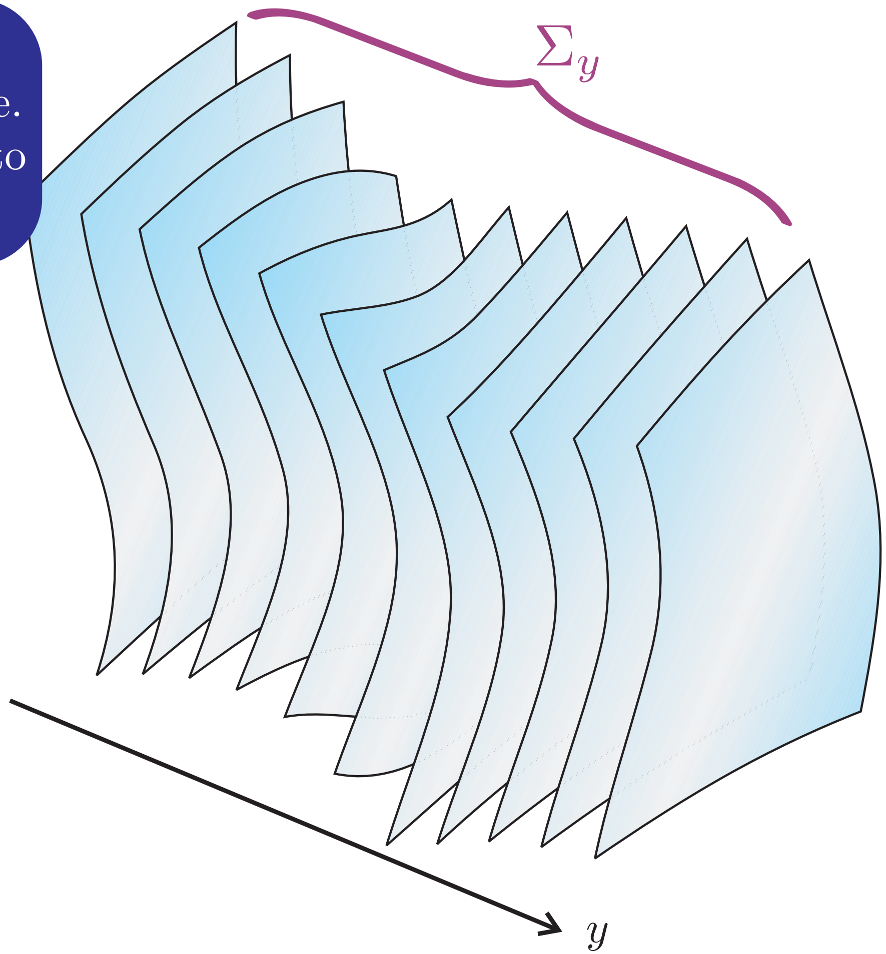

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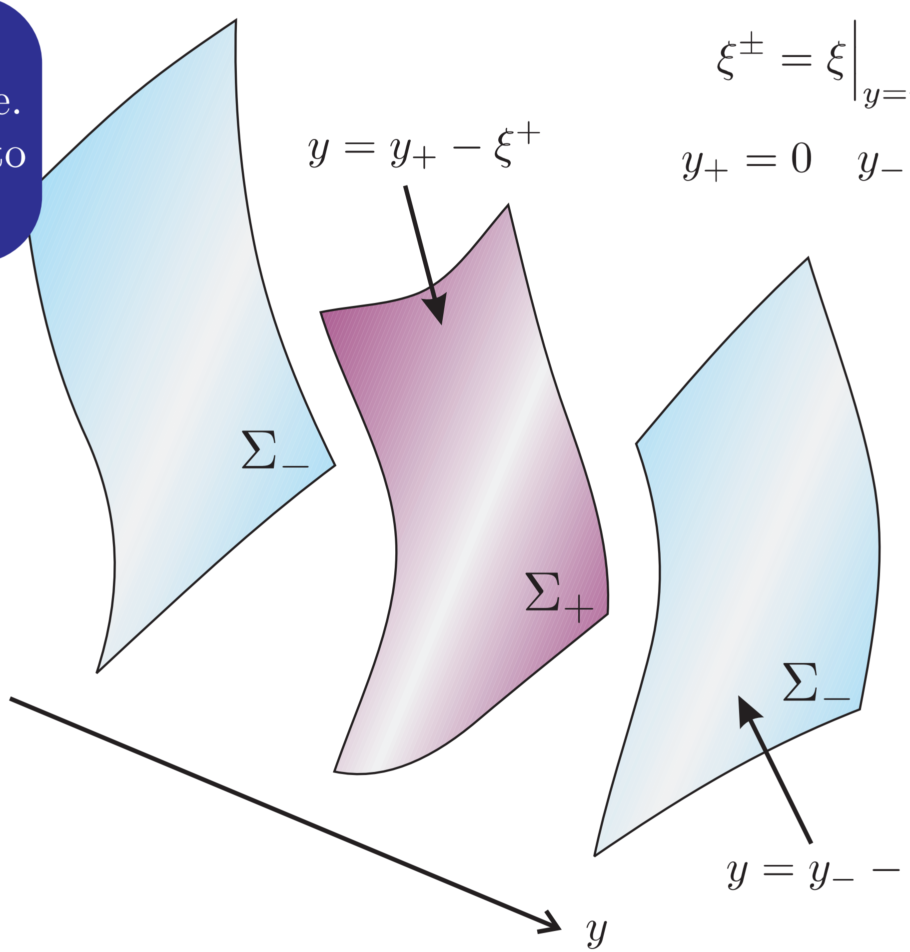
$\xi^{\pm} = \xi\Big|_{y=y_\pm}$

$y_+ = 0 \quad y_- = d$



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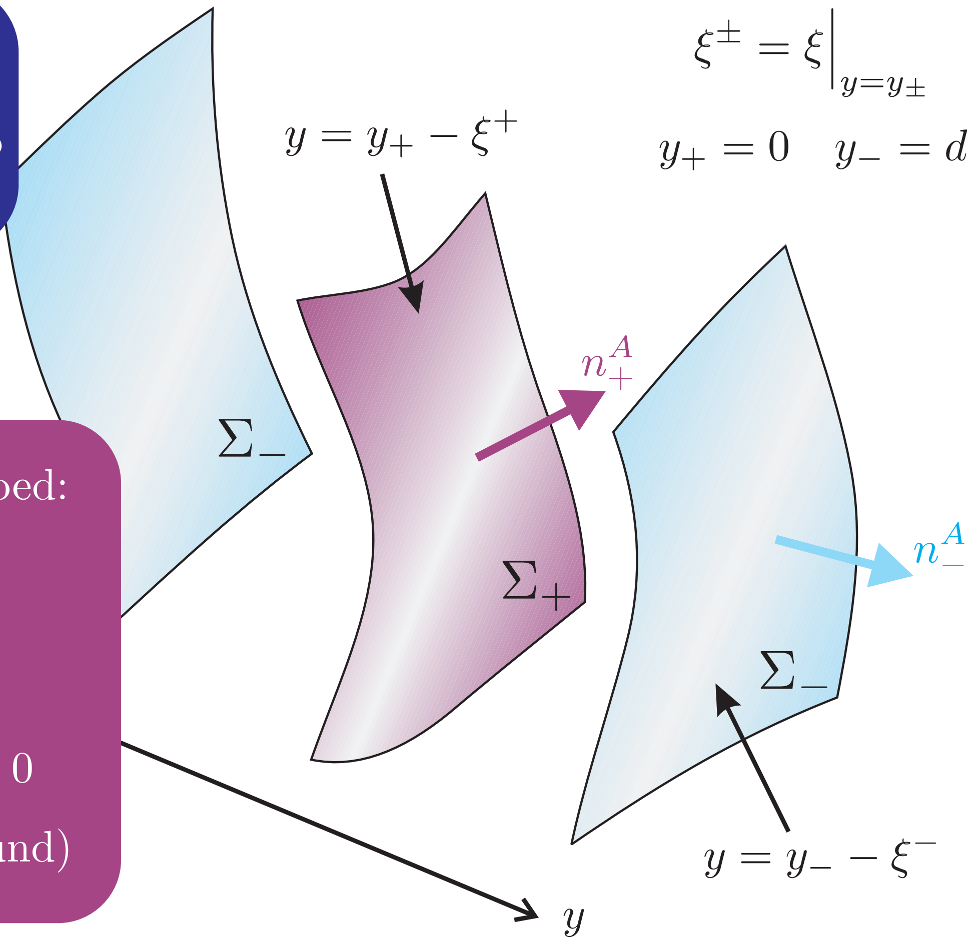

brane normals also perturbed:

$$n^A_\pm = \left(\frac{\partial^A \Phi}{\sqrt{\partial_B \Phi \, \partial^B \Phi}} \right)_\pm$$

$$\delta n^\pm_A = \nabla_A \xi^\pm \quad n^A_\pm \delta n^\pm_A = 0$$

$(n^\pm_A = \partial_A y$ in the background$)$

Perturbed junction conditions

Recall the junction conditions at each brane:

$$Q^{\pm}_{AB} = \left\{ [K_{AB}] \pm 2kq_{AB} + \kappa_5^2 (T_{AB} - \tfrac{1}{3} T q_{AB}) \right\}^{\pm} = 0$$

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$$2 q^C_A q^D_B \nabla_C \nabla_D \xi$$

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RS gauge choice

Bulk EOM:

$$\nabla^C \nabla_C h_{AB} - \nabla^C \nabla_A h_{BC} - \nabla^C \nabla_B h_{AC} + \nabla_A \nabla_B h^C{}_C - 8k^2 h_{AB} = -2\kappa_5^2 \Sigma^{\text{bulk}}_{AB}$$

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  - 15 + 2 DOFs (too many)
  - reduce number by gauge choice

RS gauge choice

- we have gauge freedom to put $h_{AB} \to h_{AB} + 2\nabla_{(A}\xi_{B)}$

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$$h_{AB} = \begin{pmatrix} \star & \star & \star & \star & \star \\ \cdot & \star & \star & \star & \star \\ \cdot & \cdot & \star & \star & \star \\ \cdot & \cdot & \cdot & \star & \star \\ \cdot & \cdot & \cdot & \cdot & \star \end{pmatrix} \begin{matrix} t \\ r \\ \theta \\ \phi \\ y \end{matrix}$$

15 DOFs

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Randall-Sundrum gauge choice

$$h_{AB} = \begin{pmatrix} \star & \star & \star & \star & 0 \\ \cdot & \star & \star & \star & 0 \\ \cdot & \cdot & \star & \star & 0 \\ \cdot & \cdot & \cdot & \star & 0 \\ \cdot & \cdot & \cdot & \cdot & 0 \end{pmatrix} \begin{matrix} t \\ r \\ \theta \\ \phi \\ y \end{matrix}$$

10 DOFs

RS gauge choice

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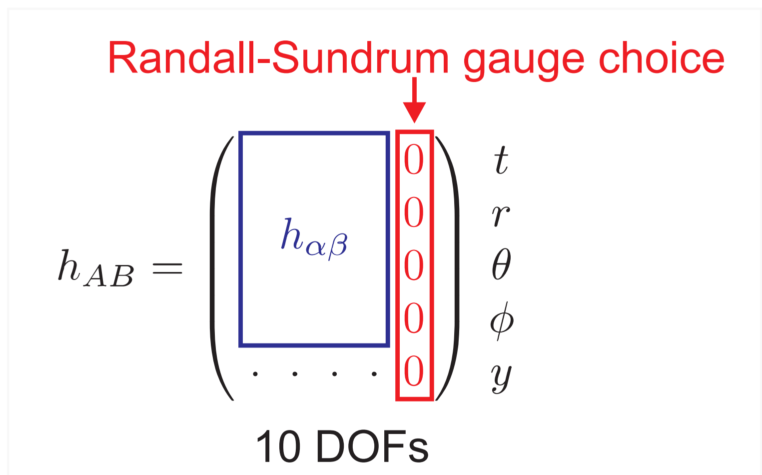

RS gauge choice

Bulk EOM in RS gauge:

$$\hat{\Delta}_{AB}{}^{CD} h_{CD} + (GMa)^2(\pounds_n^2 - 4k^2)h_{AB} = -2(GMa)^2 \kappa_5^2 \Sigma_{AB}^{\text{bulk}},$$

where we have defined the operator

$$\hat{\Delta}_{AB}{}^{CD} = (GMa)^2[q^{MN}\nabla_M q_N^P q_A^C q_B^D \nabla_P + 2\,{}^{(4)}R_A{}^C{}_B{}^D]$$

$$= (GMa)^2 e_A^\alpha e_B^\beta \left[\delta_\alpha^\gamma \delta_\beta^\delta \nabla^\rho \nabla_\rho + 2R_\alpha{}^\gamma{}_\beta{}^\delta\right]_q e_\gamma^C e_\delta^D$$

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- curvature tensor on Σ_y:
  $${}^{(4)}R_{MNPQ} = q_M^A q_N^B q_P^C q_Q^D R_{ABCD} + 2 K_{M[P} K_{Q]N}$$

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- curvature tensor on Σ_y: ${}^{(4)}R_{MNPQ} = q_M^A q_N^B q_P^C q_Q^D R_{ABCD} + 2K_{M[P}K_{Q]N}$
- induced metric on Σ_y: $q_{\alpha\beta} = a^2 g_{\alpha\beta}$ where $g_{\alpha\beta}$ is Schwarzschild

RS gauge choice

Bulk EOM in RS gauge:

$$\hat{\Delta}_{AB}{}^{CD} h_{CD} + (GMa)^2(\pounds_n^2 - 4k^2) h_{AB} = -2(GMa)^2 \kappa_5^2 \Sigma_{AB}^{\text{bulk}},$$

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- $[\cdots]_{q \text{ or } g}$ means calculate with $q_{\alpha\beta}$ or $g_{\alpha\beta}$

RS gauge consequences

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Junction conditions:

$$\delta Q_{AB}^{\pm} = \Big\{2q_A^C q_B^D \nabla_C \nabla_D \xi + \tfrac{1}{2}[\pounds_n h_{AB}] \\ \pm 2kh_{AB} + \kappa_5^2 \left(T_{AB} - \tfrac{1}{3}Tq_{AB}\right)\Big\}_0^{\pm} = 0$$

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- LHS of bulk EOM manifestly traceless and orthogonal to y

$$\Sigma_{AB}^{\text{bulk}} = e_A^\alpha e_B^\beta \Sigma_{\alpha\beta}^{\text{bulk}} \quad q^{\alpha\beta}\Sigma_{\alpha\beta}^{\text{bulk}} = 0$$

RS gauge not general!

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  RS gauge not general!
  - just like radiation gauge in GR (see Wald)
- trace of junction conditions yields an EOM for $\xi^{\pm}$

$$q^{AB} \nabla_A \nabla_B \xi^{\pm} = \tfrac{1}{6} \kappa_5^2 T^{\pm}$$

RS gauge consequences

Principal equations in RS gauge:

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- Generalization of seminal Garriga & Tanaka (1999) result to bulk with $C_{ABCD} \neq 0$ and arbitrary bulk coordinates
- we will set $\Sigma_{AB}^{\text{bulk}} = T_{AB}^{\text{bulk}} = 0$ from now on

Streamlined bulk equation of motion

- can incorporate junction conditions into bulk EOM by introducing distributional sources

$$\hat{\Delta}_{AB}{}^{CD} h_{CD} - \hat{\mu}^2 h_{AB} = -2(GMa)^2 \kappa_5^2 \sum_{\epsilon=\pm} \delta(y - y_\epsilon) \Sigma^\epsilon_{AB}$$

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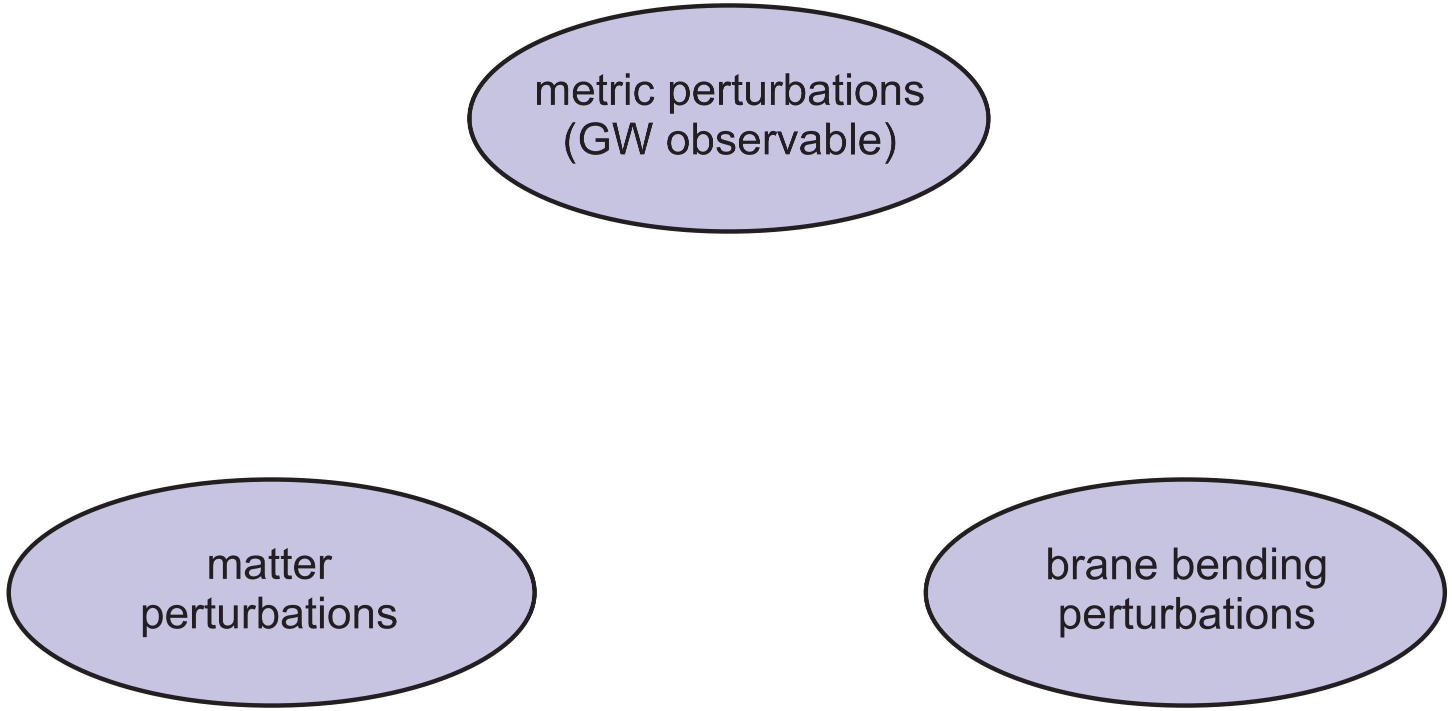
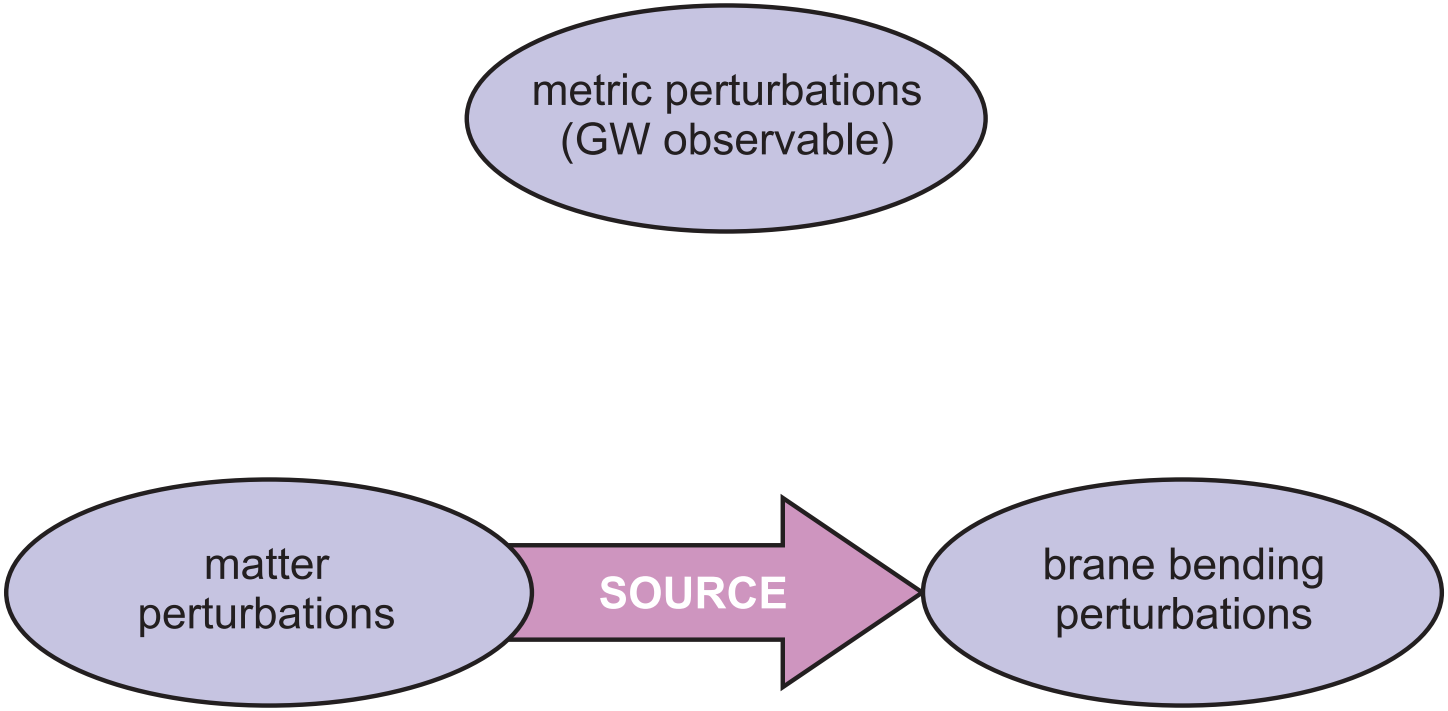
- recover previous formulae by integrating over small y interval across each brane

Summary



metric perturbations
(GW observable)
matter
perturbations
brane bending
perturbations

Summary


metric perturbations
(GW observable)
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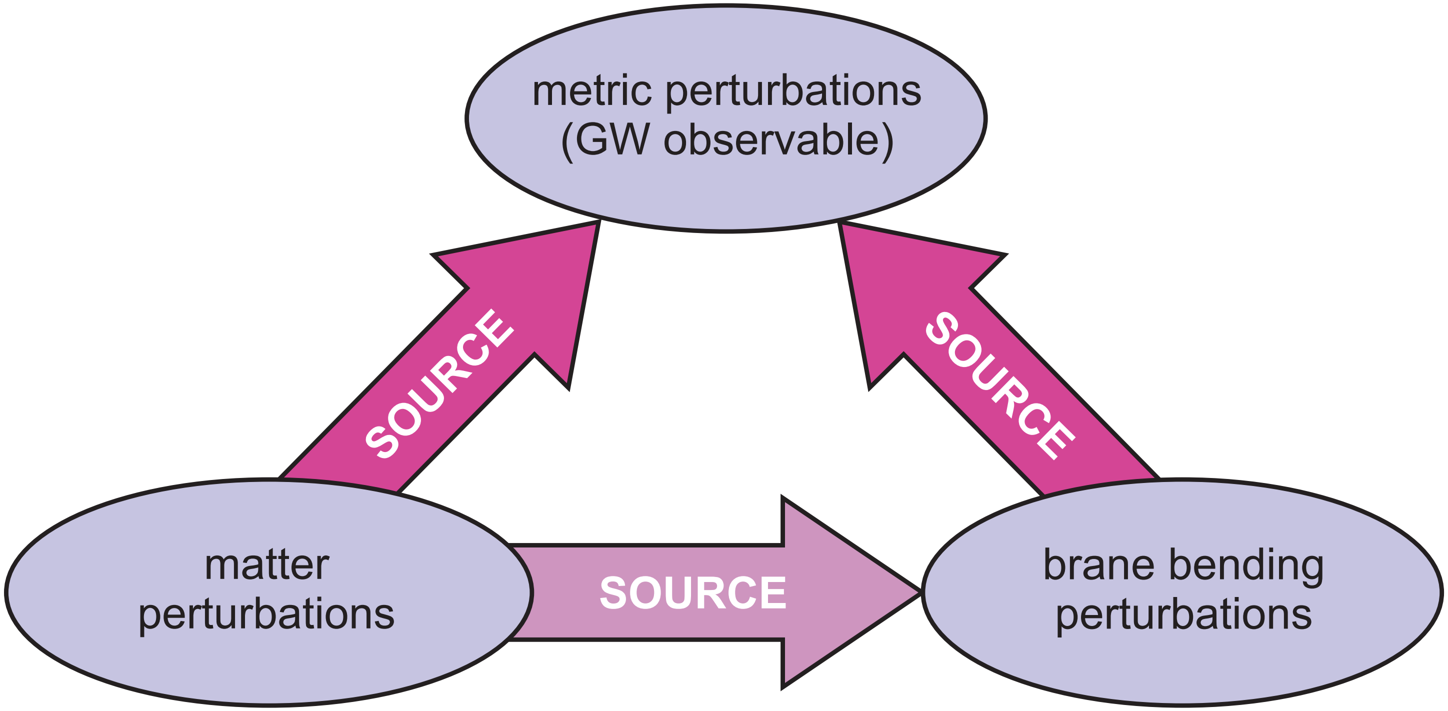
Summary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(GW observable)
SOURCE
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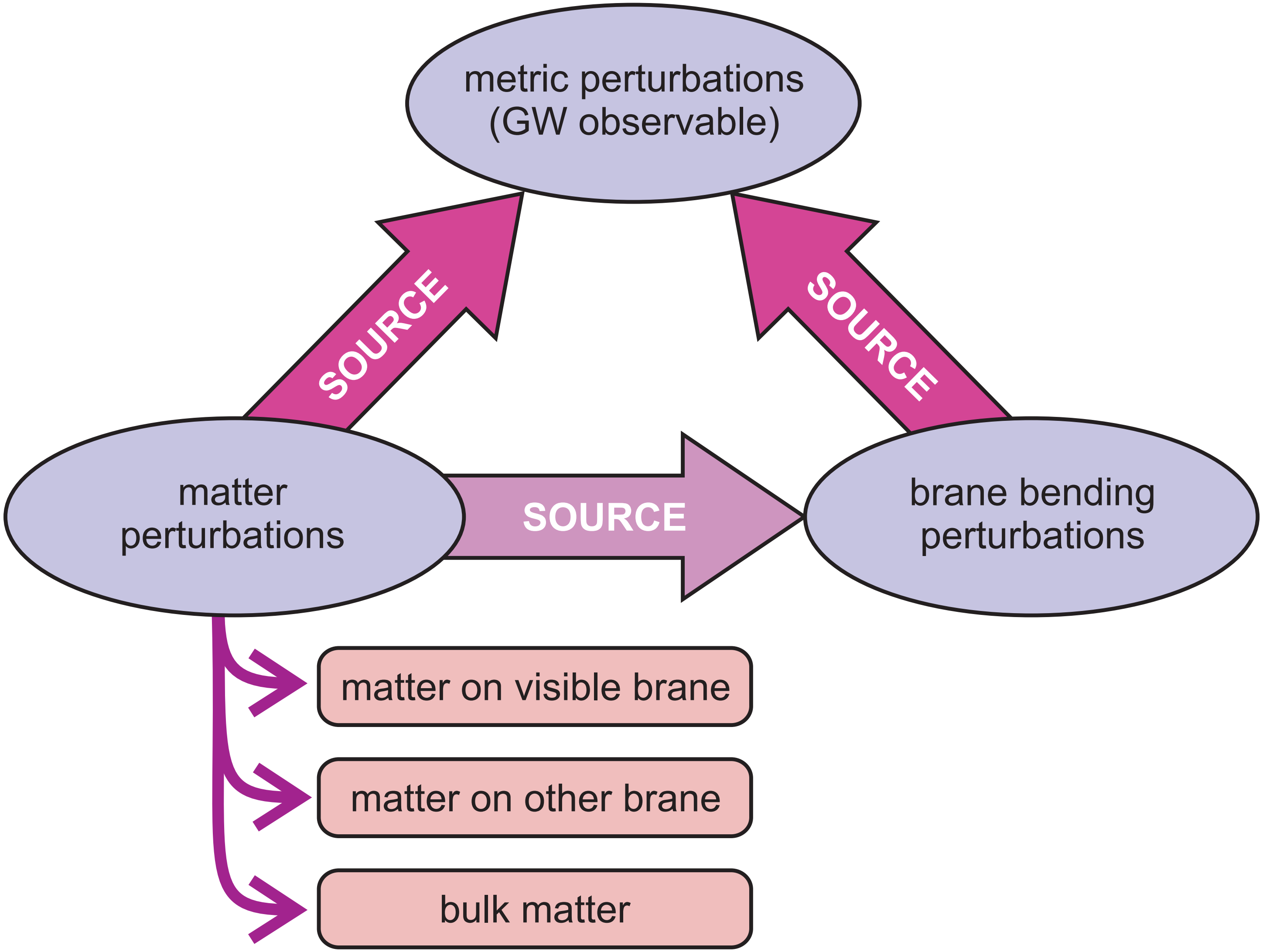
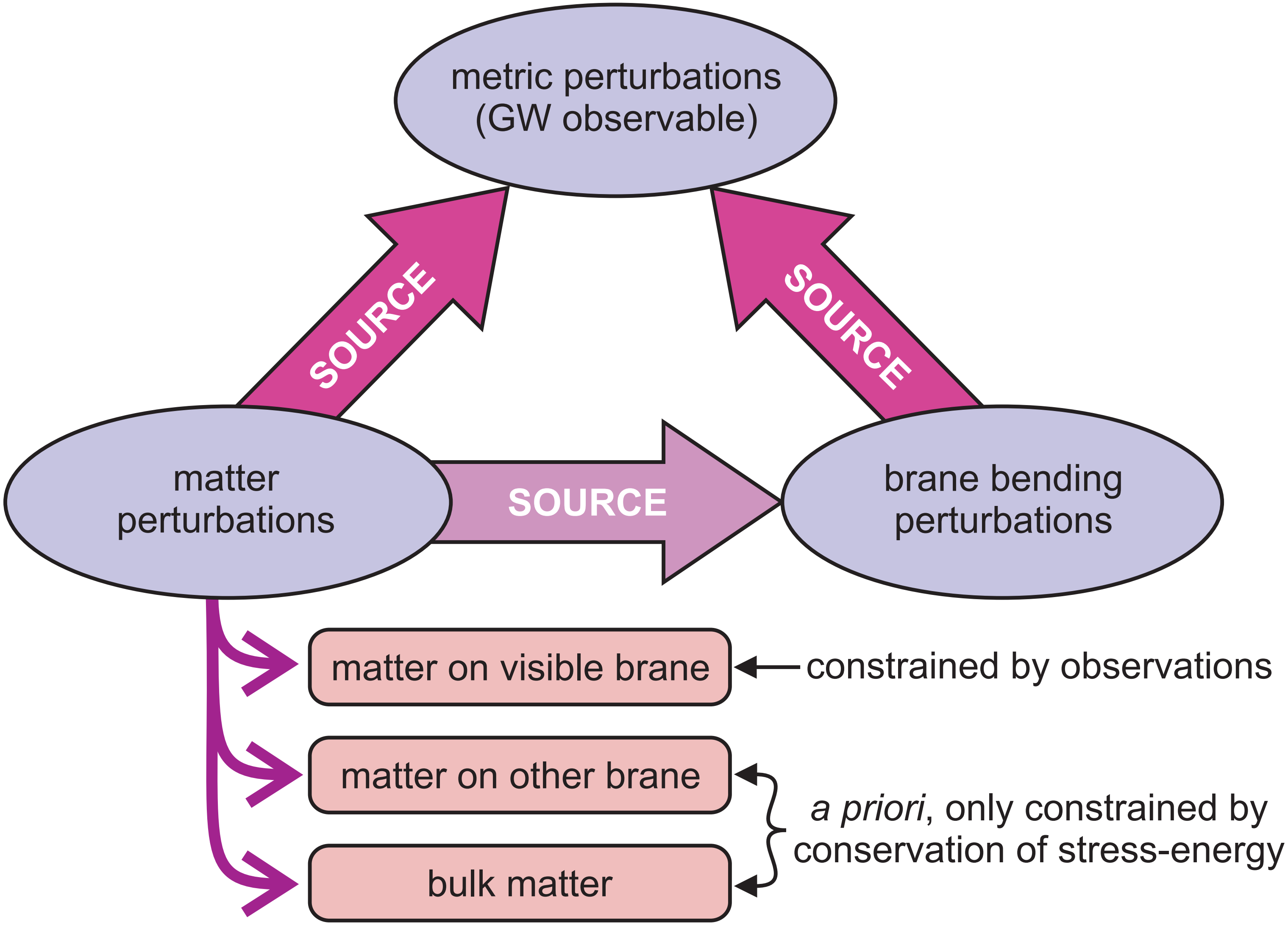


metric perturbations
(GW observable)
SOURCE
SOURCE
matter
perturbations
SOURCE
brane bending
perturbations
matter on visible brane
matter on other brane
bulk matter

Summary



metric perturbations (GW observable)
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SOURCE
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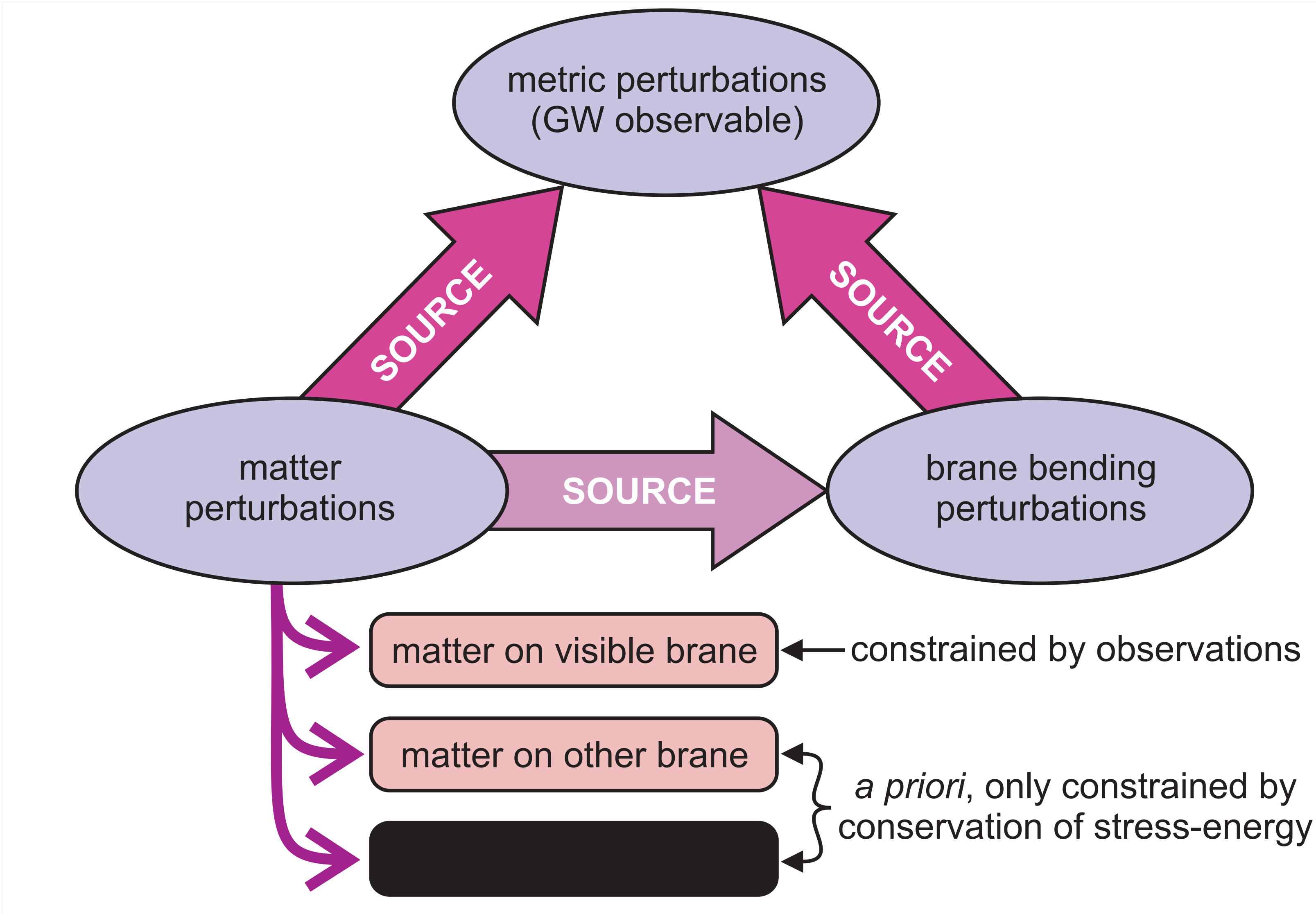
matter on visible brane
constrained by observations
matter on other brane
bulk matter
a priori, only constrained by conservation of stress-energy

Summary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The massive modes

Kaluza-Klein decomposition

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- they formally commute $[\hat{\Delta}_{AB}{}^{CD}, \hat{\mu}^2] h_{CD} = 0$
- hence we can seek a solution by separation of variables

$$h_{AB}(z^\alpha, y) = Z(y)\tilde{h}_{AB}(z^\alpha) \quad \hat{\mu}^2 Z(y) = \mu^2 Z(y)$$

i.e.: $Z(y)$ is a eigenfunction of $\hat{\mu}^2$ with eigenvalue μ^2

Even parity eigenfunctions

- if we assume that $Z = Z(y)$ is reflection symmetric about each brane, the eigenvalue problem reduces to

$$m^2 Z(y) = -a^2(y) \left[\partial_y^2 + 4k \sum_{\epsilon=\pm} \epsilon\delta(y - y_\epsilon) - 4k^2 \right] Z(y),$$

where $\mu = GMm$

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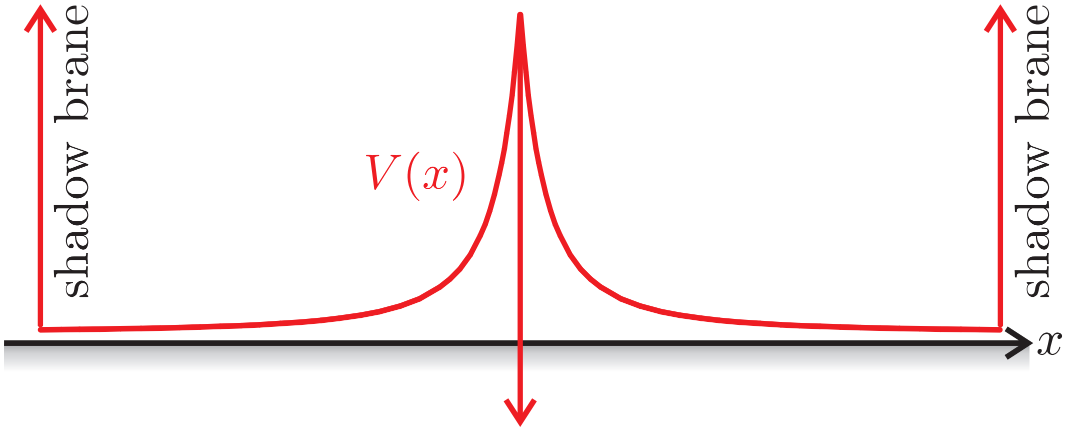
- transform into Schrödinger form with $x = \ell e^{ky}$ and $\psi = x^{1/2} Z$

$$m^2 \psi = -\psi'' + V(x)\psi$$



Even parity eigenfunctions

- there is a discrete spectrum of solutions labeled by the positive integers $n = 1, 2, 3 \ldots$:

$$Z_n(y) = \alpha_n^{-1}[Y_1(m_n\ell)J_2(m_n\ell e^{k|y|}) - J_1(m_n\ell)Y_2(m_n\ell e^{k|y|})]$$

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$$\delta_{mn} = \int_{-d}^{d} dy\, a^{-2}(y) Z_m(y) Z_n(y)$$

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$$\delta_{mn} = \int_{-d}^{d} dy\, a^{-2}(y) Z_m(y) Z_n(y)$$

- there is also a solution corresponding to $m_0 = \mu_0 = 0$, which is known as the zero-mode:

$$Z_0(y) = \alpha_0^{-1} e^{-2k|y|}, \quad \alpha_0 = \sqrt{\ell}(1 - e^{-2kd})^{1/2}.$$

Odd parity eigenfunctions

- there are also odd parity harmonics satisfying

$$m^2 Z(y) = -a^2(y)(\partial_y^2 - 4k^2)Z(y) \quad 0 = Z(y_+) = Z(y_-)$$

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- we can verify

$$m_1 < m_{3/2} < m_2 < m_{5/2} < \cdots$$

Source-free solution

- in the absence of sources, the general solution to the bulk EOM is

$$h_{AB} = e^\alpha_A e^\beta_B \sum_{n=0}^{\infty} \left[\mathcal{A}_n Z_n h^{(n)}_{\alpha\beta} + \mathcal{B}_{n+3/2} Z_{n+3/2} h^{(n+3/2)}_{\alpha\beta} \right]$$

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- each of the $h^{(n)}_{\alpha\beta}$ behaves like a massive graviton on the Schwarzschild background

$$\nabla^{\gamma} \nabla_{\gamma} h^{(n)}_{\alpha\beta} + 2 R_{\alpha}{}^{\gamma}{}_{\beta}{}^{\delta} h^{(n)}_{\gamma\delta} - m_n^2 h^{(n)}_{\alpha\beta} = 0$$

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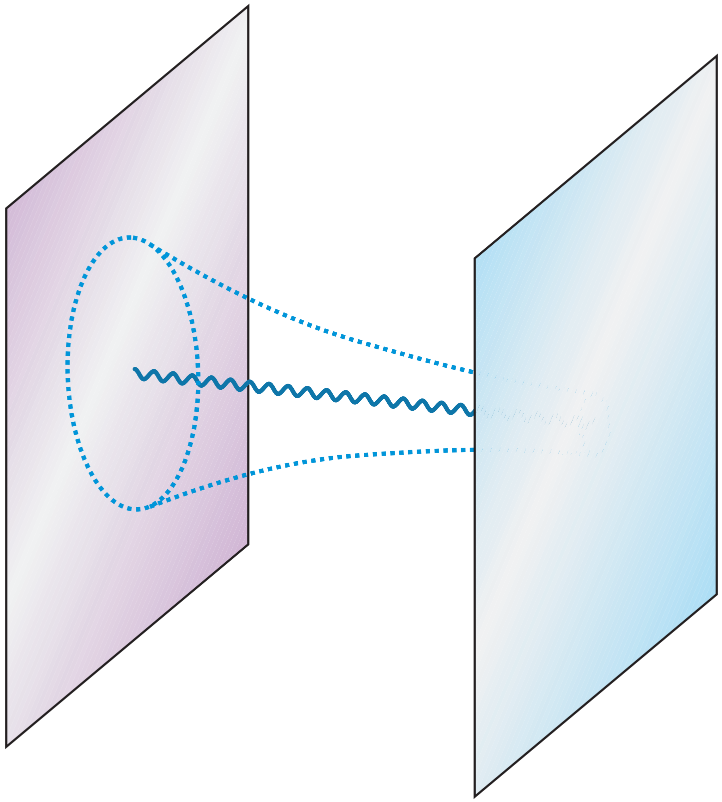
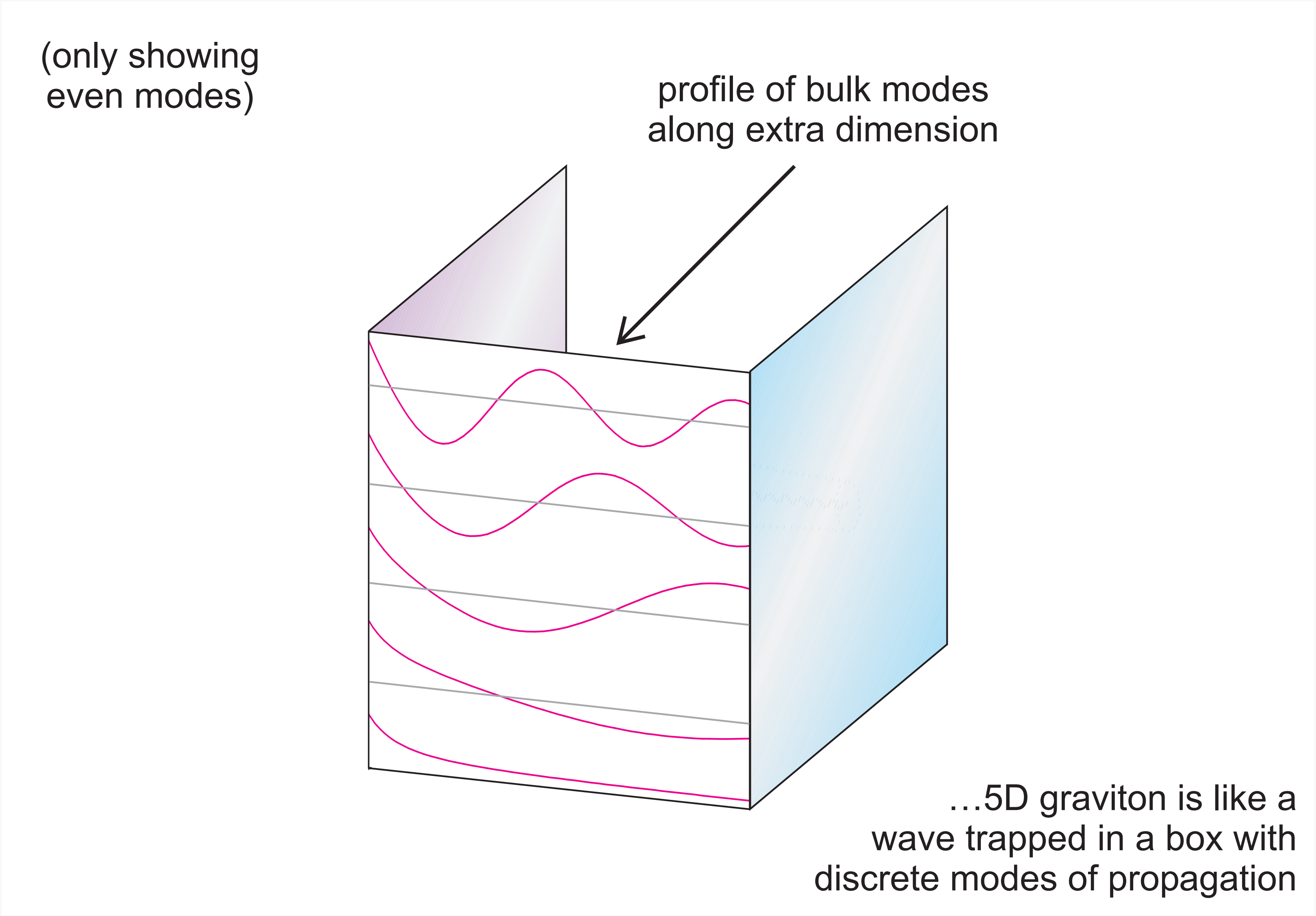
- since $m_0 = 0$, the zero-mode behaves exactly like massless graviton in GR

Graviton in a box



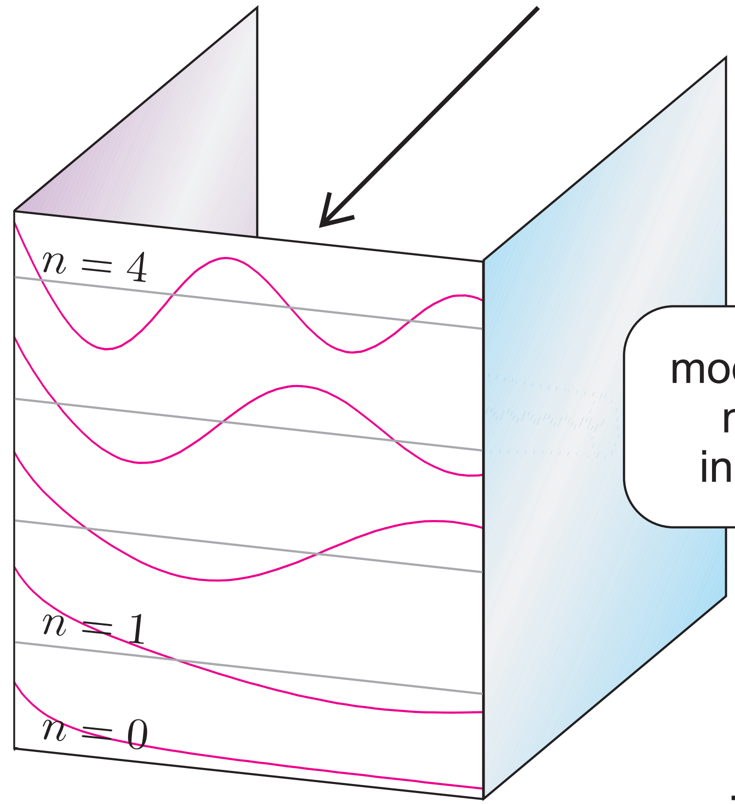
Graviton in a box



Graviton in a box

(only showing even modes)

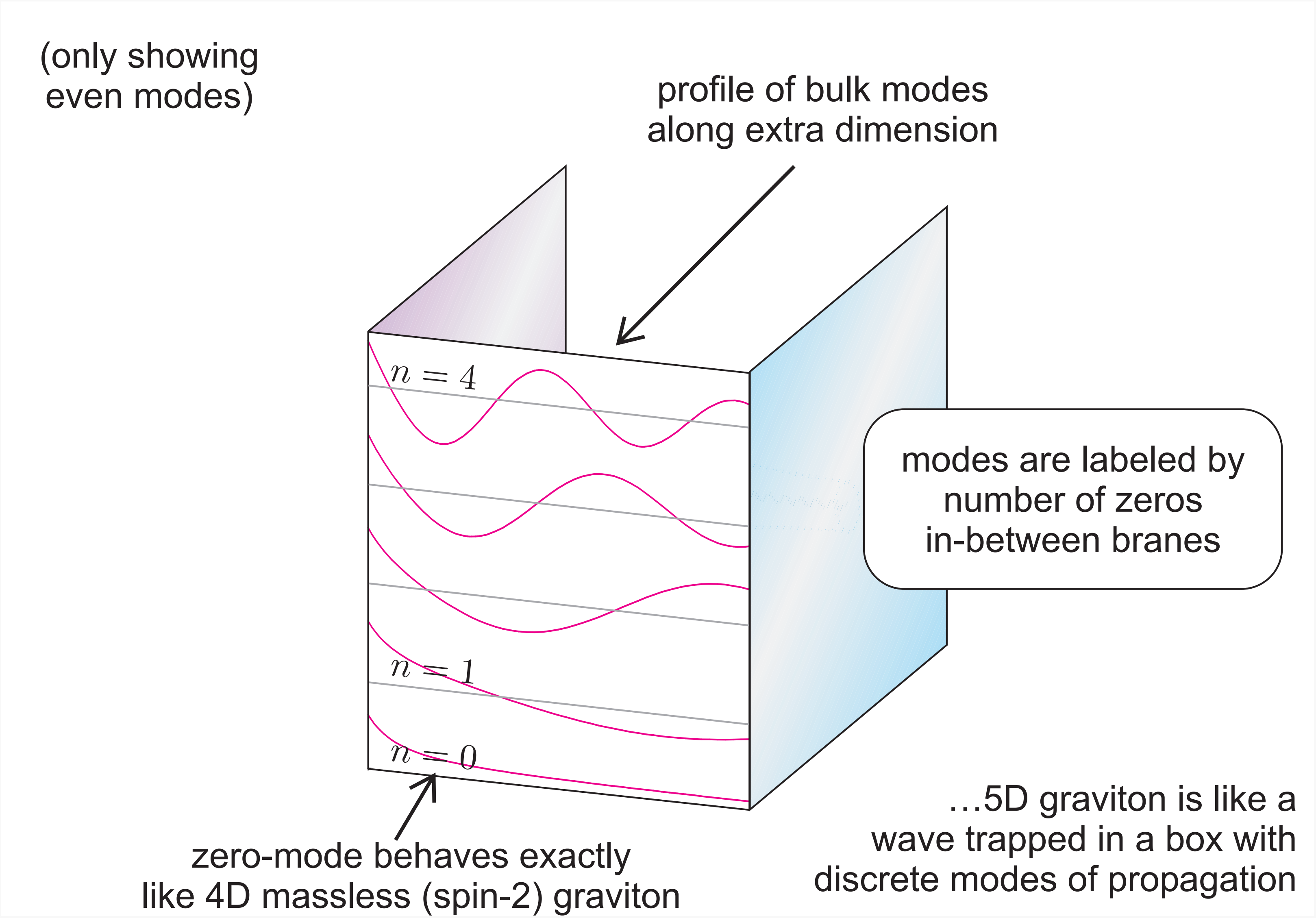
profile of bulk modes along extra dimension



modes are labeled by number of zeros in-between branes

…5D graviton is like a wave trapped in a box with discrete modes of propagation

Graviton in a box



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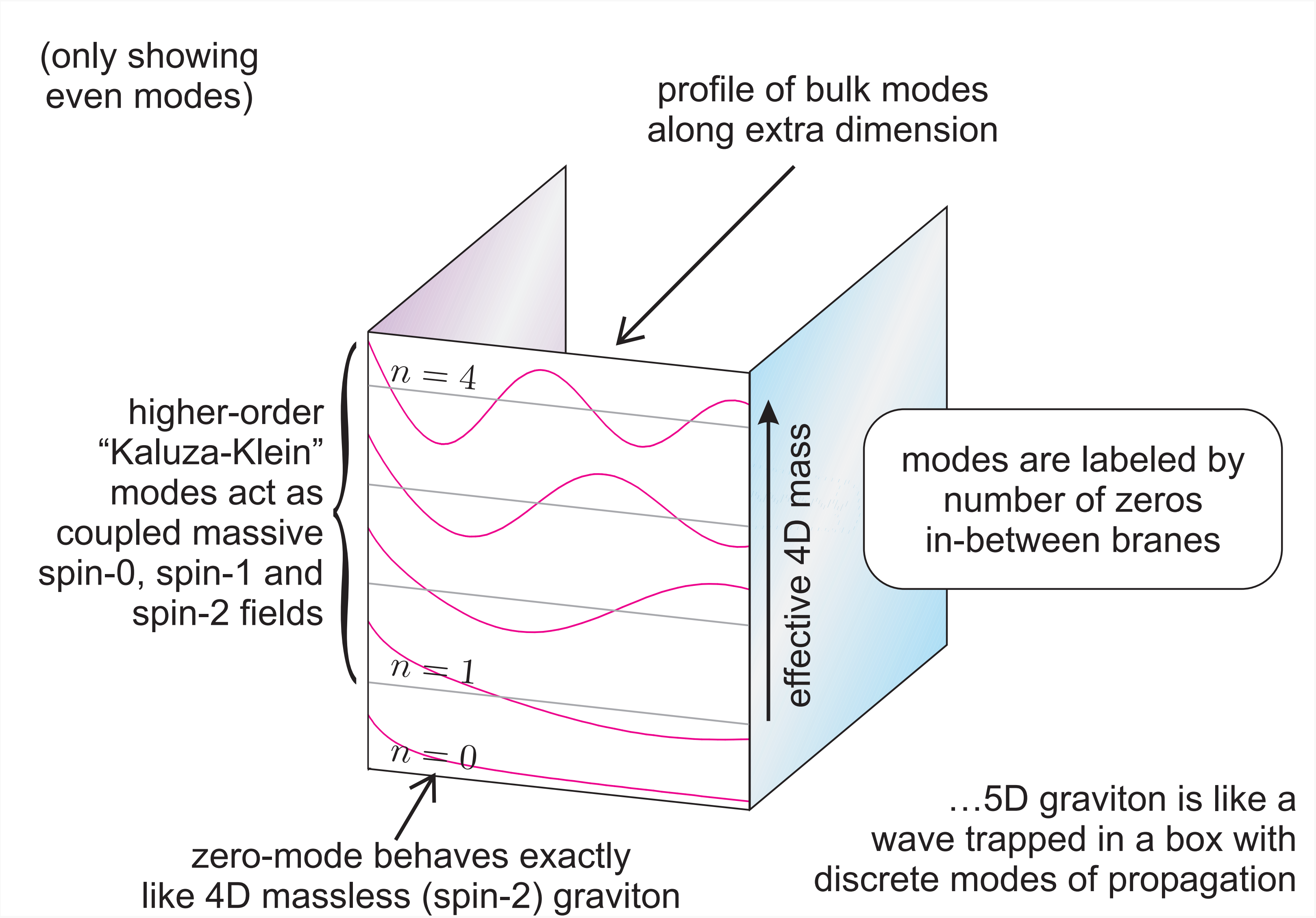

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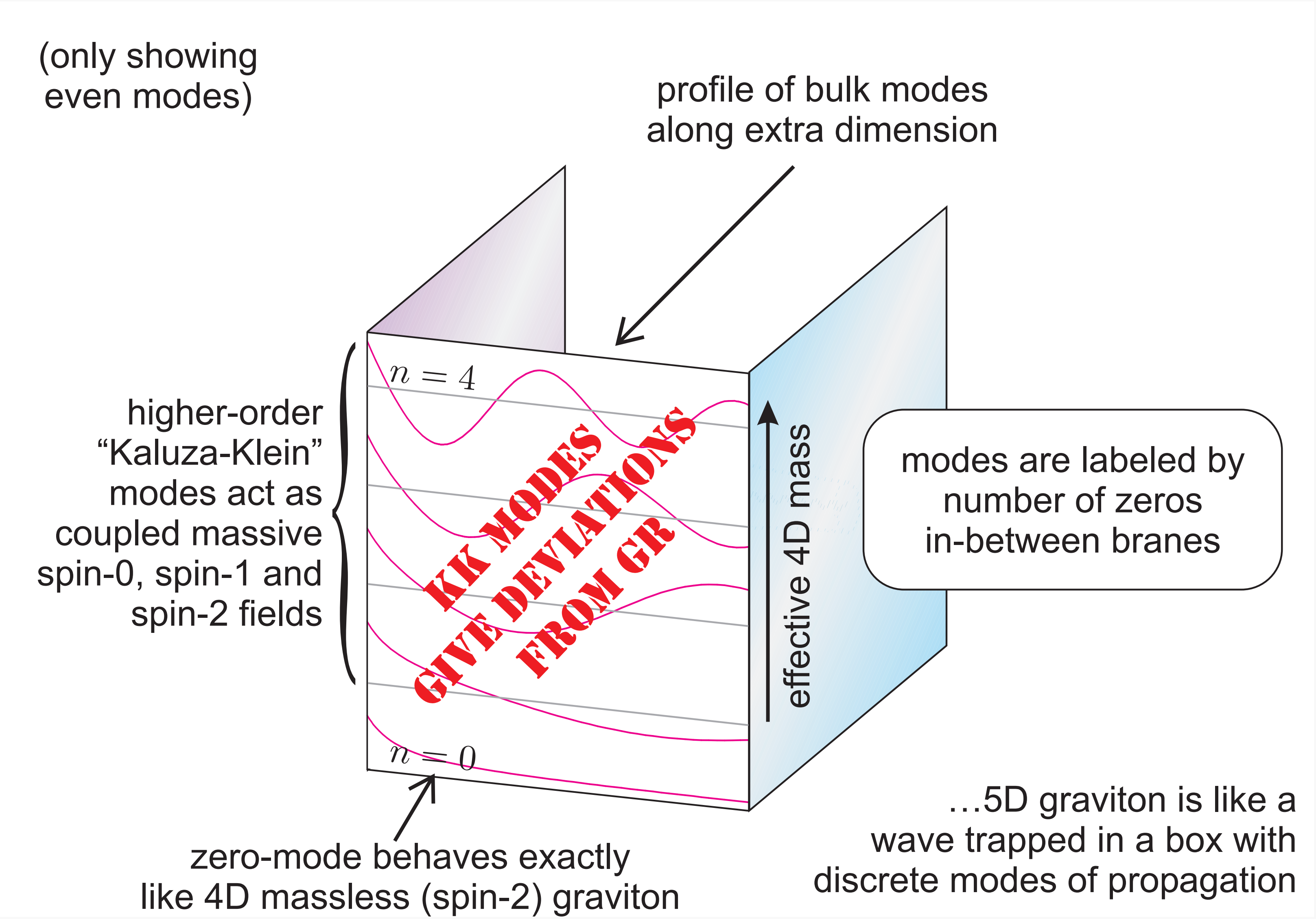

Recovering GR

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- **zero-mode truncation:** retain only the first term in the series

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- this justifies zero-mode truncation (sort-of)

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- similar to our previous calculation of perturbed junction conditions, we find

$$\begin{aligned} \delta q^{+}_{AB} \equiv \bar{h}^{+}_{AB} &= \left\{ \frac{\delta q_{AB}}{\delta \Phi} \delta\Phi + \frac{\delta q_{AB}}{\delta n_C} \delta n_C + \frac{\delta q_{AB}}{\delta g_{CD}} \delta g_{CD} \right\}^{+}_{0} \\ &= h^{+}_{AB} + 2k\xi^{+} q^{+}_{AB} - (n_A \nabla_B + n_B \nabla_A)\xi^{+} \end{aligned}$$

Brans-Dicke theory

- define

$$\bar{h}^{+}_{\alpha\beta} = e^A_\alpha e^B_\beta \bar{h}^{+}_{AB} \quad T^{+}_{\alpha\beta} = e^A_\alpha e^B_\beta T^{+}_{AB} \quad Z^2_+ = Z^2_0(y_+) = \frac{k}{1 - e^{-2kd}}$$

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- equation of motion for $\bar{h}^+_{\alpha\beta}$:

$$\nabla^\gamma \nabla_\gamma \bar{h}_{\alpha\beta} + \nabla_\alpha \nabla_\beta \bar{h}^\gamma{}_\gamma - \nabla^\gamma \nabla_\alpha \bar{h}_{\beta\gamma} - \nabla^\gamma \nabla_\beta \bar{h}_{\alpha\gamma} =$$
$$-2Z^2_+ \kappa^2_5 \left[T_{\alpha\beta} - \frac{1}{3} \left(1 + \frac{k}{2Z^2_+} \right) T^\gamma{}_\gamma g_{\alpha\beta} \right] + (6k - 4Z^2_+) \nabla_\alpha \nabla_\beta \xi$$

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- still have the gauge freedom

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- Brans-Dicke parameter and Newton constant related to brane separation

$$\omega_{\text{BD}} = \frac{3}{2}(e^{2d/\ell} - 1) \qquad G = \frac{\kappa_5^2}{8\pi\ell(1 - e^{-2d/\ell})}$$

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- we need to adopt this limit on brane separation to model black holes in the nearby universe as black strings

Angular harmonic decomposition

Tensor harmonics

Let us move beyond the zero mode approximation...

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helpful to further decompose them in terms of tensor spherical harmonics $[Y_{lm}]_{AB}$

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Recall background geometry: $ds^2 = a^2(y)[-f\,dt^2 + f^{-1}\,dr^2 + r^2\,d\Omega^2] + dy^2$

(see also Kokkotas lecture)

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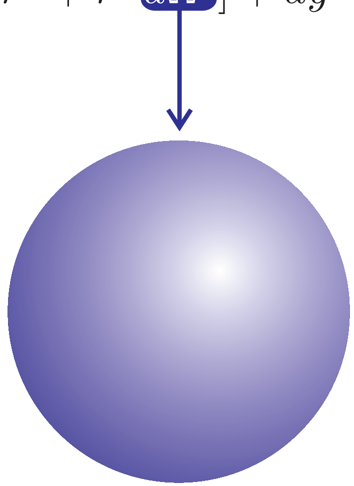

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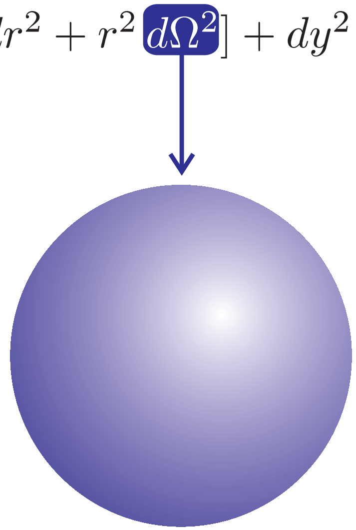

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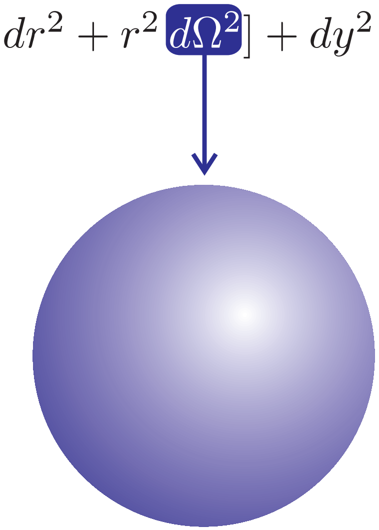

2-sphere

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rank 0 (S)	Y_{lm}	
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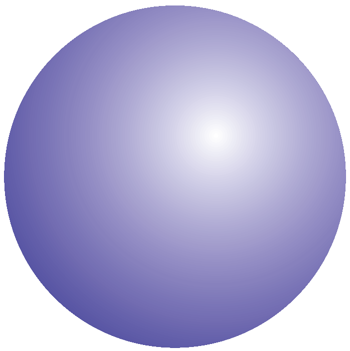

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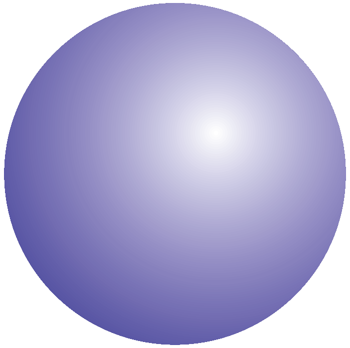

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$$\quad t \quad r \quad (\theta,\phi) \quad y$$

for fixed (l, m):

- six S degrees of freedom
- six V degrees of freedom
- three T degrees of freedom

fifteen total DOFs

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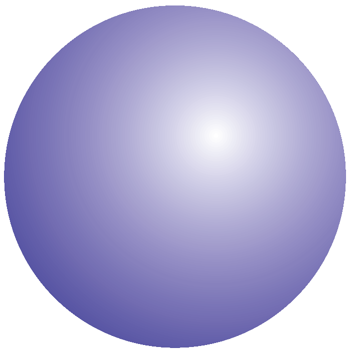

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RS gauge

$$h_{AB} = \begin{pmatrix} S & S & V & \cancel{S} \\ & S & V & \cancel{S} \\ & & T & \cancel{V} \\ & & & \cancel{S} \end{pmatrix} \begin{matrix} t \\ r \\ (\theta,\phi) \\ y \end{matrix}$$

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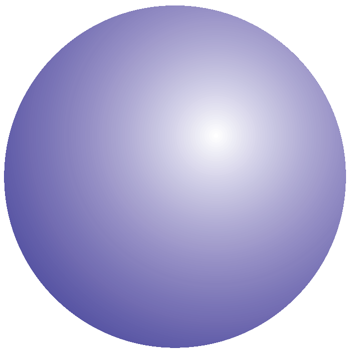

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RS gauge

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$t \quad r \quad (\theta, \phi) \quad y$

for fixed (l, m):

- ~~six~~ S degrees of freedom
- ...es of freedom
- ...egrees of freedom

~~fifteen~~ ten total DOFs

- **polar:** transforms as $(-1)^l$ under $\mathbf{x} \to -\mathbf{x}$
- **axial:** transforms as $(-1)^{l+1}$ under $\mathbf{x} \to -\mathbf{x}$

Tensor harmonics

...focus on RS gauge

Separation of variables:

$$h_{\alpha\beta} = \sum_{n=0}^{\infty} Z_n(y) h_{\alpha\beta}^{(n)}(x^\mu)$$

Randall-Sundrum gauge choice

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↓

$$h^{(nlm)}_{\alpha\beta} = \underbrace{\sum_{i=1}^{7} \mathcal{P}^{ni}_{lm}(t,r)\mathbb{P}^{(ilm)}_{\alpha\beta}}_{\text{polar}} + \underbrace{\sum_{j=1}^{3} \mathcal{A}^{nj}_{lm}(t,r)\mathbb{A}^{(jlm)}_{\alpha\beta}}_{\text{axial}}$$

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rank 0 (S)	Y_{lm}	
rank 1 (V)	$D_a Y_{lm}$	$\epsilon_a{}^b D_b Y_{lm}$
rank 2 (T)	$D_a D_b Y_{lm}$ and $Y_{lm}\gamma_{ab}$	$-\epsilon_{c(a} D_{b)} D^c Y_{lm}$

$$h^{(nlm)}_{\alpha\beta} = \sum_{i=1}^{7} \mathcal{P}^{ni}_{lm}(t,r)\mathbb{P}^{(ilm)}_{\alpha\beta} + \sum_{j=1}^{3} \mathcal{A}^{nj}_{lm}(t,r)\mathbb{A}^{(jlm)}_{\alpha\beta}$$

"standard" tensor spherical harmonics

Tensor harmonics

...focus on RS gauge

Randall-Sundrum gauge choice

$$h_{AB} = \begin{pmatrix} h_{\alpha\beta} & \begin{matrix} 0 \\ 0 \\ 0 \\ 0 \end{matrix} \\ \cdot\ \cdot\ \cdot\ \cdot & 0 \end{pmatrix} \begin{matrix} t \\ r \\ \theta \\ \phi \\ y \end{matrix}$$

10 DOFs

Separation of variables:

$$h_{\alpha\beta} = \sum_{n=0}^{\infty} Z_n(y) h^{(n)}_{\alpha\beta}(x^\mu)$$

spherical harmonics

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dynamics of these expansion coefficients determined by linearized field equations... polar and axial sets are decoupled

Master wave equations

$$\nabla^\gamma \nabla_\gamma h_{\alpha\beta}^{(nlm)} + 2R_\alpha{}^\gamma{}_\beta{}^\delta h_{\gamma\delta}^{(nlm)} - m_n^2 h_{\alpha\beta}^{(nlm)} = \text{sources}$$

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- general idea is to solve dynamical equations for master variables $\{\psi\}$ and then obtain metric perturbation $h^{(nlm)}$ by algebra/differentiation/quadrature
- actual process of finding master variables tedious and often involves trial and error

Master wave equations

- essential DOFs are described by (vector-valued) master variables $\mathbf{\Psi}^a_{nlm} = \mathbf{\Psi}^a_{nlm}(\tau, x)$, which satisfy wave equations

$$[\partial_\tau^2 - \partial_x^2 + \mathbf{V}_l^a(x, m_n)]\mathbf{\Psi}^a_{nlm} = \text{source}(\text{matter}, \xi^\pm)$$

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$n = 0$	polar	axial
$l = 0$	n/a	n/a
$l = 1$	n/a	n/a
$l \geq 2$	$\dim \mathbf{\Psi}^a_{nlm} = 1$	$\dim \mathbf{\Psi}^a_{nlm} = 1$

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  - $a = (\text{polar}, \text{axial})$
  - number of radiative DOFs in **massive mode** sector:

$n \geq 1$	polar	axial
$l = 0$	$\dim \mathbf{\Psi}^a_{nlm} = 1$	n/a
$l = 1$	$\dim \mathbf{\Psi}^a_{nlm} = 2$	$\dim \mathbf{\Psi}^a_{nlm} = 1$
$l \geq 2$	$\dim \mathbf{\Psi}^a_{nlm} = 3$	$\dim \mathbf{\Psi}^a_{nlm} = 2$

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- also as in 4D, the remaining coefficients are gauge invariant

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- 4D observers interpret these perturbations as a massive graviton coupled to a massive graviphoton

Gregory-Laflamme instability

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(see also Kodama and Obler lectures)

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- black string background has structure $\mathbb{R}^3 \times S^2$
  - Birkhoff's theorem does not apply
  - can have radiative s-wave ($l = 0$) GWs

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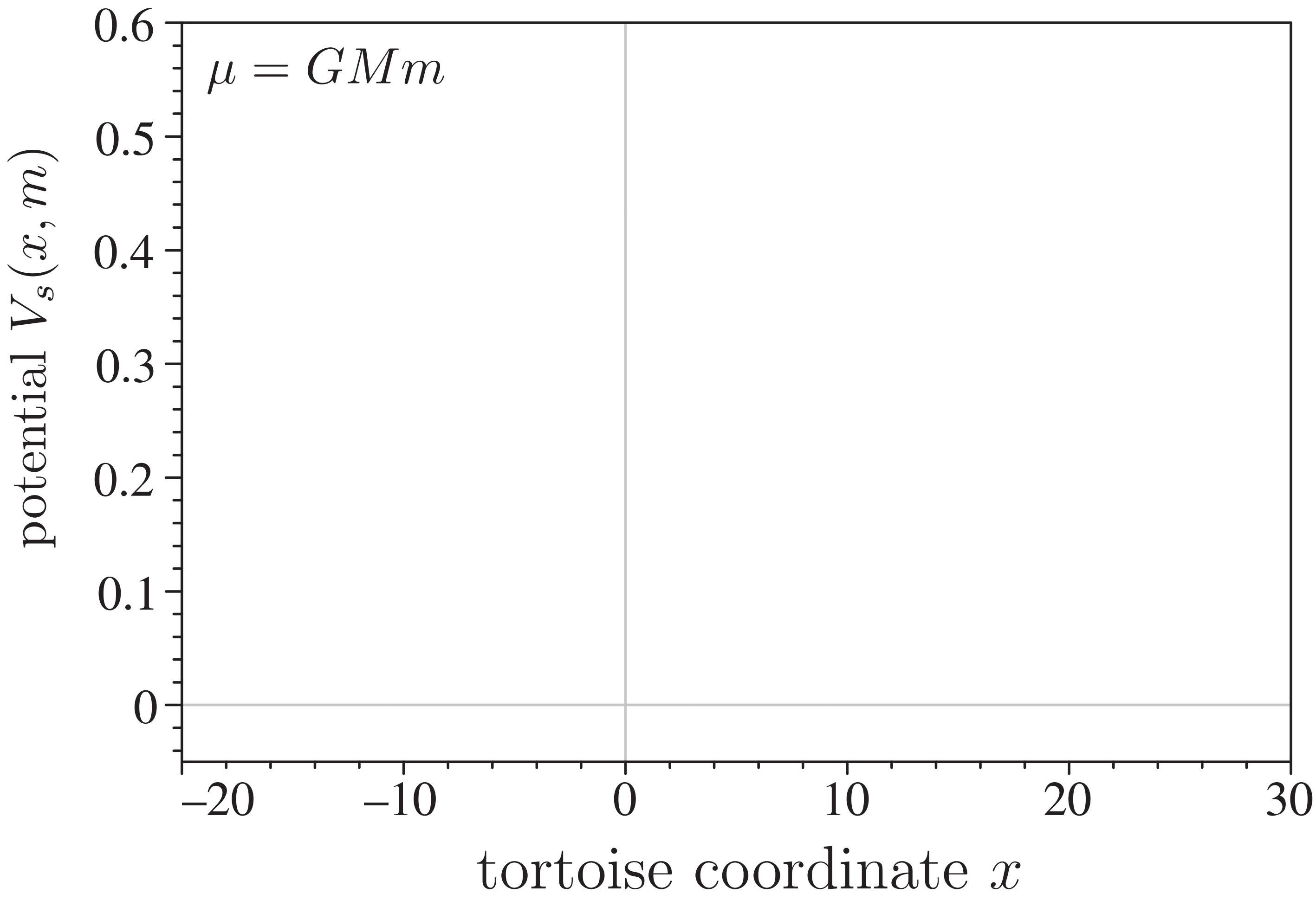
- does the s-wave potential actually support a bound state?

Potential in the s-wave equation



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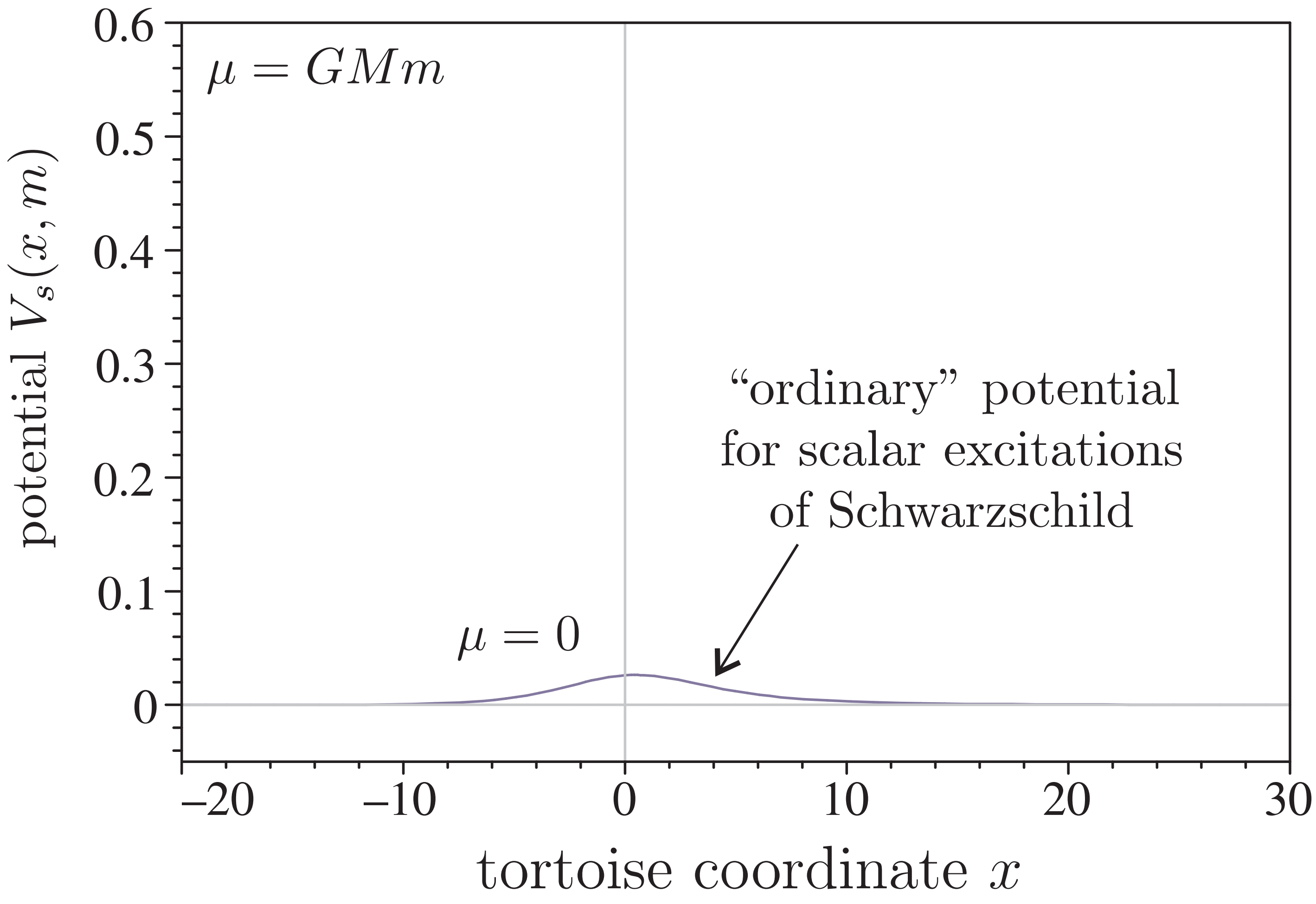

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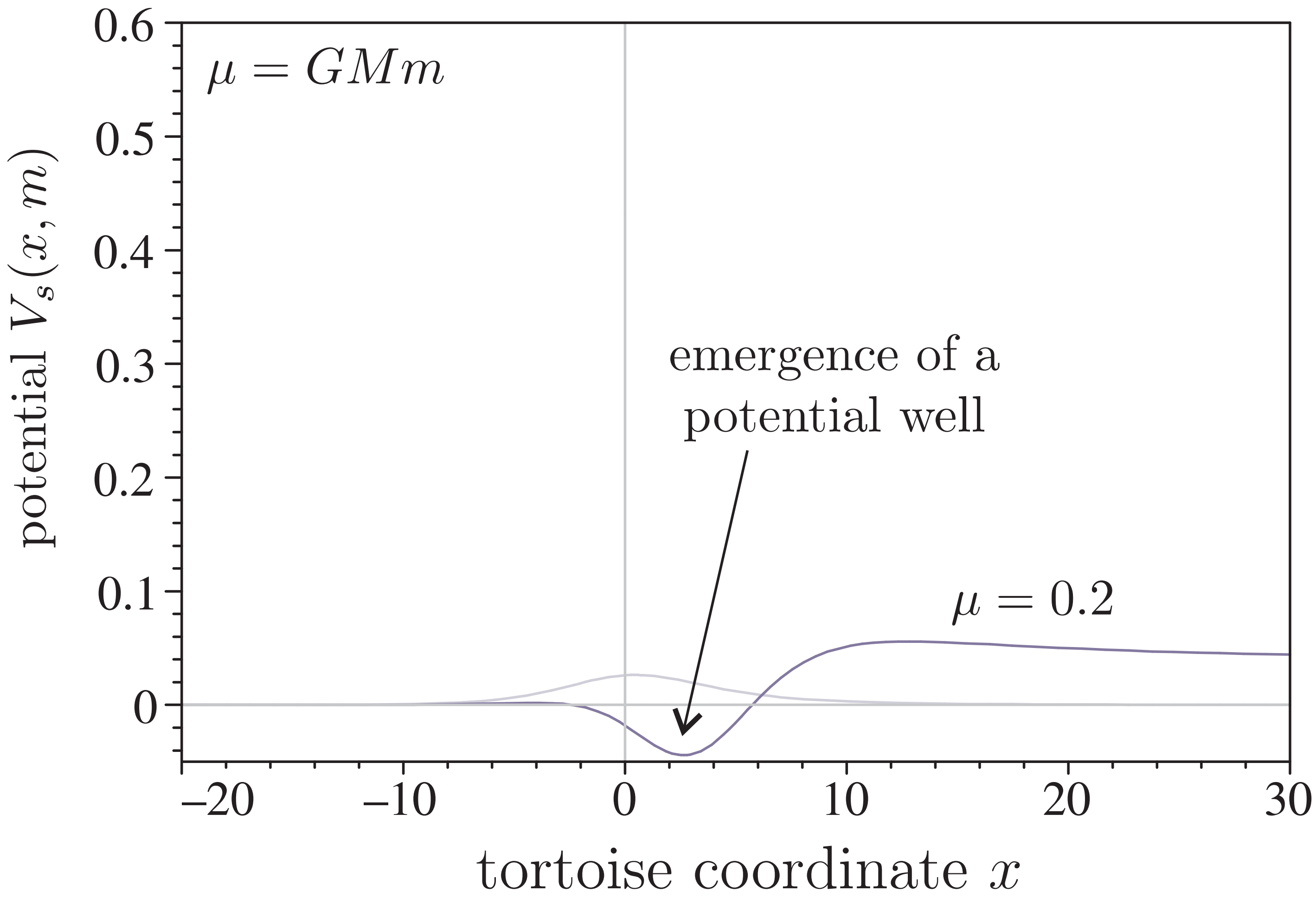

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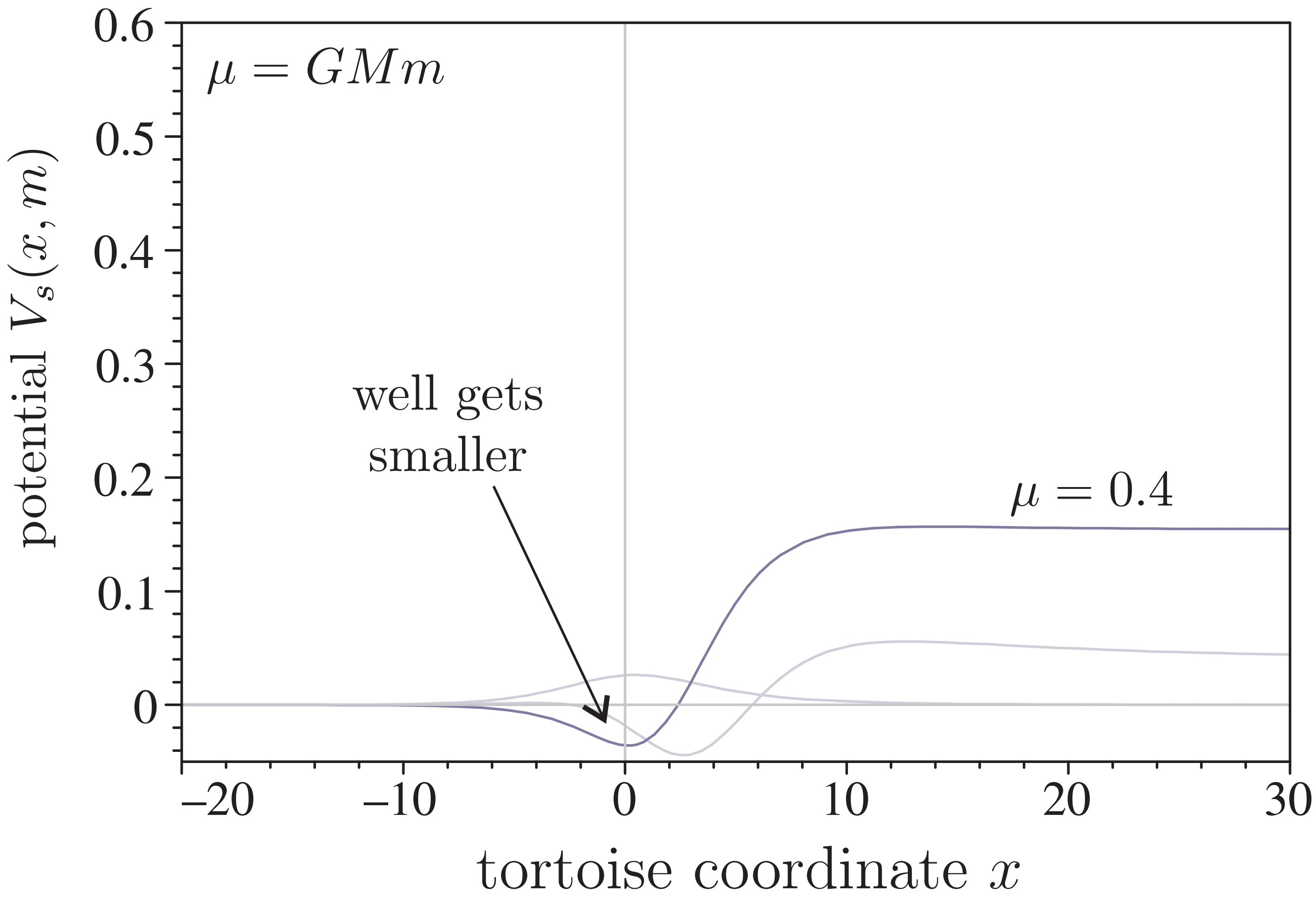

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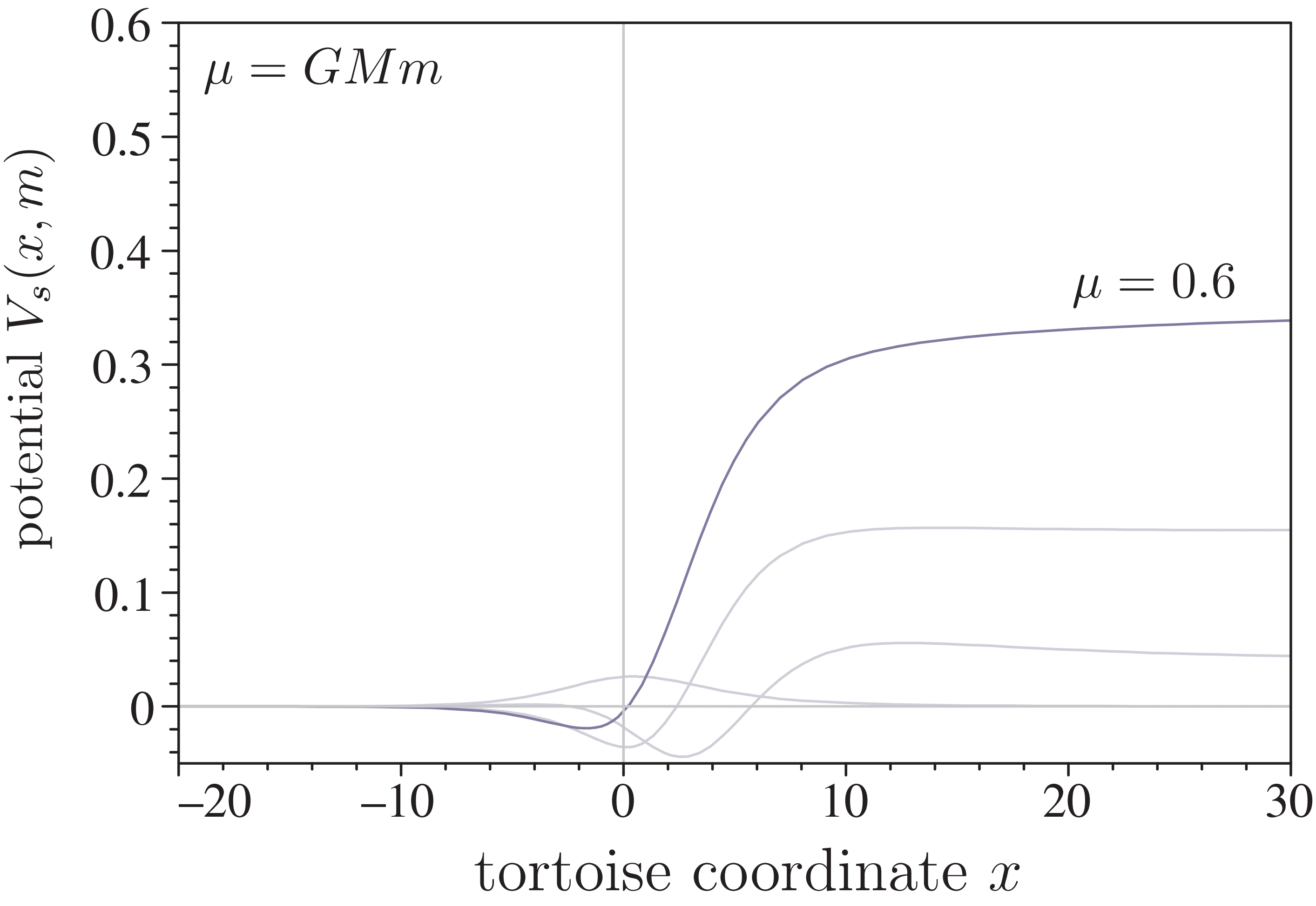

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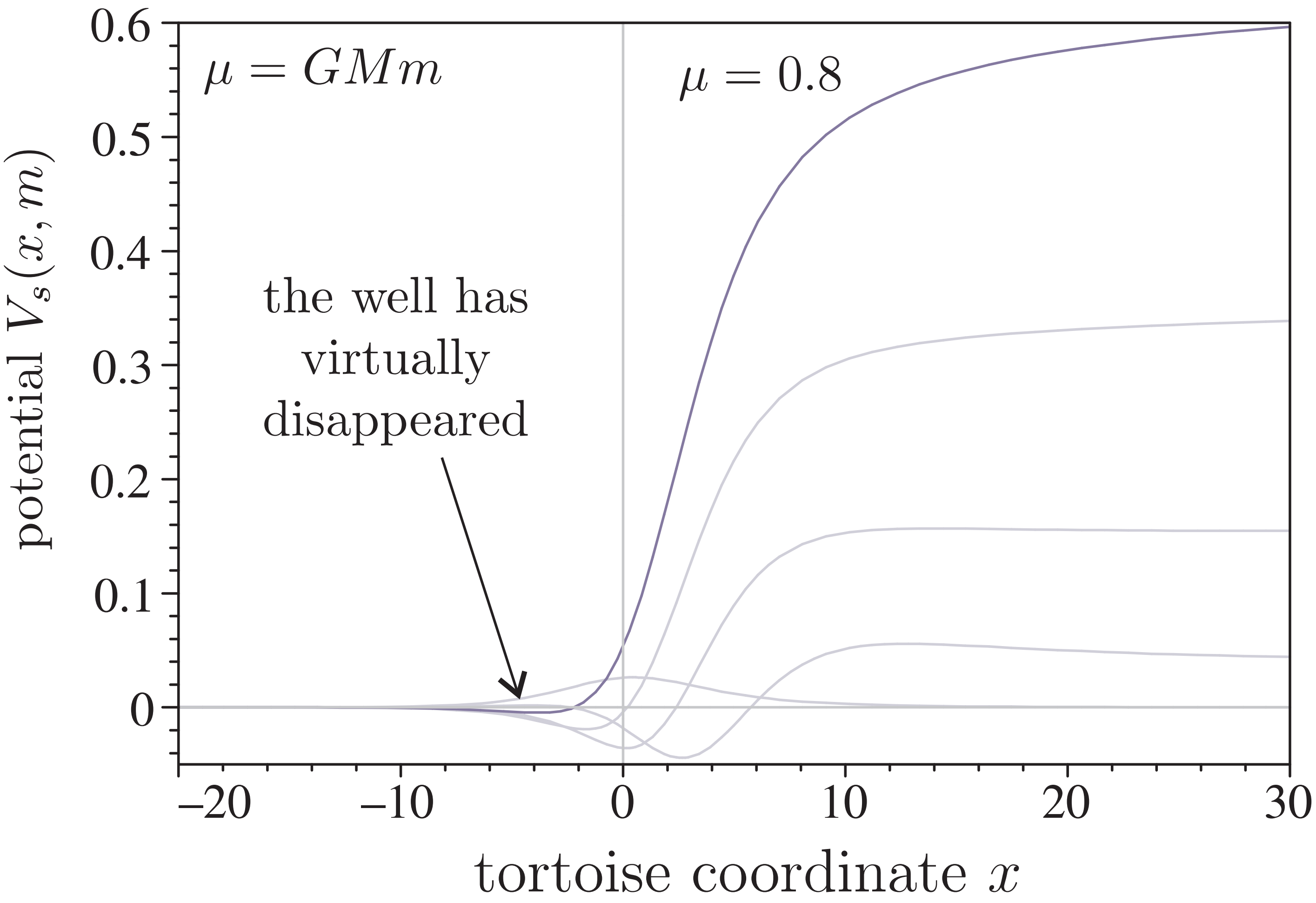

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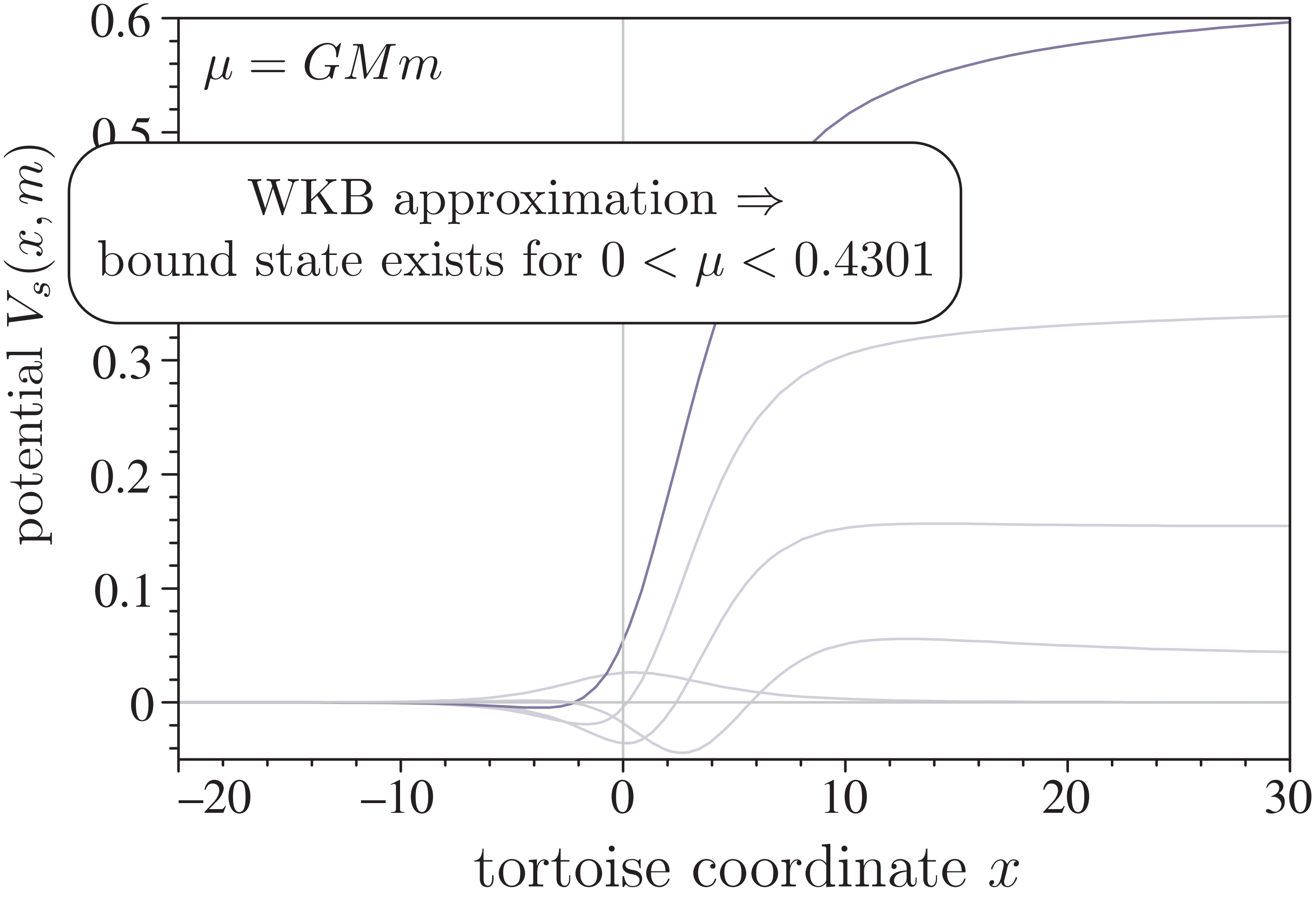
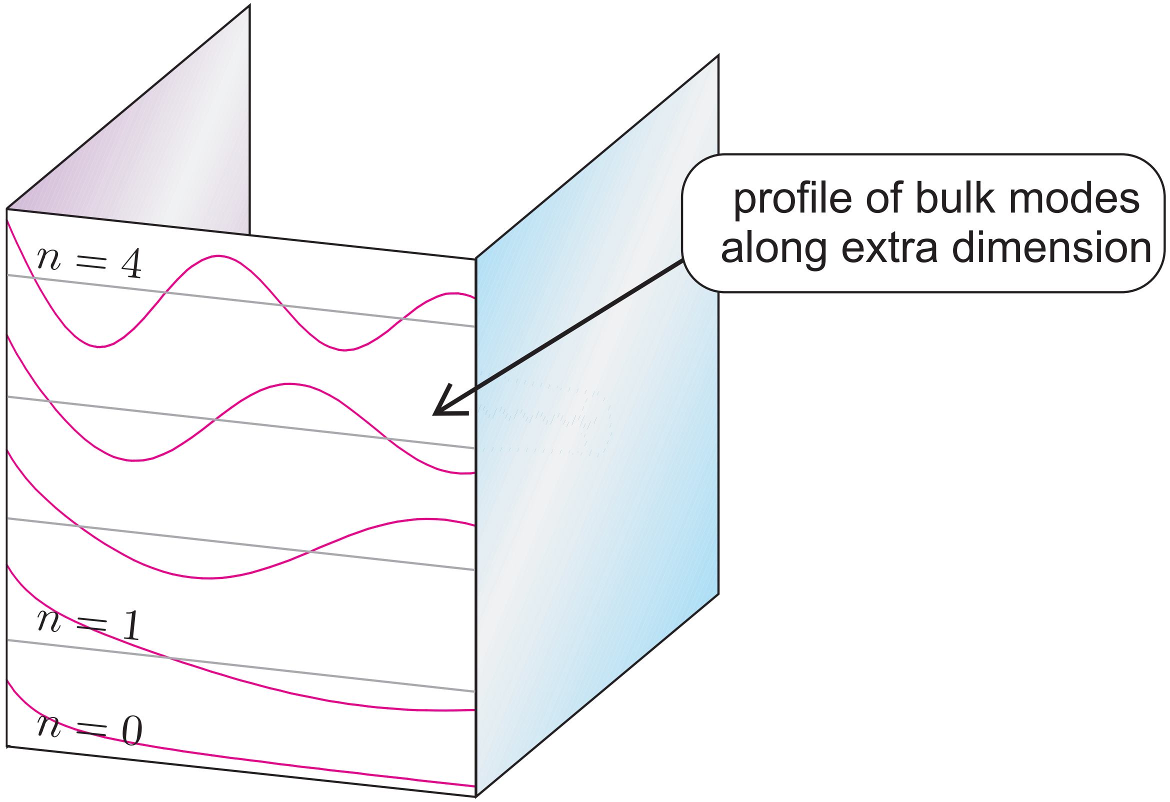


WKB approximation $\Rightarrow$
bound state exists for $0 < \mu < 0.4301$

Potential in the s-wave equation

instability: $0 < GMm_n < 0.4301$ for $n = 1, 3/2, 2, 5/2, \ldots$



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wavelength in the extra dimension $\sim m_n^{-1}$
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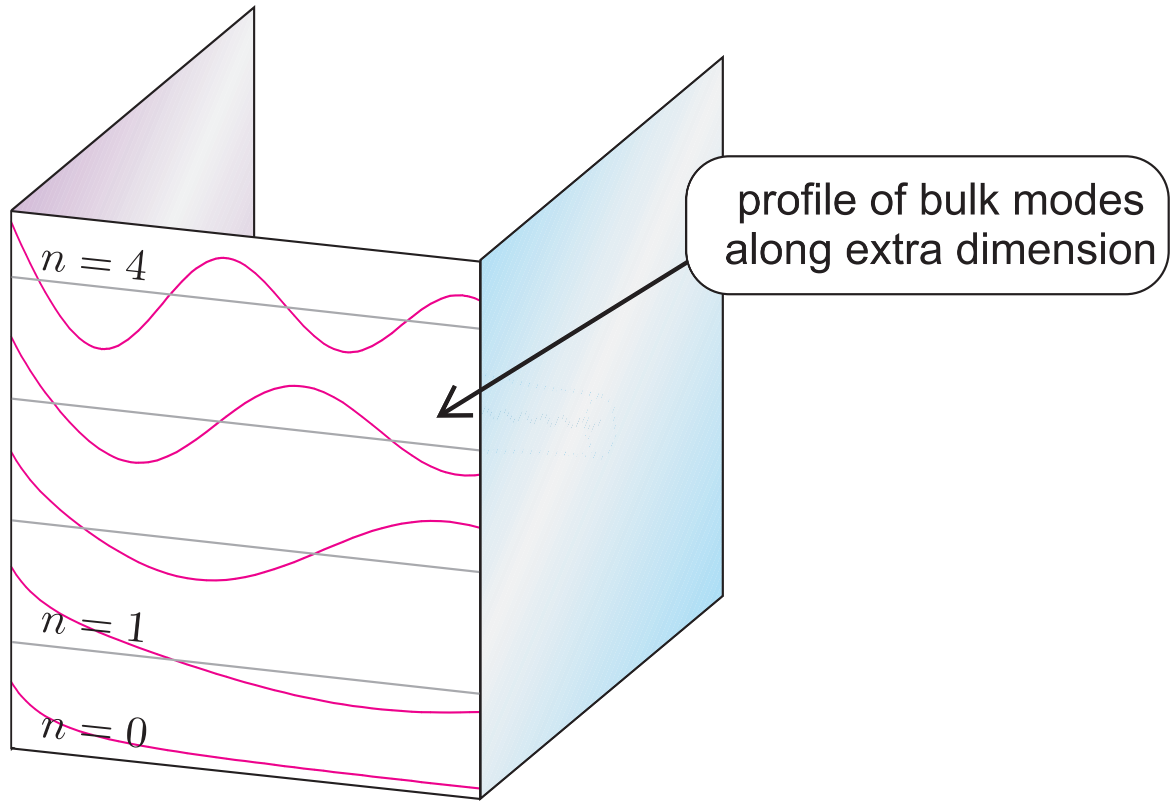

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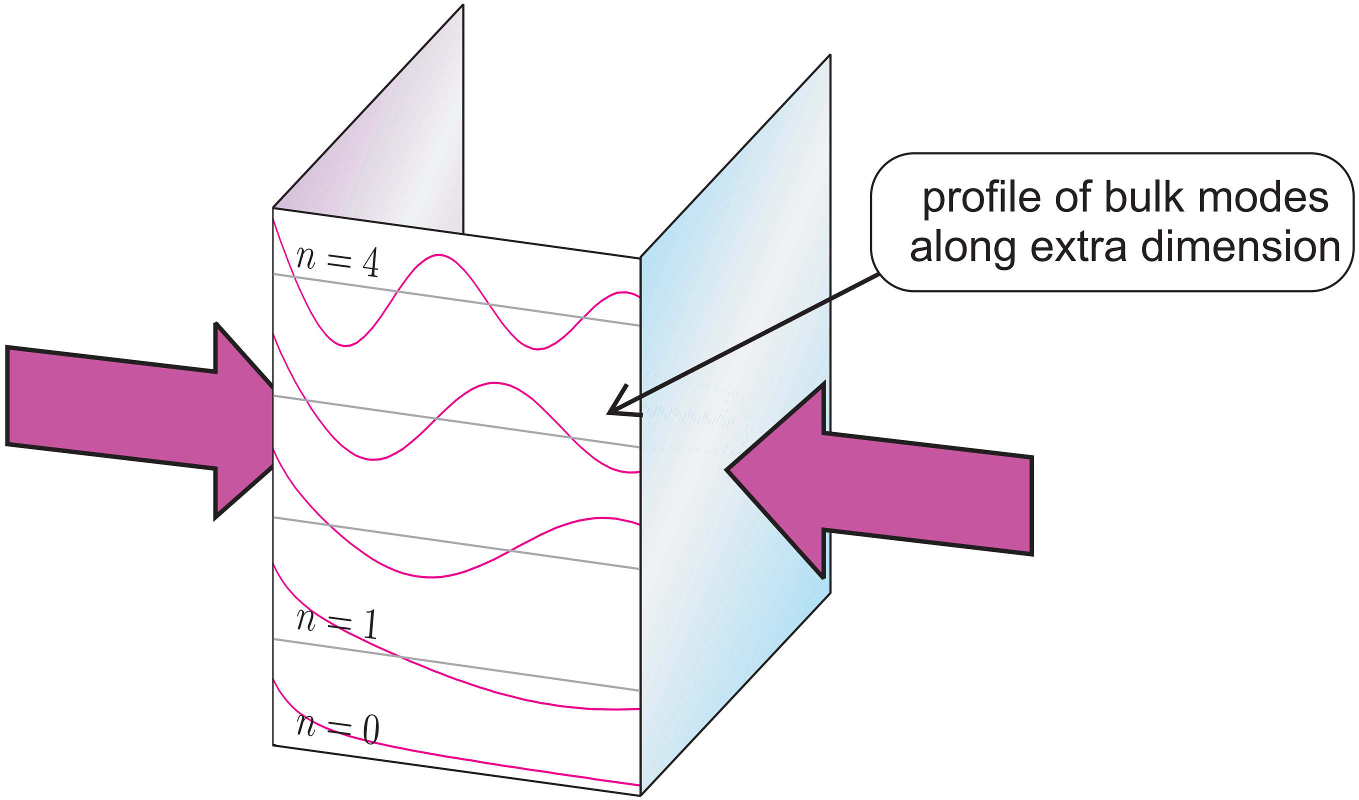

$\Rightarrow$ avoid instability by $\uparrow M$ or "squeezing" branes

Braneworld stability criteria

- first KK mass $m_1 = \mu_1/GM$ was smallest solution of

$$Y_1(m_n\ell)J_1(m_n\ell e^{kd}) = J_1(m_n\ell)Y_1(m_n\ell e^{kd})$$

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- useful stability criterion when $e^{kd} \gg 1$:

$$\frac{M}{M_\odot} \gtrsim 1.1 \times 10^{-6} \left(\frac{\ell}{0.1\,\text{mm}} \right) e^{(d-5\ell)/\ell}$$

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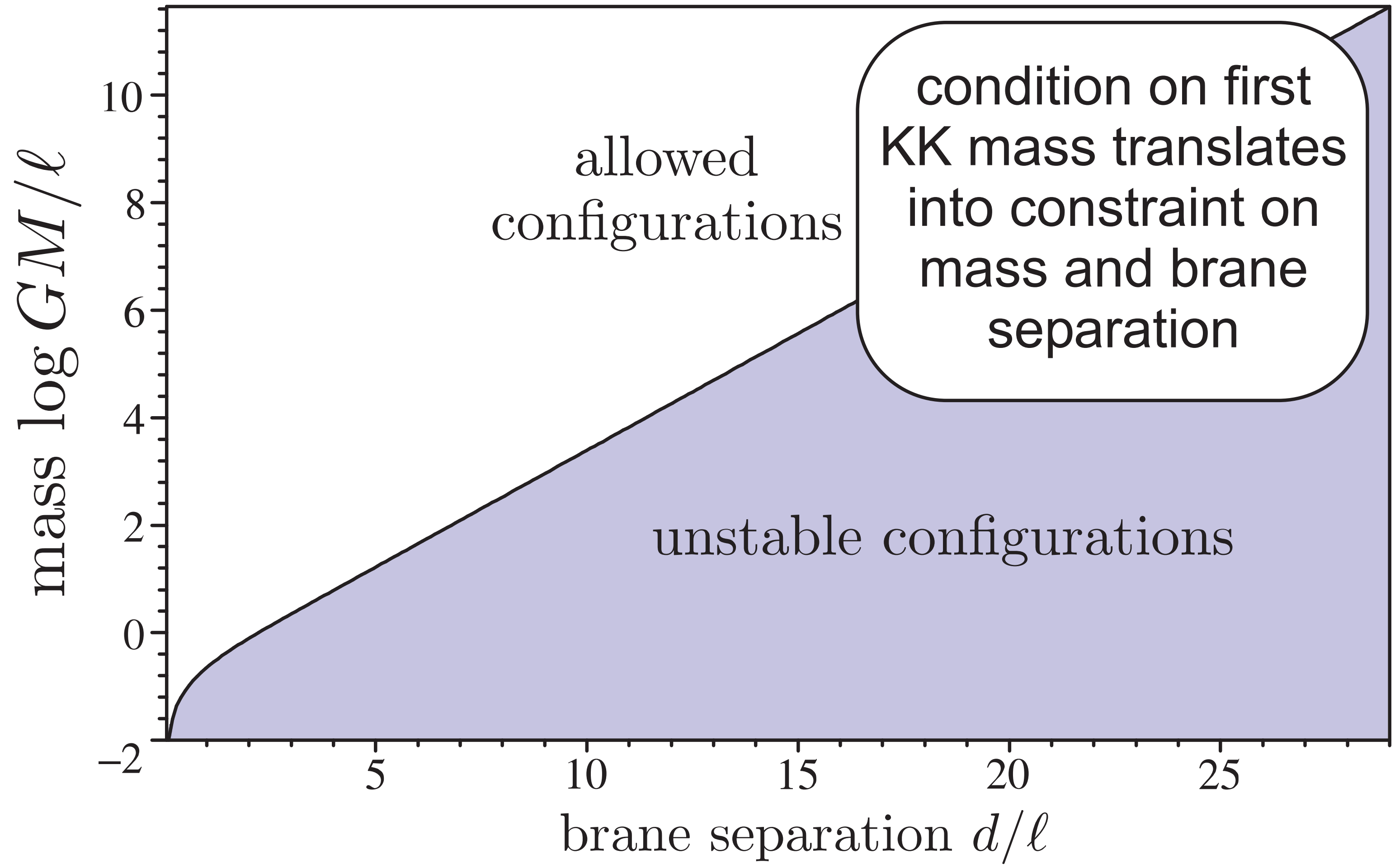

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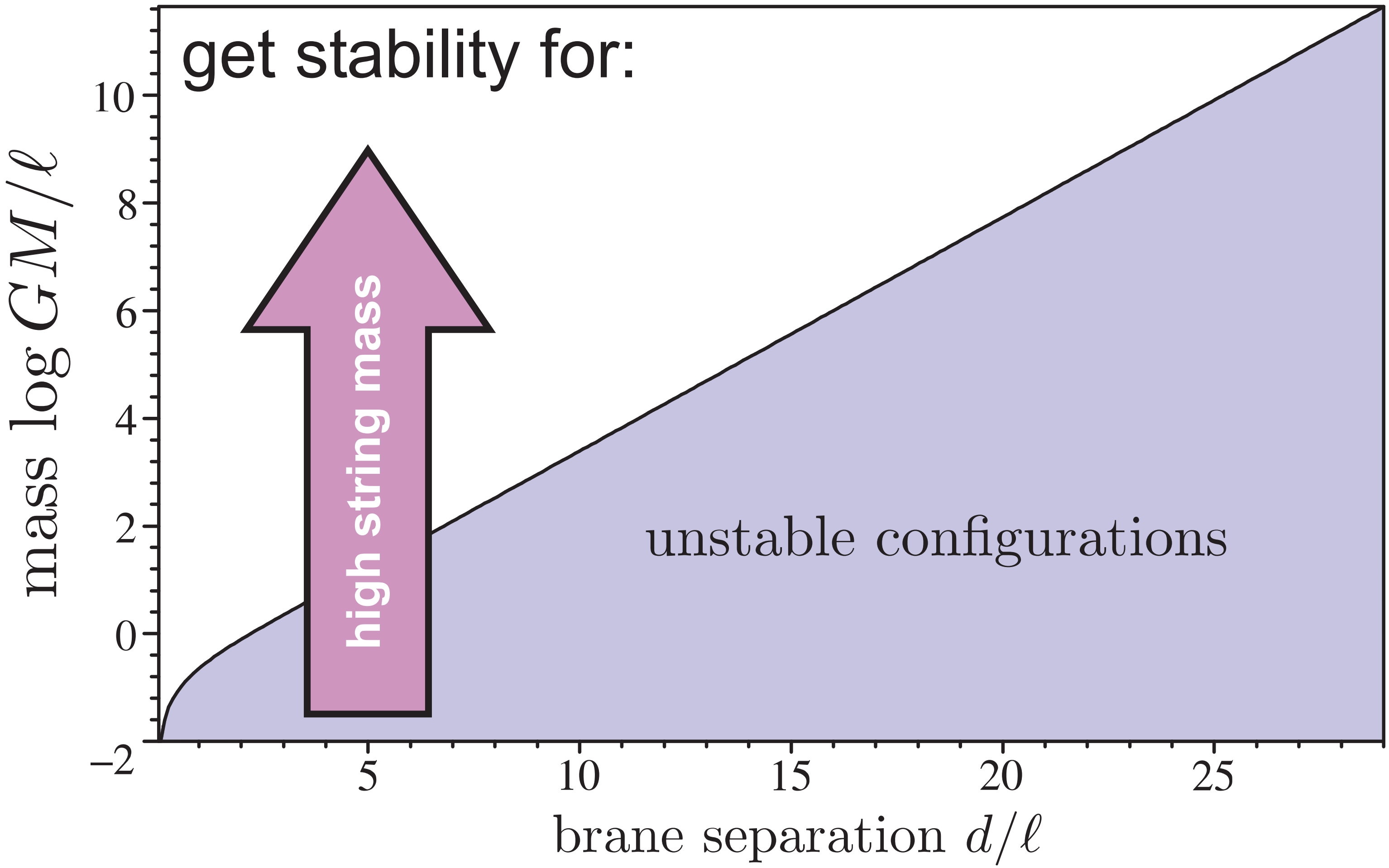

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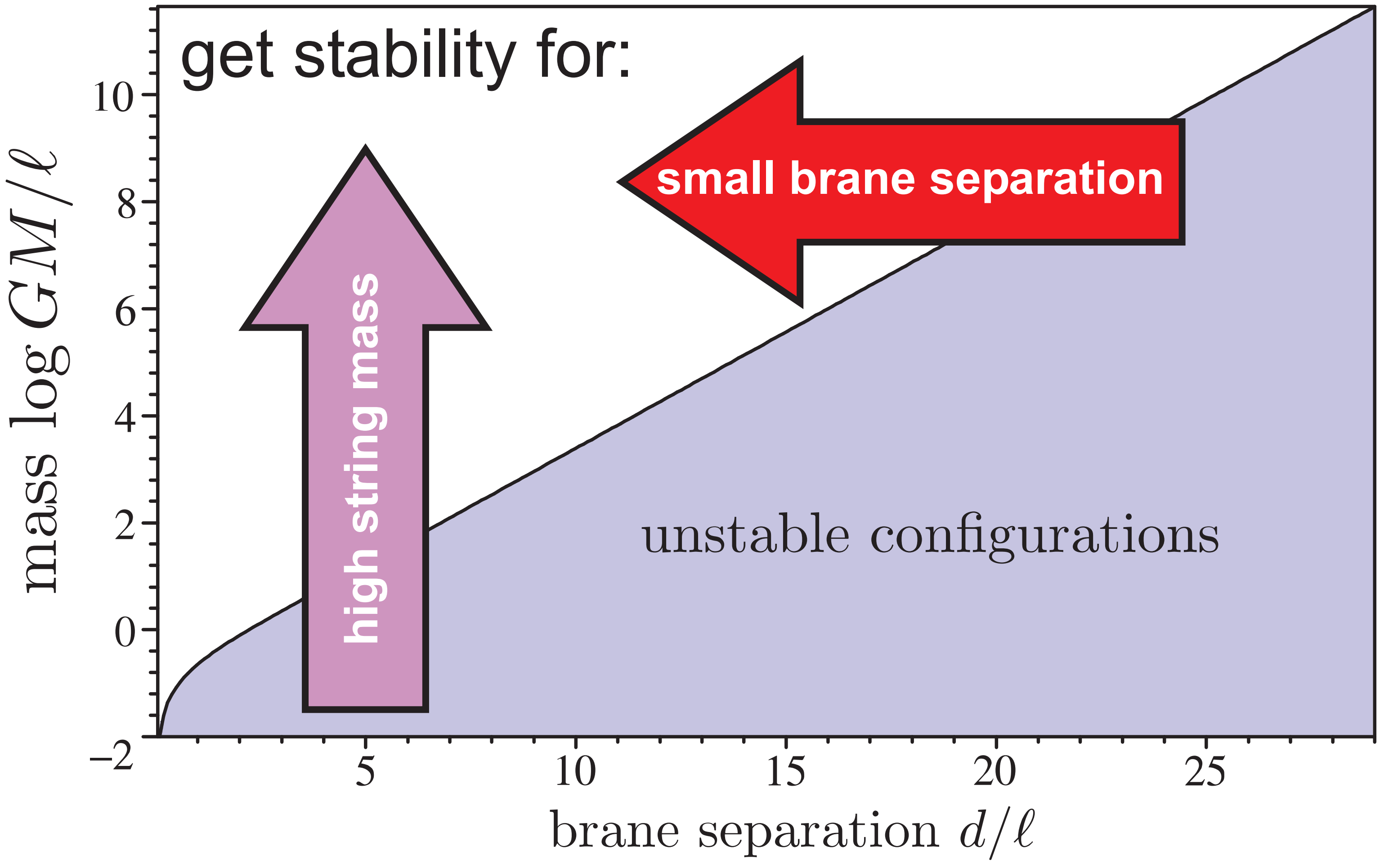

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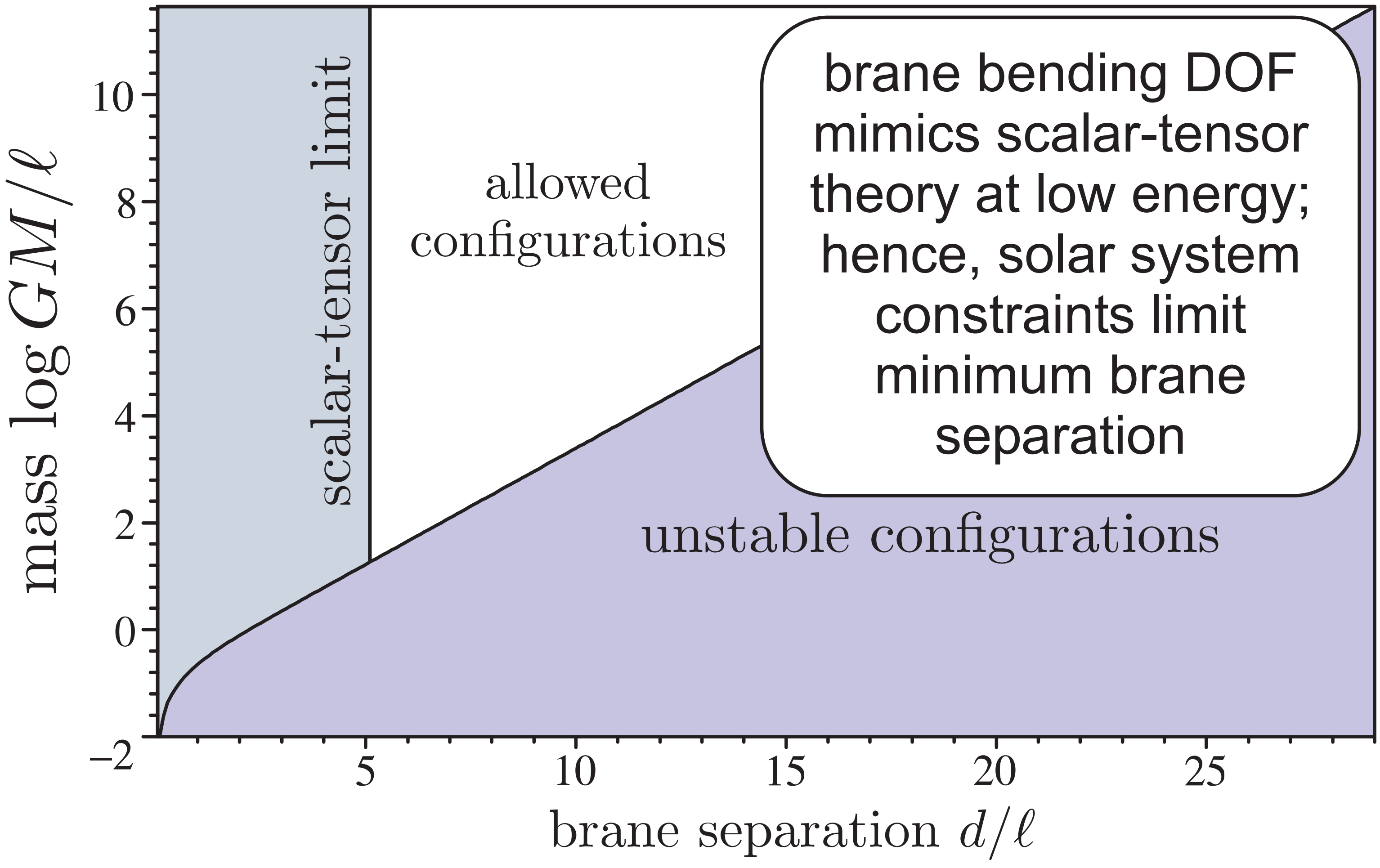

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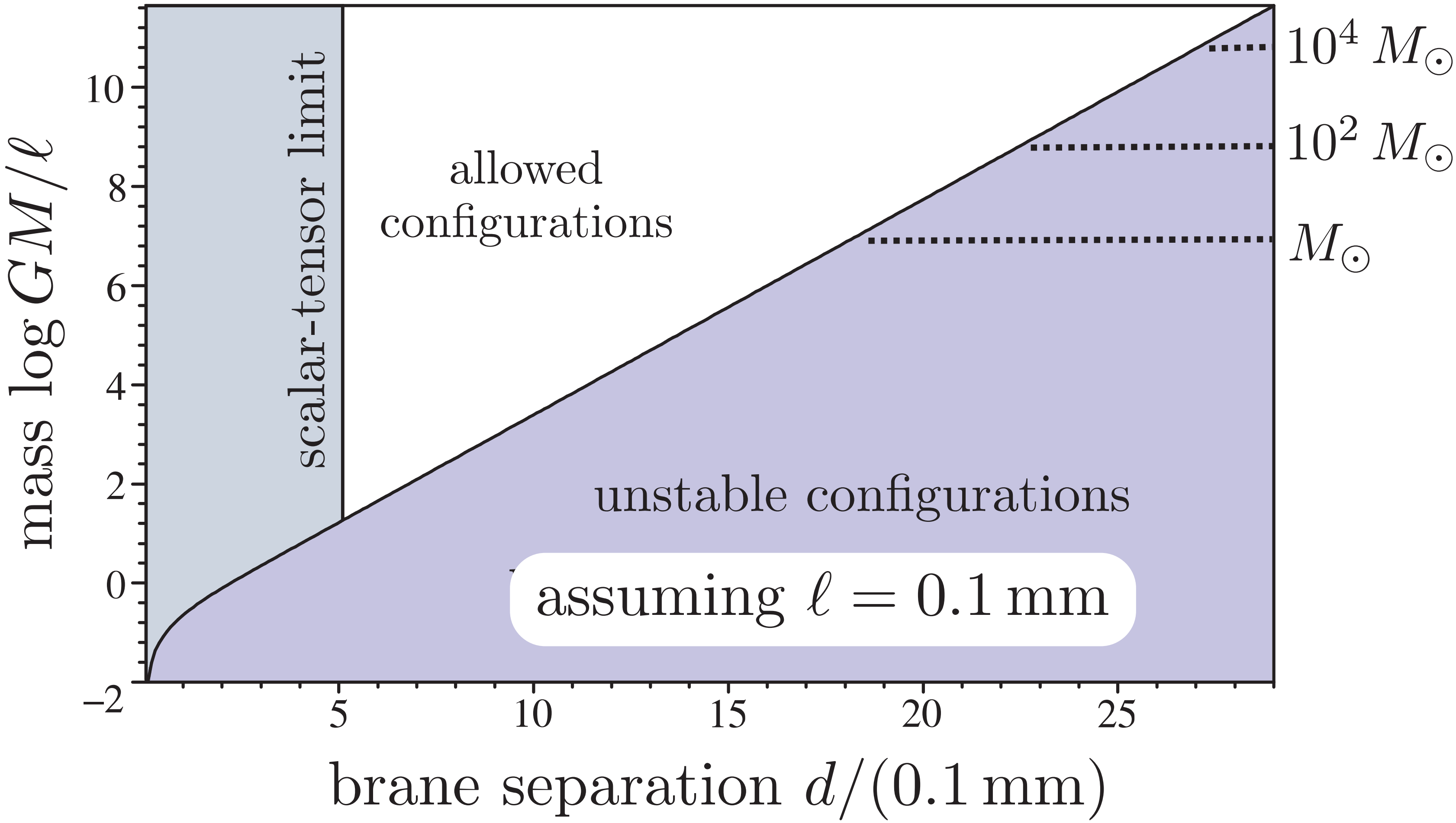

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Source-free GWs

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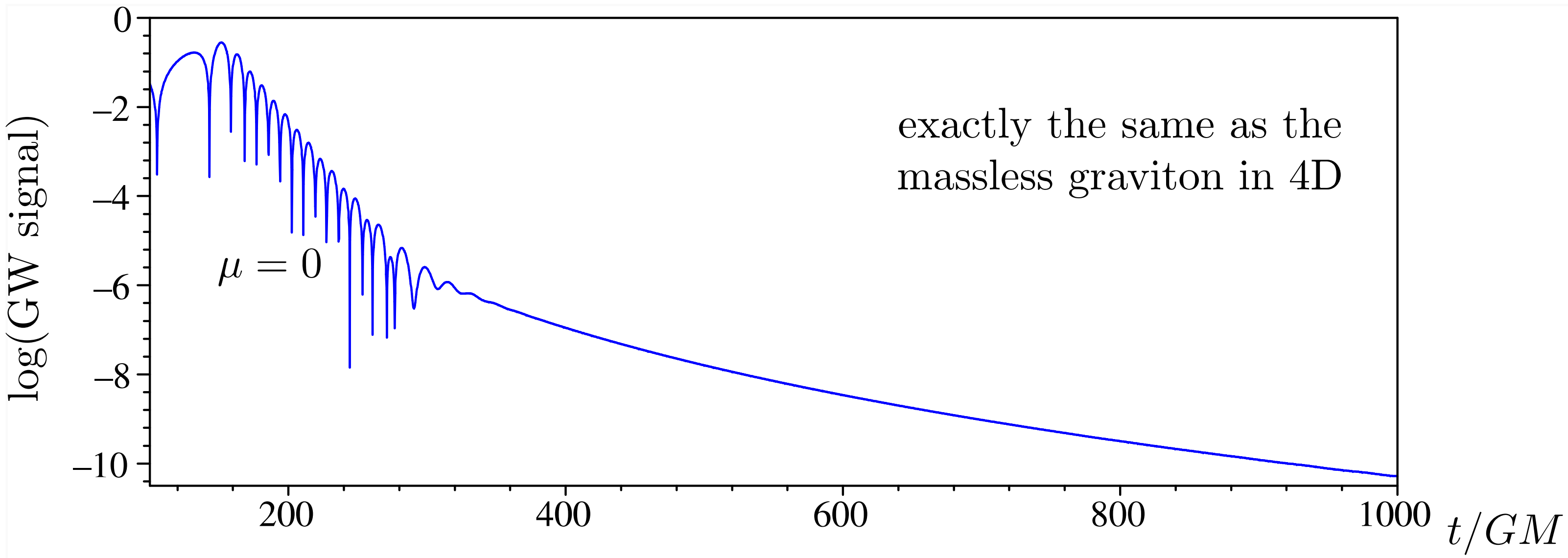

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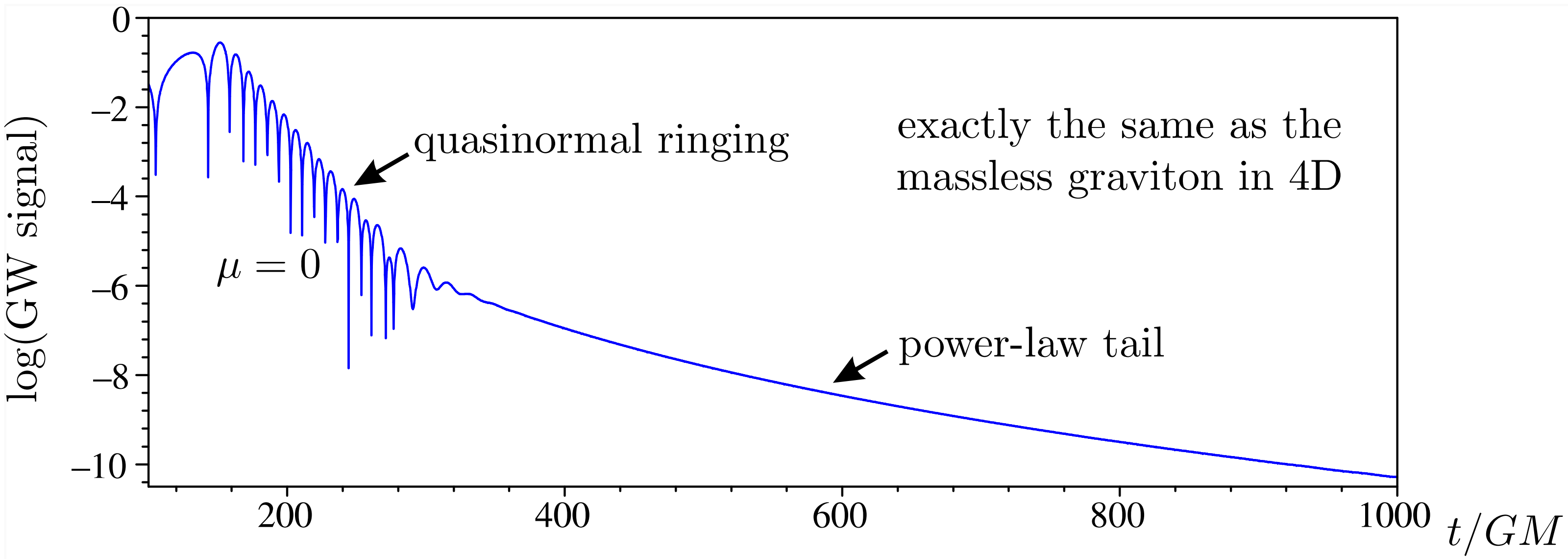

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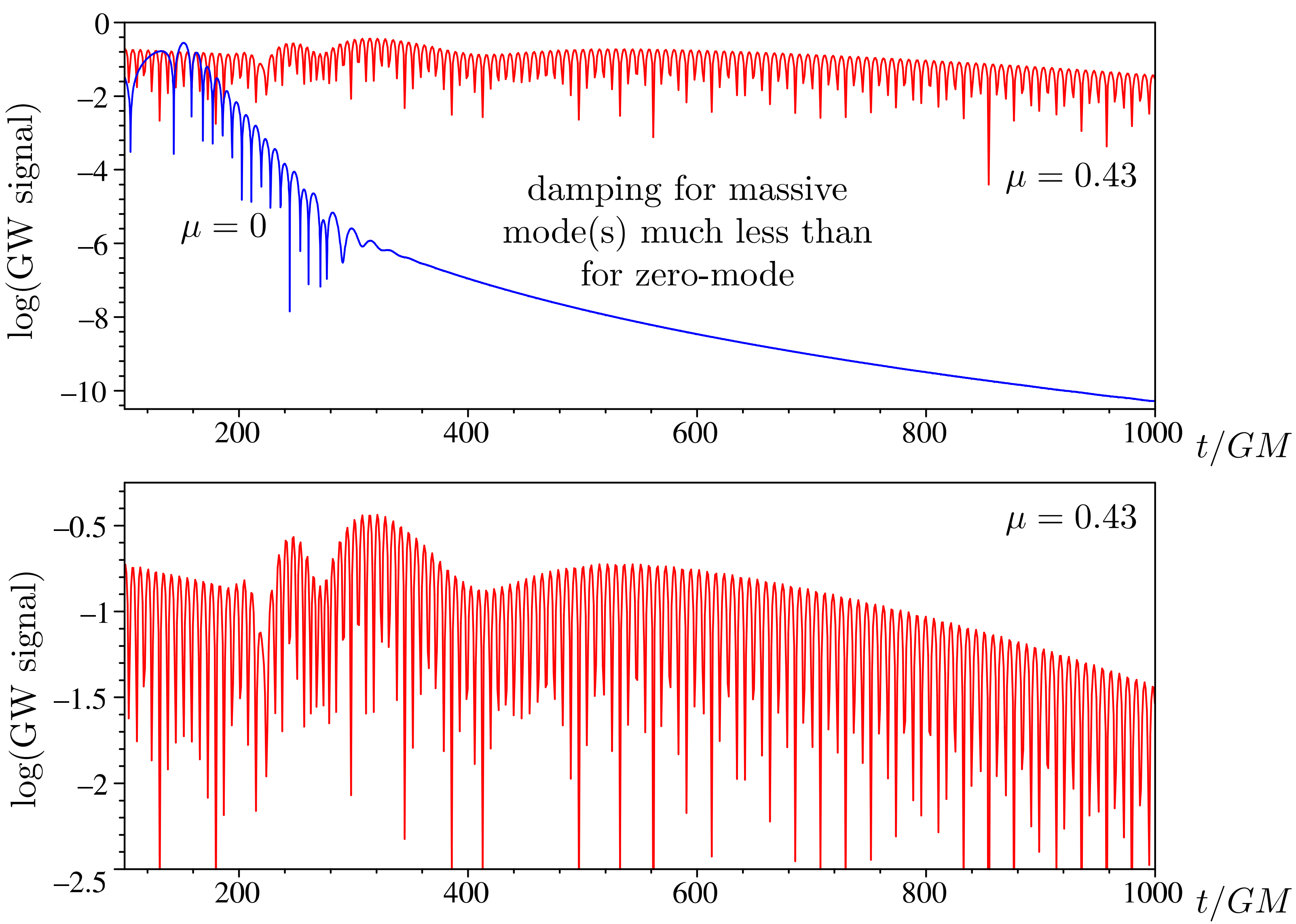

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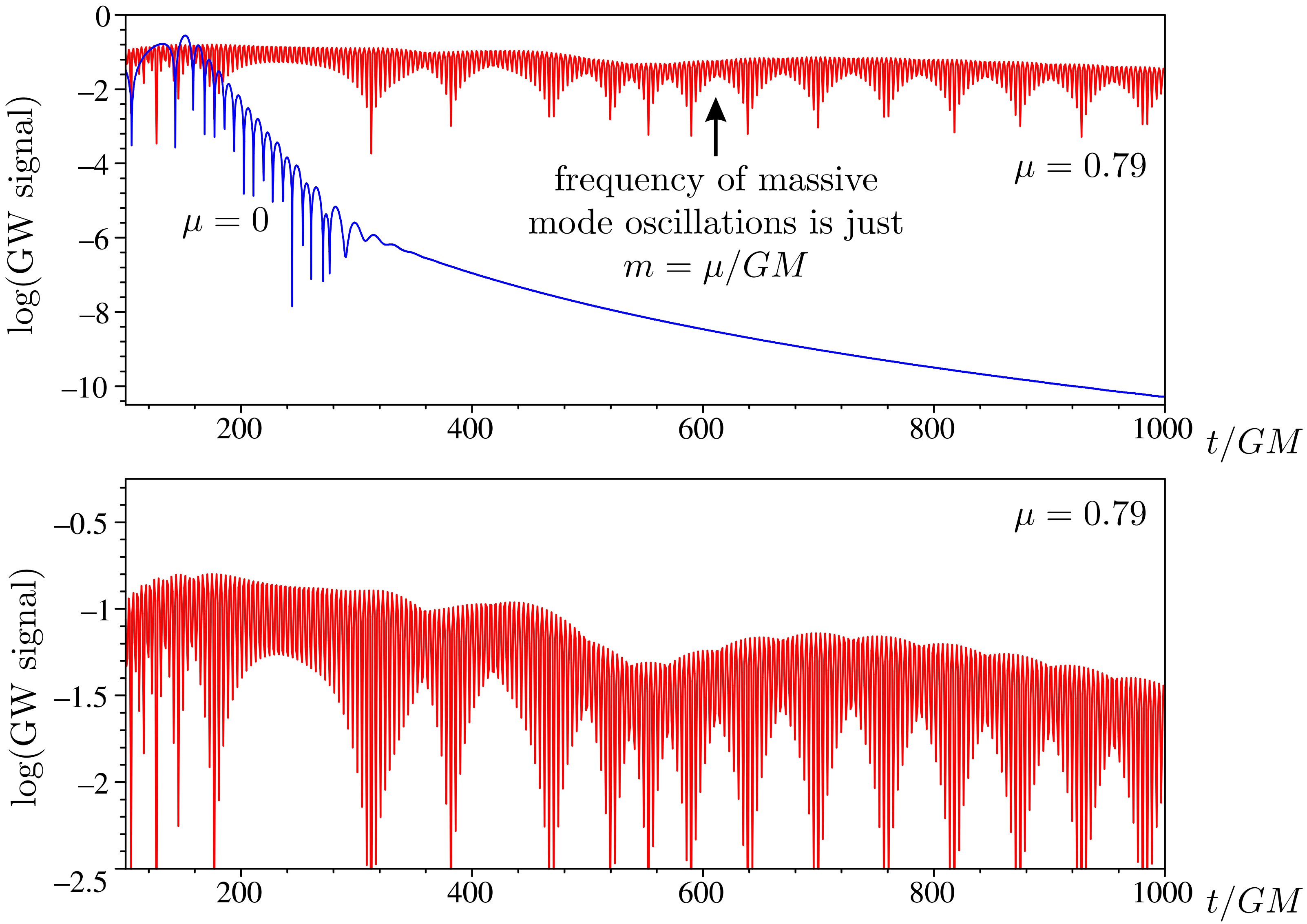

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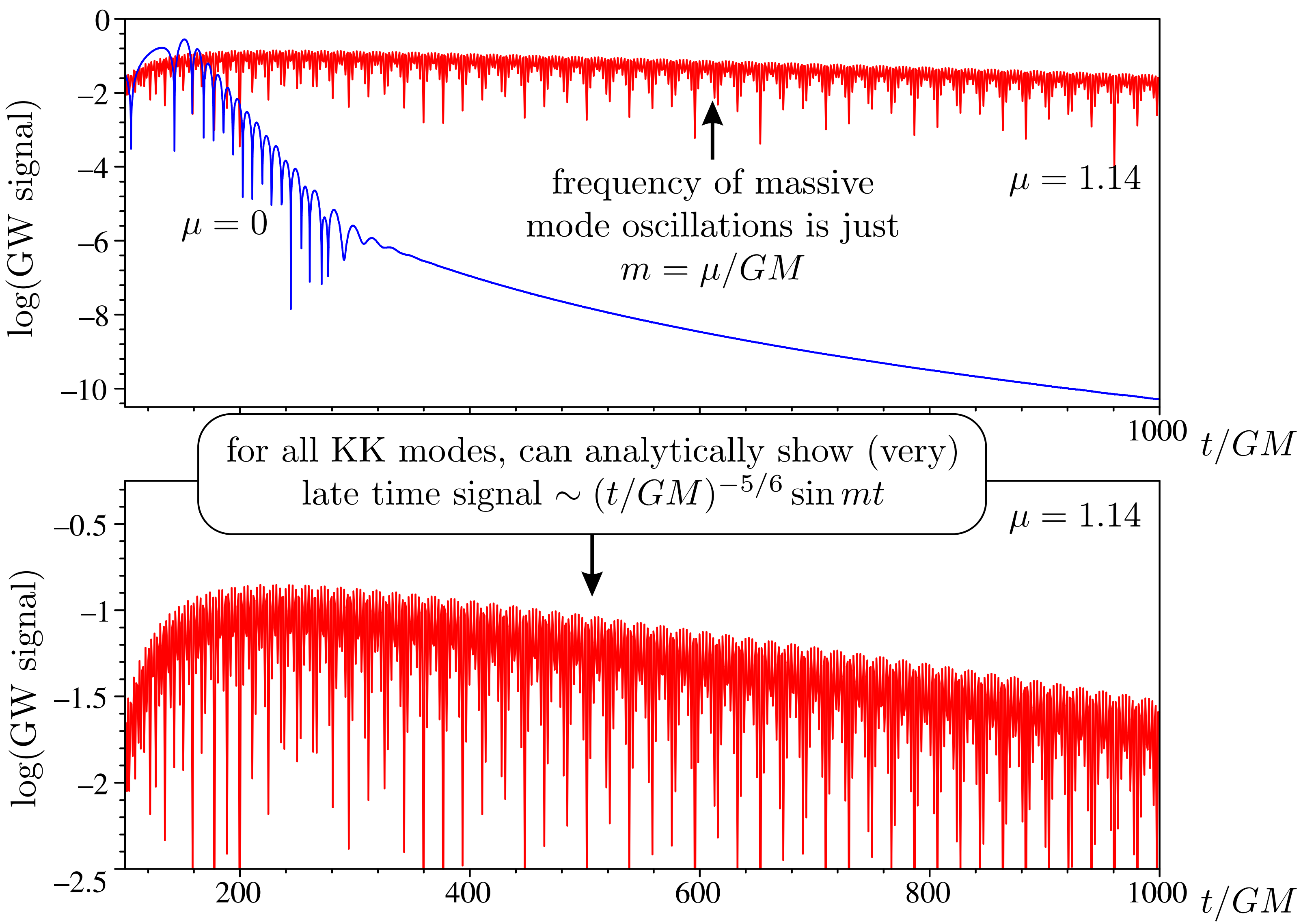

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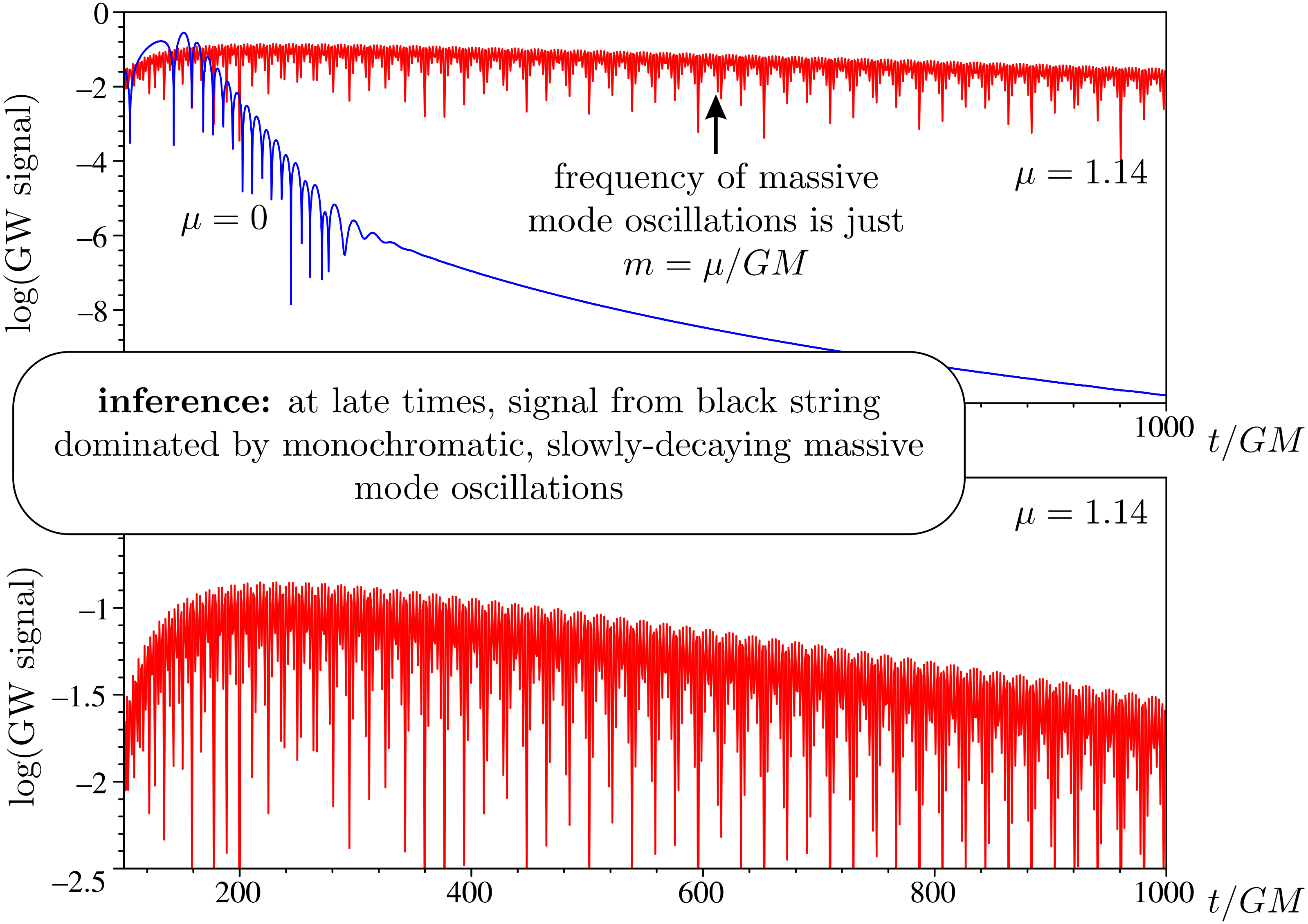

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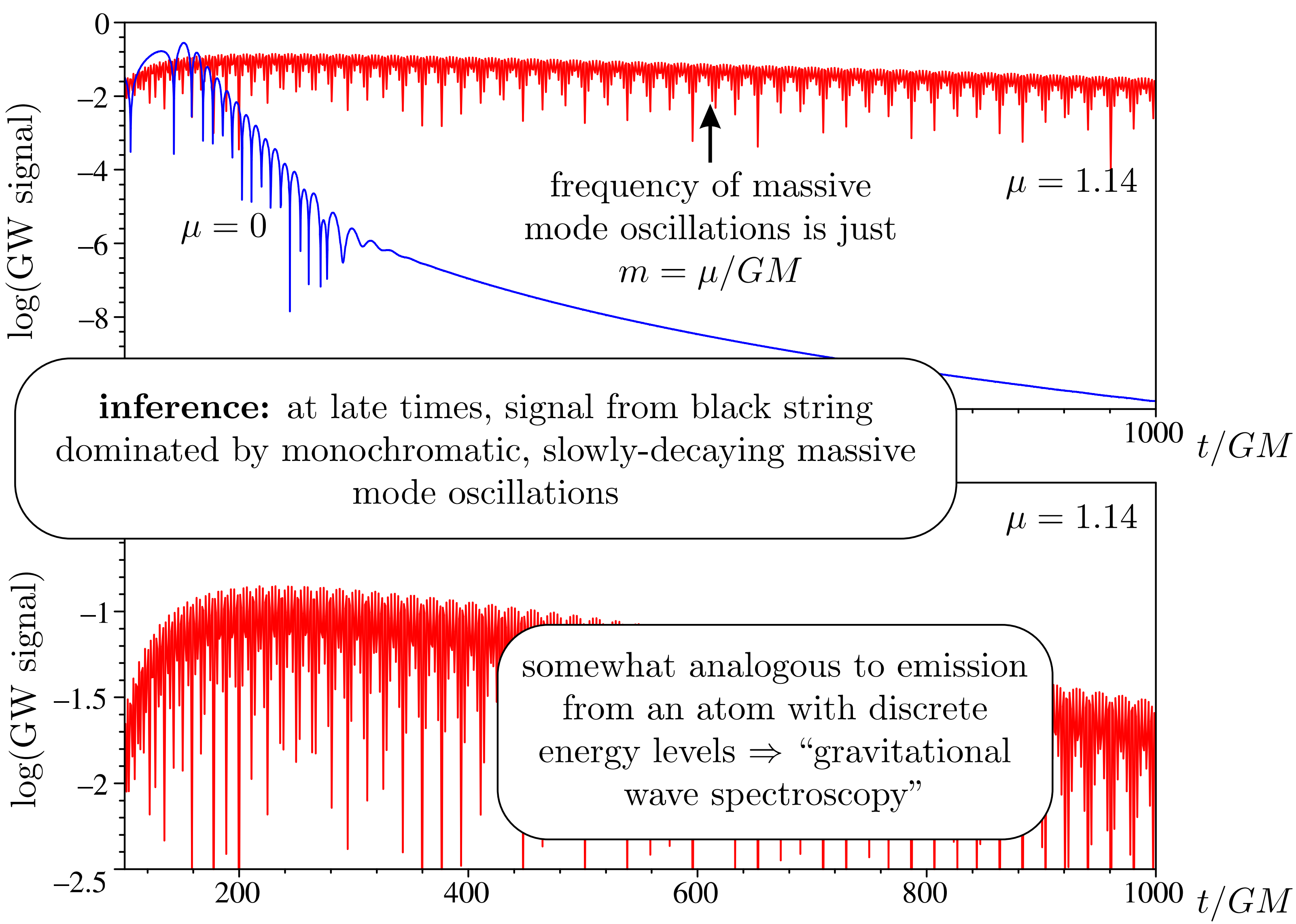

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somewhat analogous to emission from an atom with discrete energy levels ⇒ "gravitational wave spectroscopy"

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Composite gravitational wave signal

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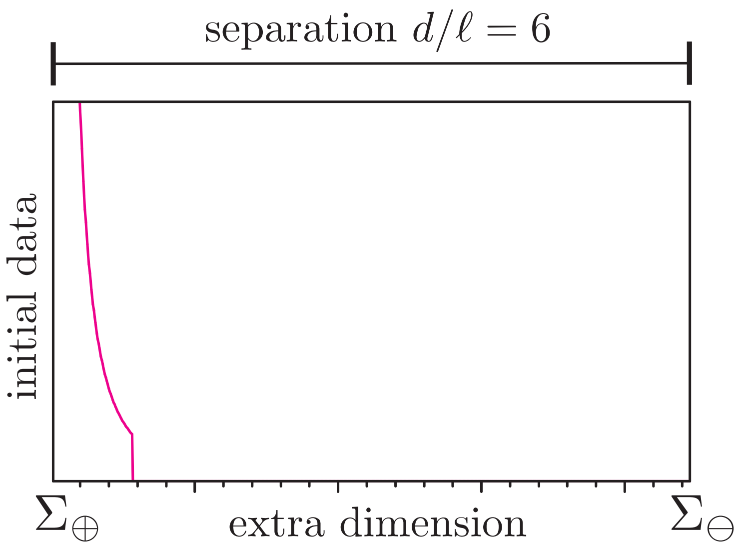

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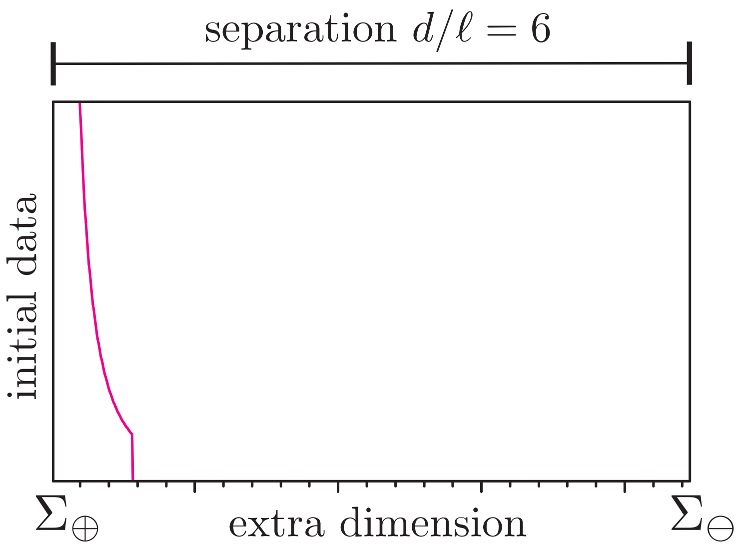

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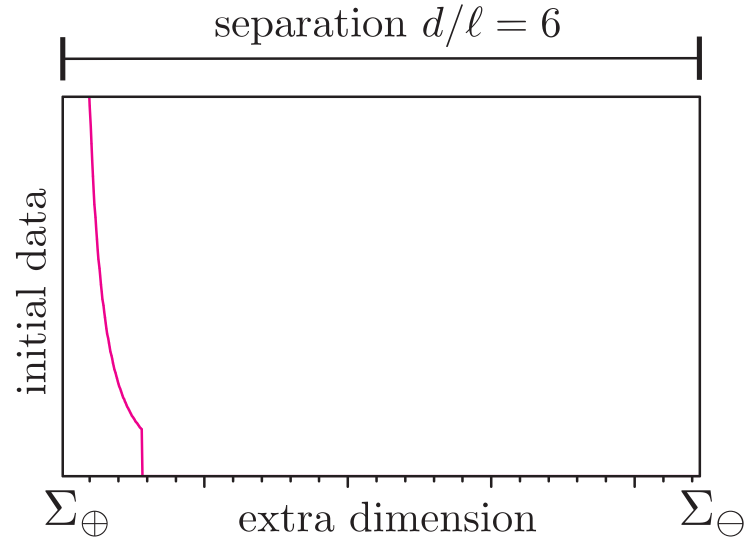

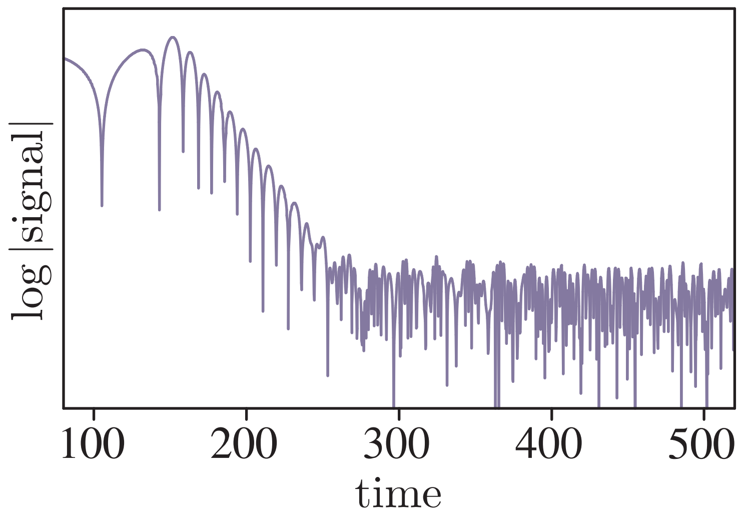

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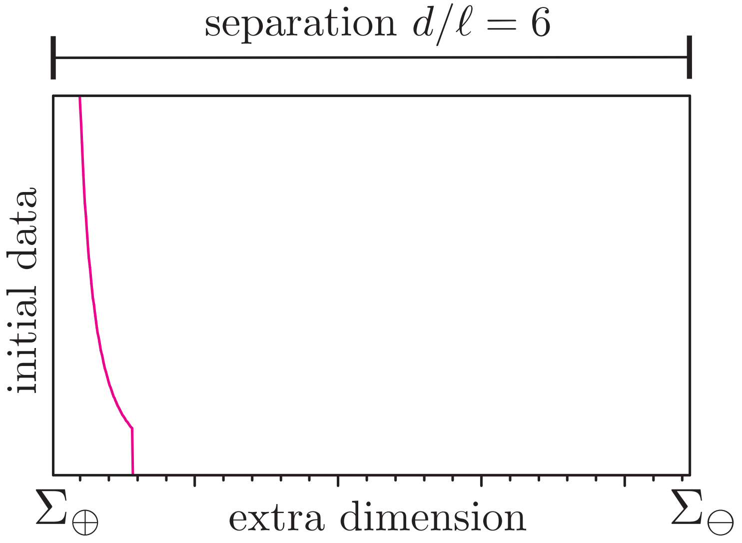

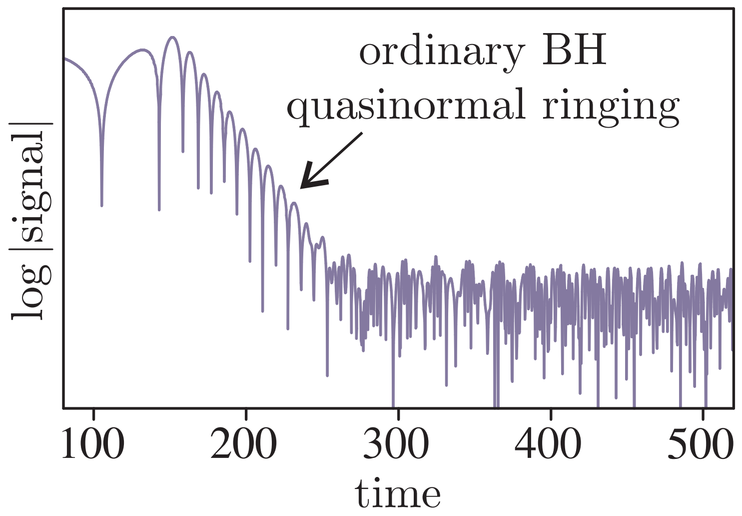

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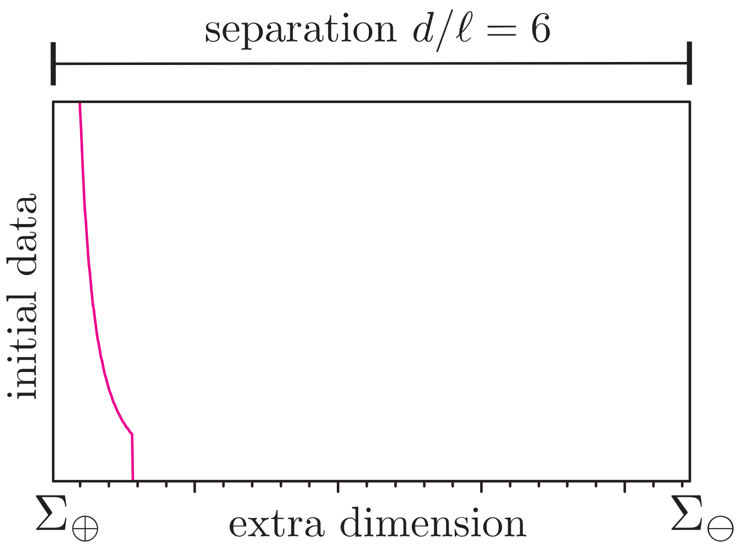

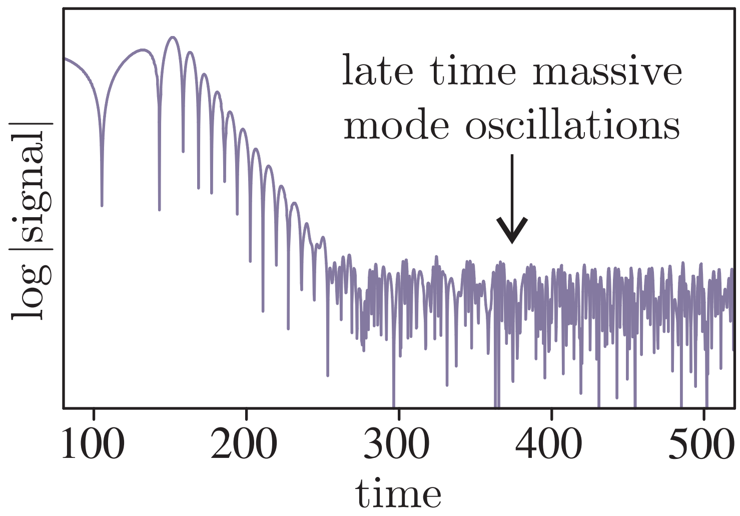

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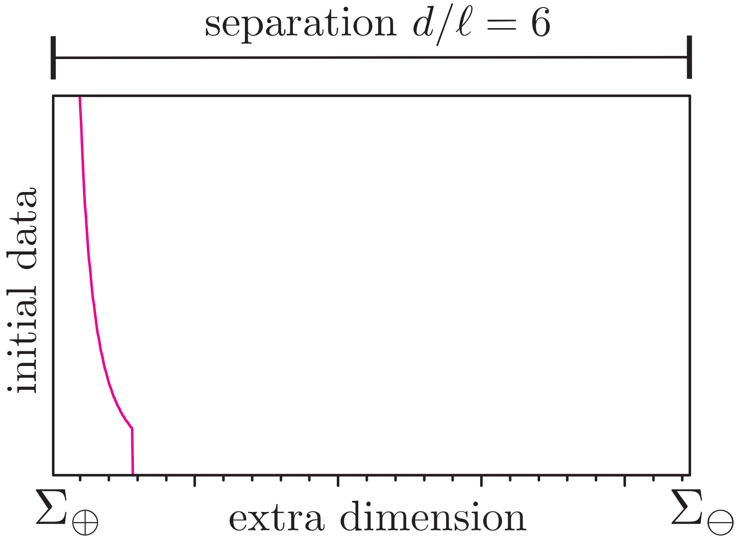

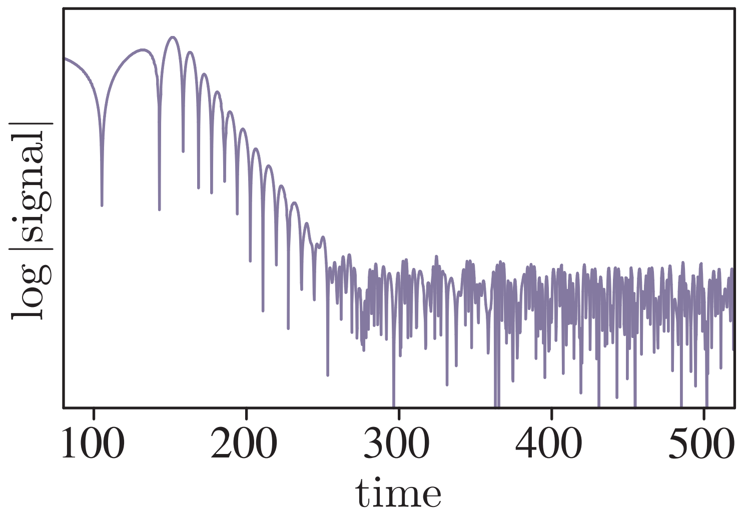

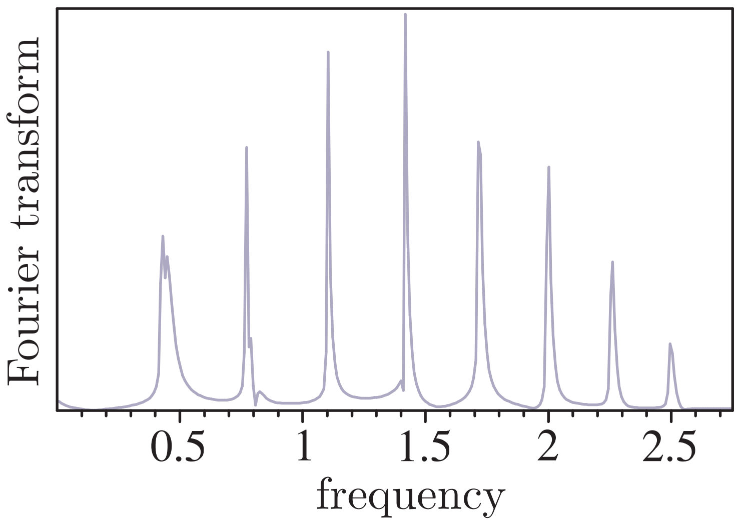

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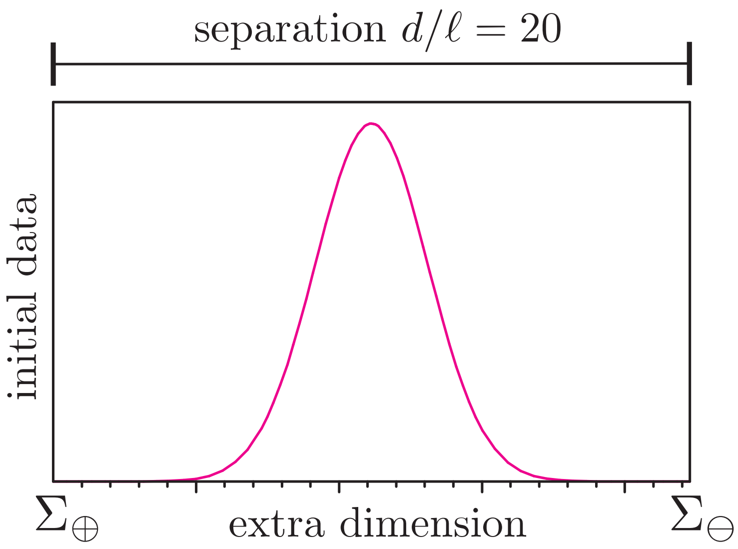

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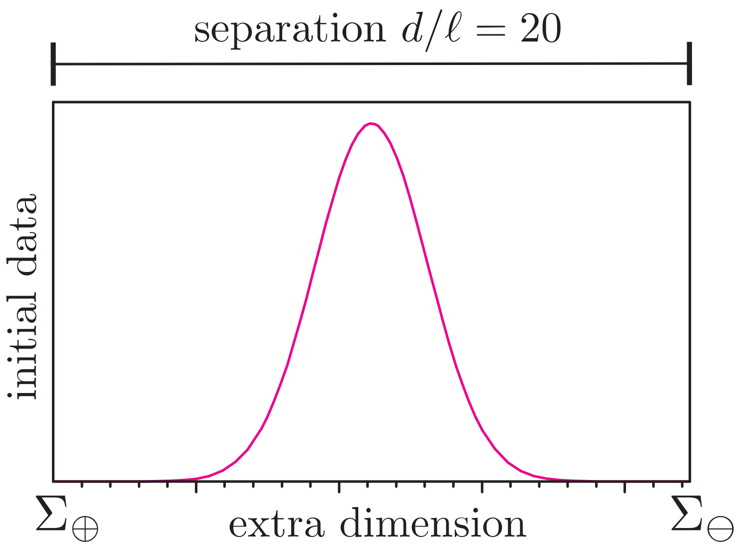

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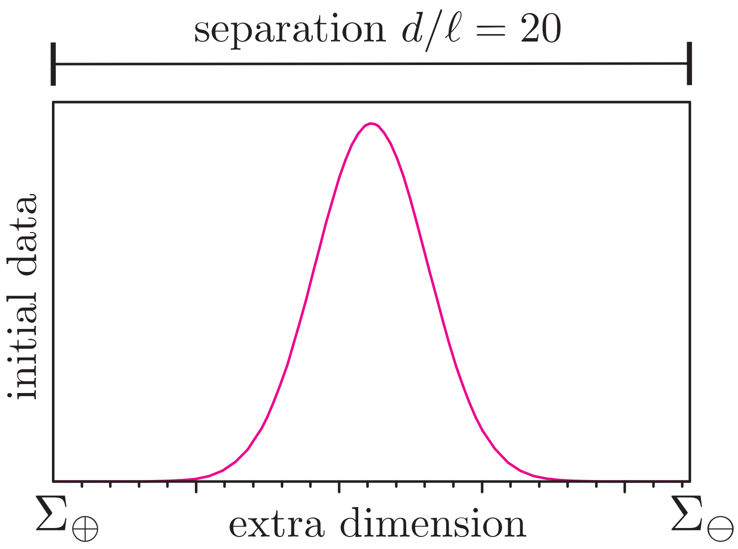

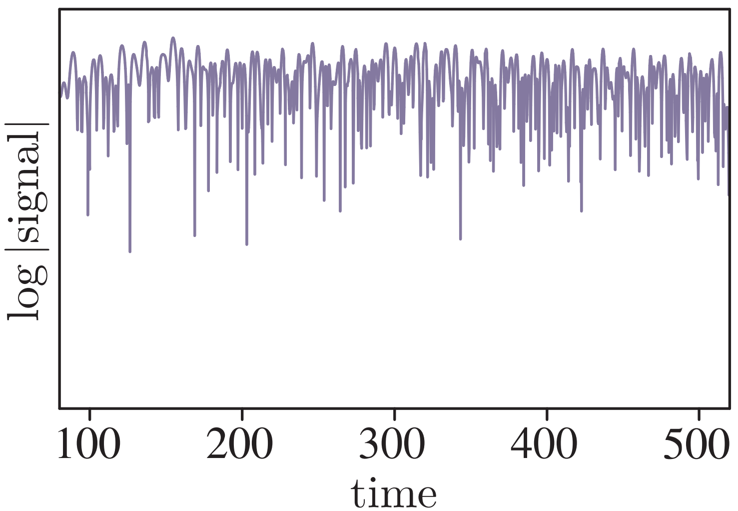

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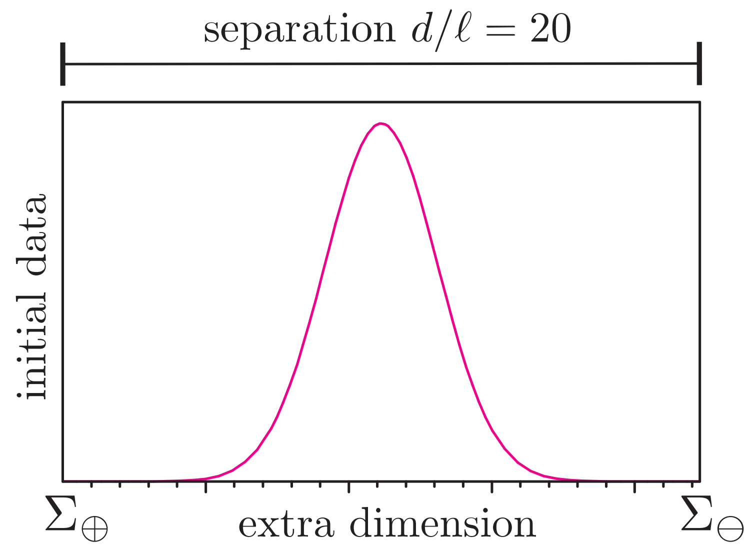

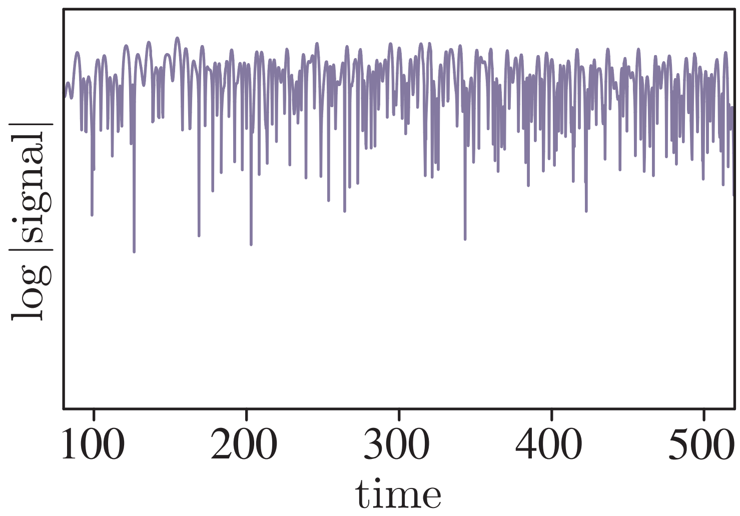

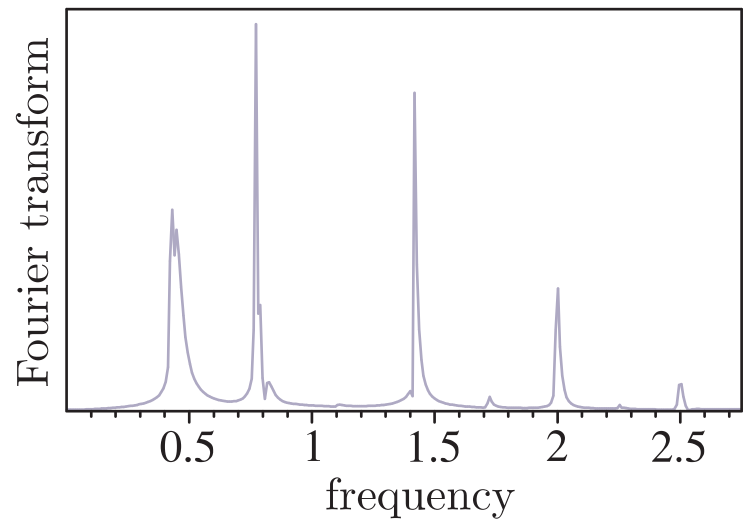

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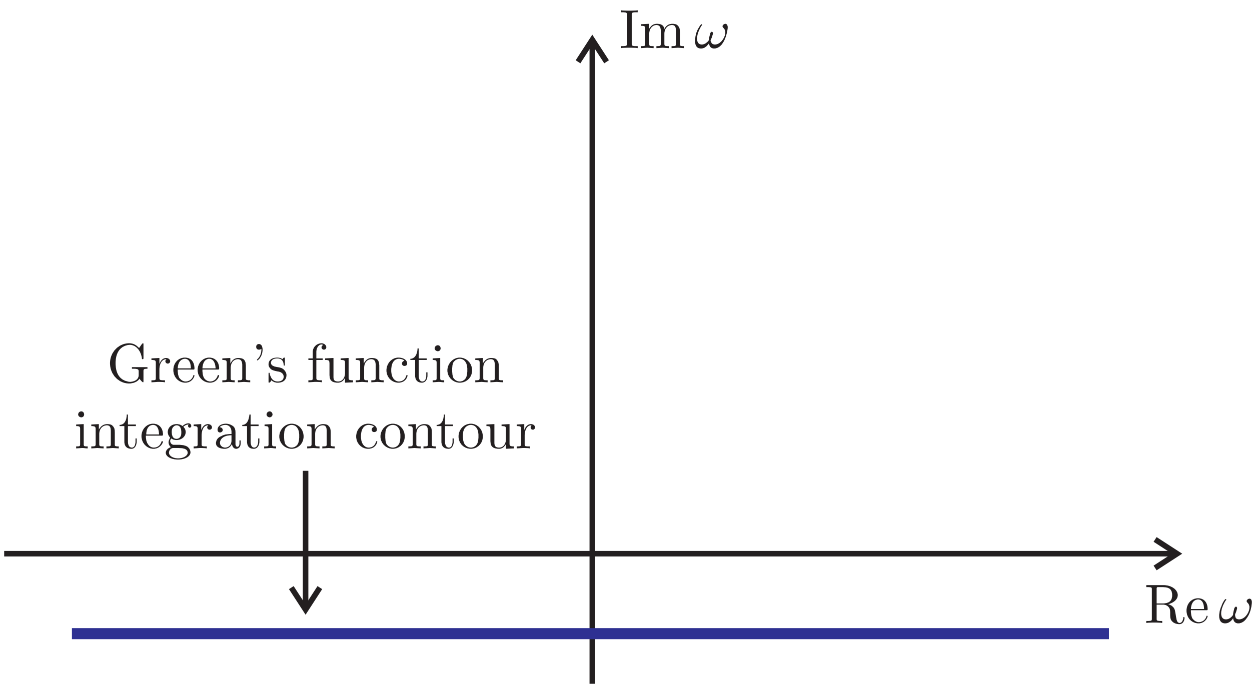

The complex frequency plane

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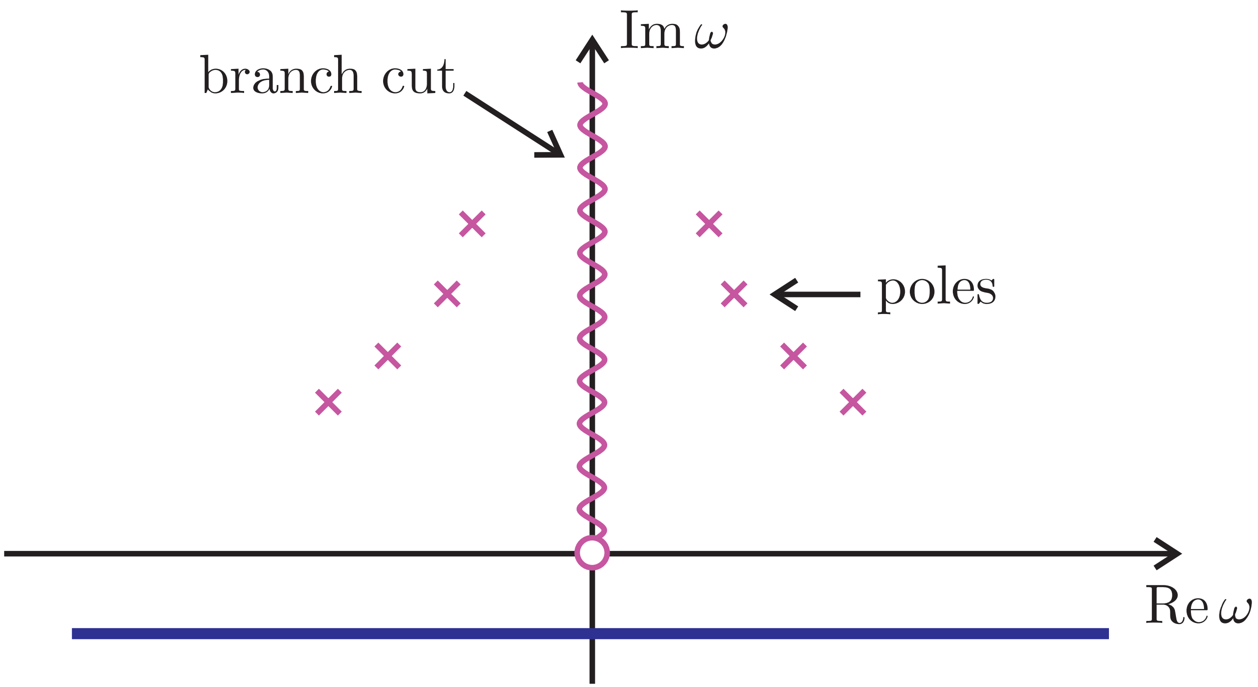

Non-trivial structure of the (massless) Green's function

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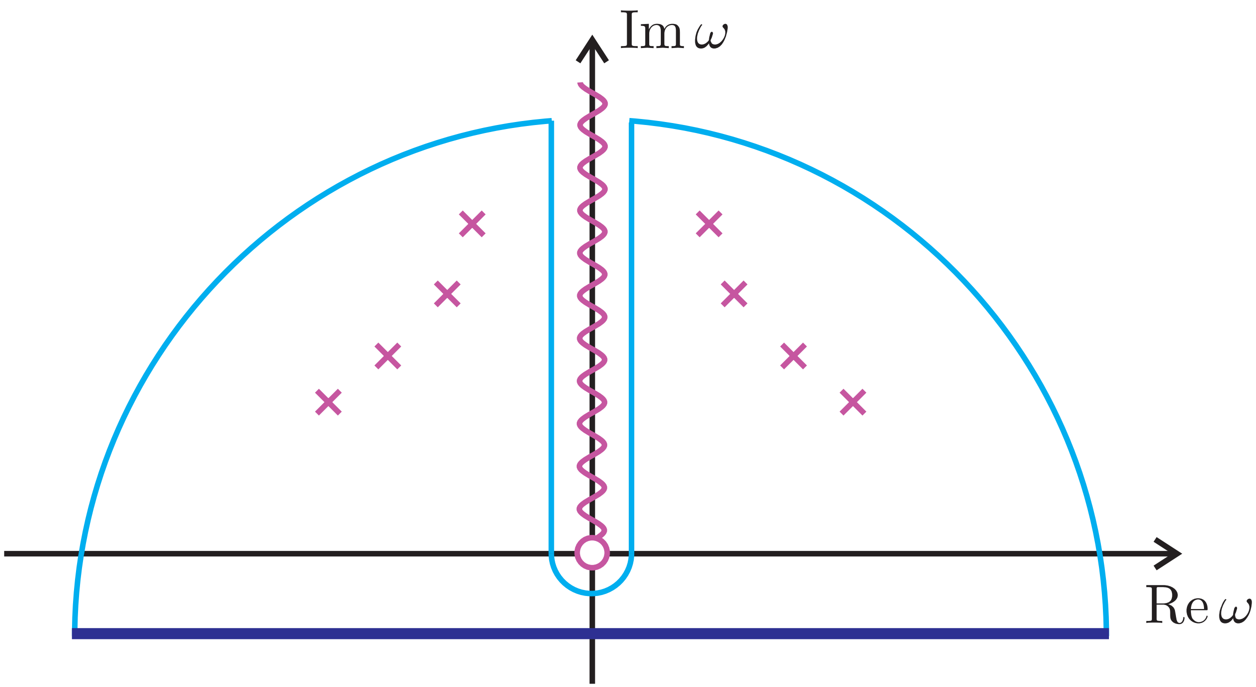
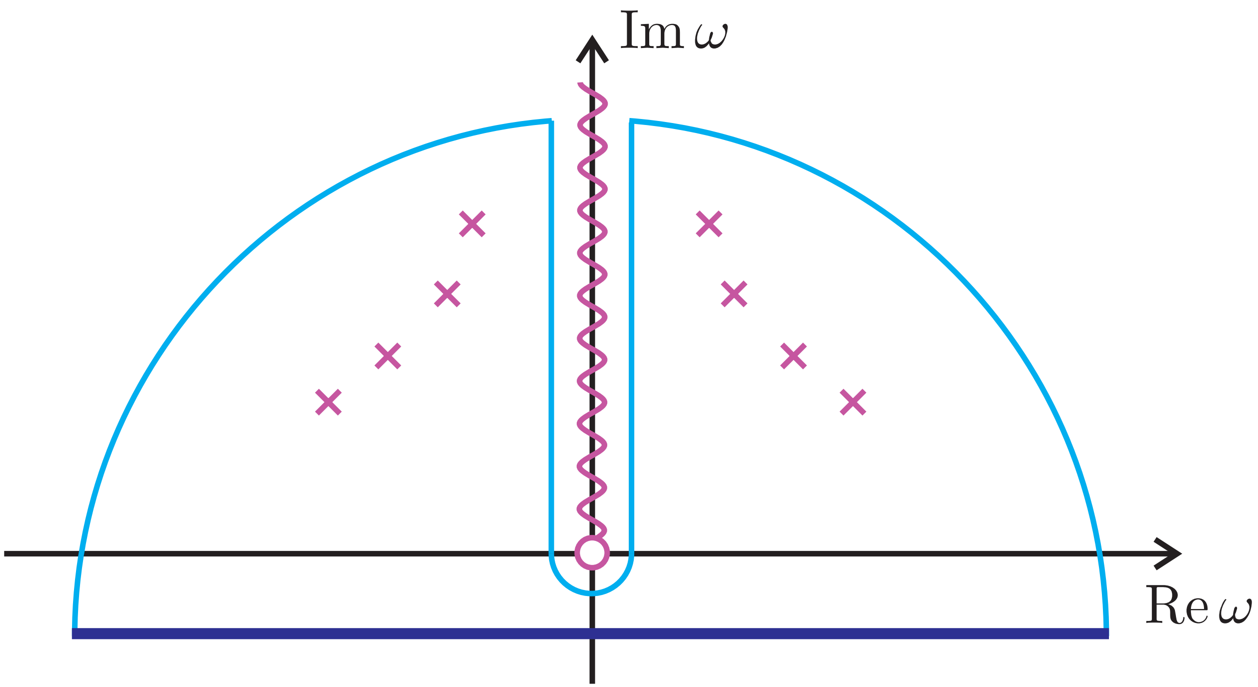


To compute the integral, we complete the contour as shown

Contour integration



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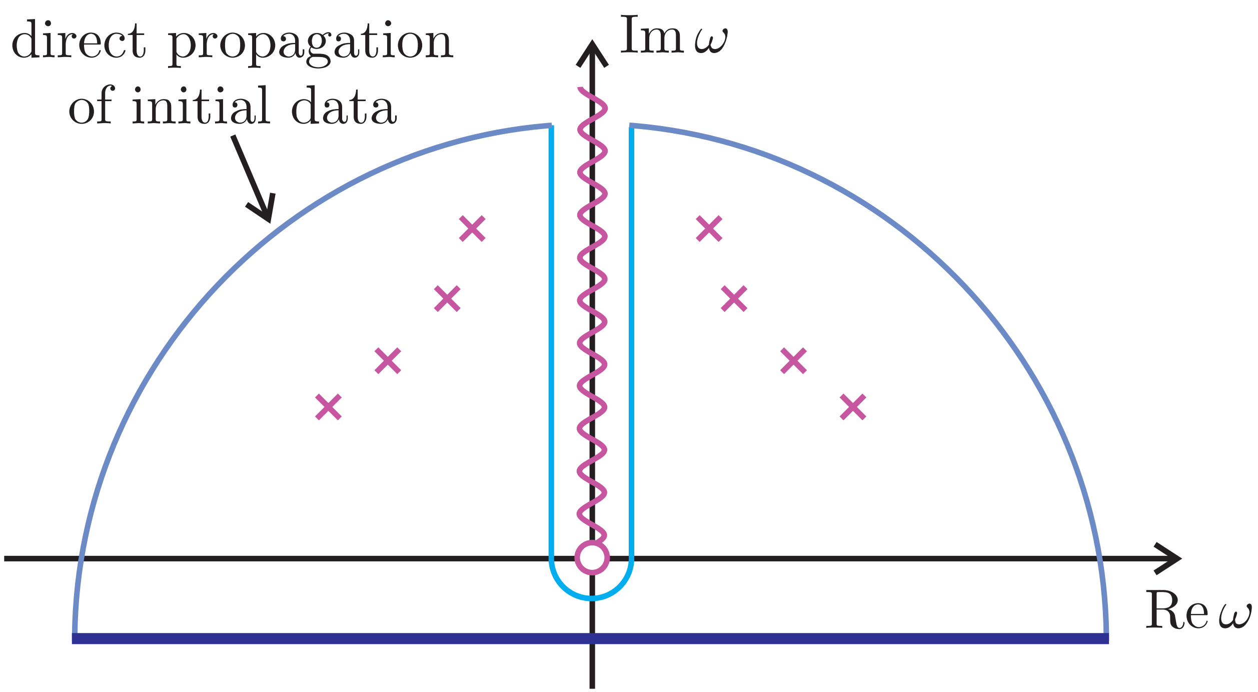

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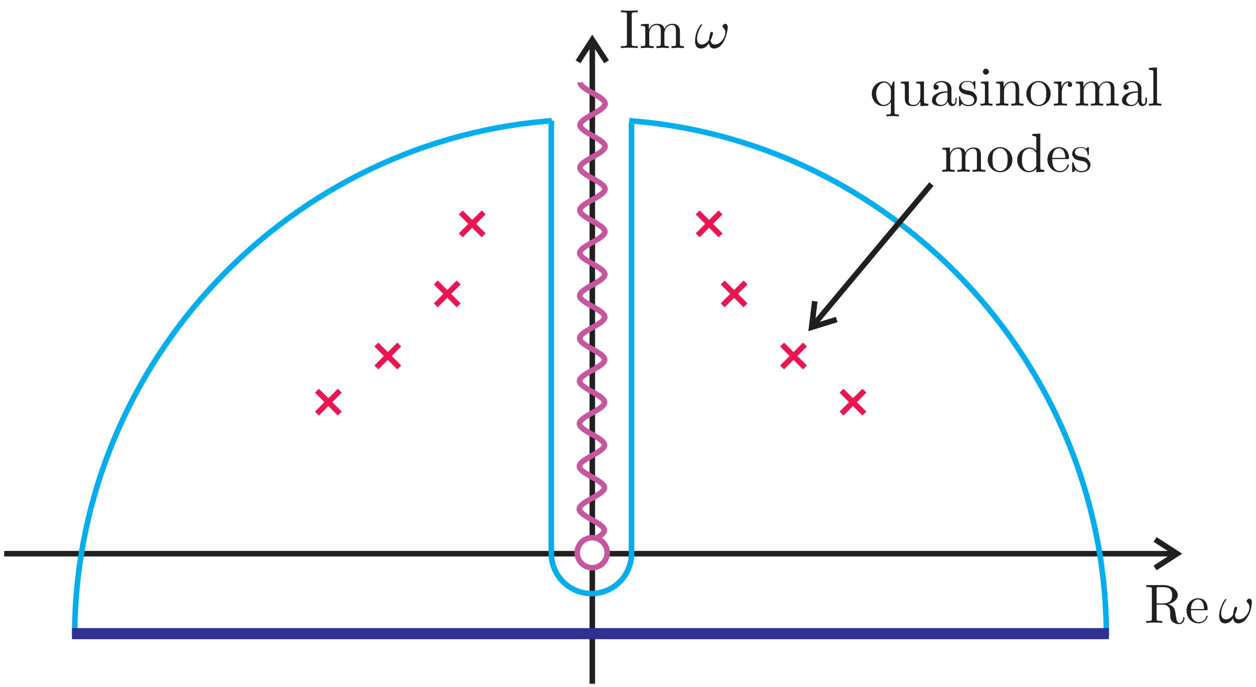

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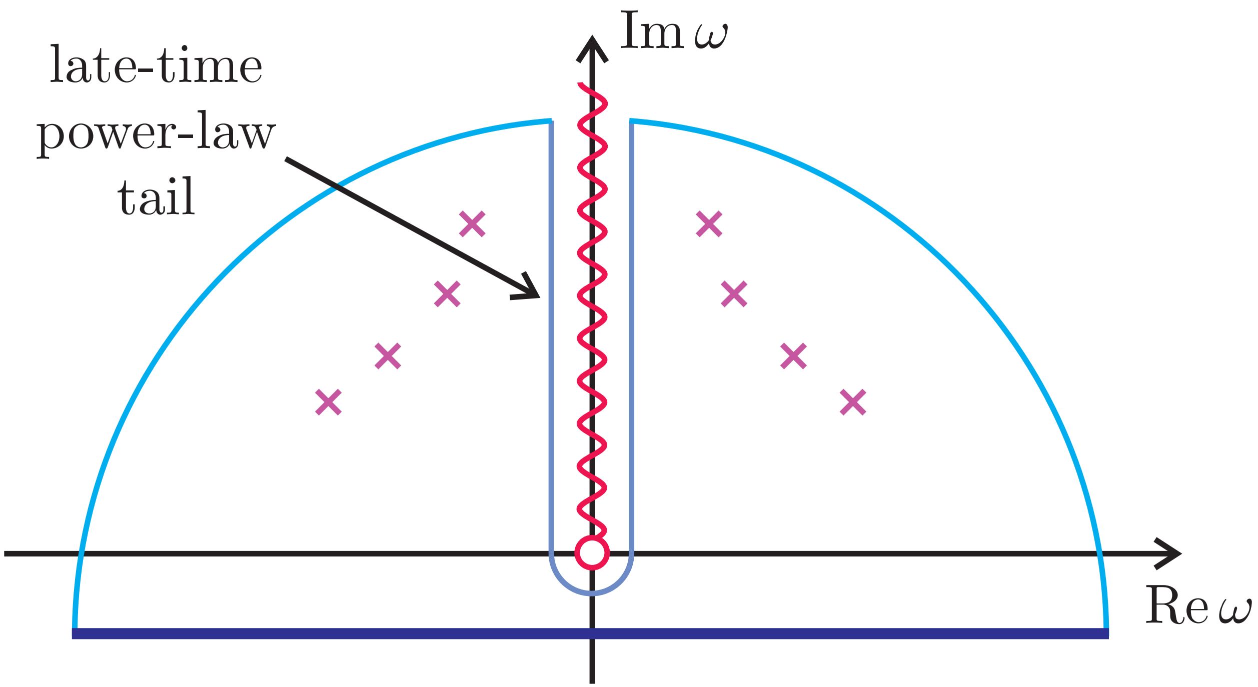

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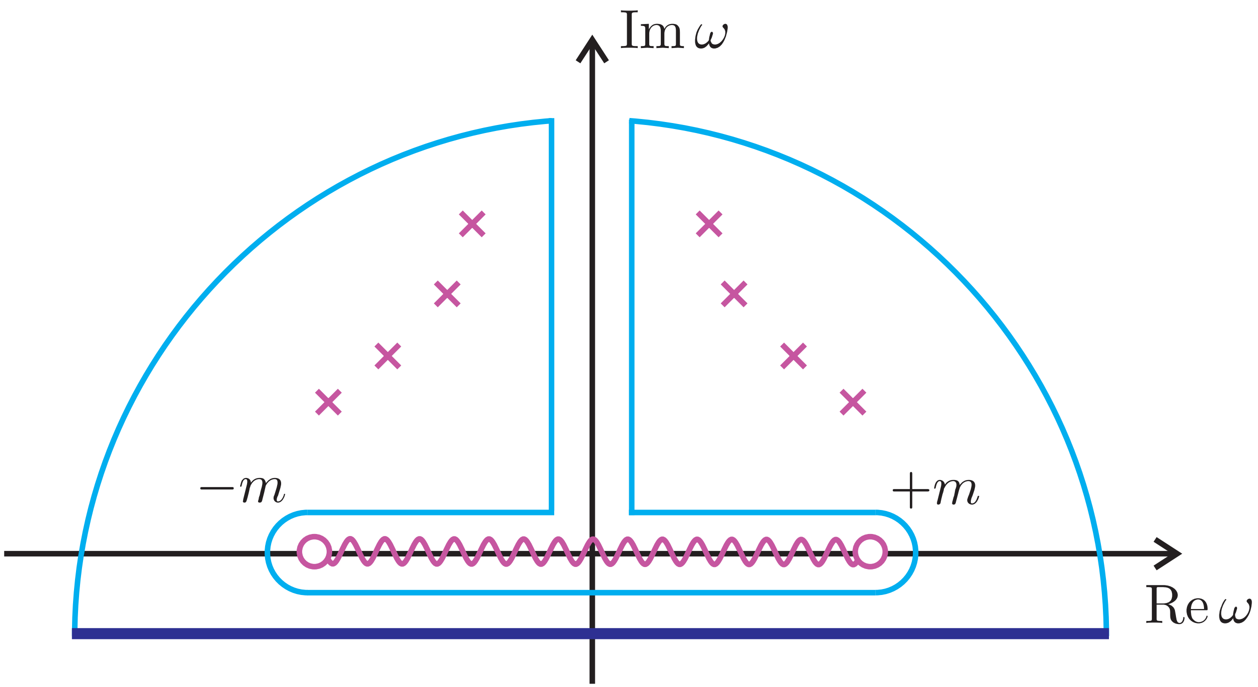
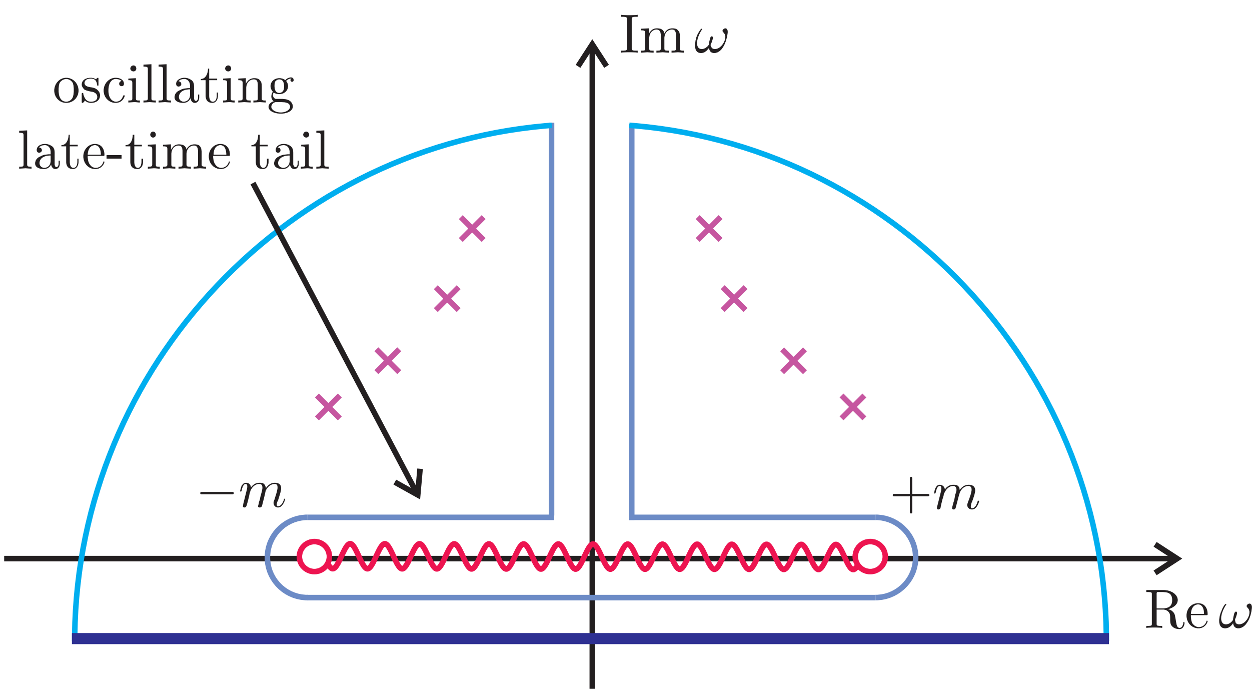


A non-zero field mass changes the branch cut and integration contour



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- we'll use this later

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- assuming $\ell = 0.1$ mm:

d/ℓ	f_n	eg. detector	string mass
33	$\gtrsim 10^{-2}$ Hz	LISA	$\gtrsim 10^{+6}\ M_\odot$
24	$\gtrsim 10^{+2}$ Hz	LIGO	$\gtrsim 10^{+2}\ M_\odot$
10	$\gtrsim 10^{+8}$ Hz	B'ham (?)	$\gtrsim 10^{-4}\ M_\odot$

GWs from point sources

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  - analogous to extreme mass ratio inspirals (EMRIs) in GR

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- let's use the delta function approximation anyways, should give an upper limit on the amplitude of GWs from these events

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$$\Sigma^{\text{bulk}}_{AB} = 0 \text{ and } \Sigma^{+}_{AB} = 0 \text{ or } \Sigma^{-}_{AB} = 0$$

- we decompose h_{AB} as

$$h_{AB} = \frac{\kappa_5^2 (GM)^2}{\mathcal{C}} e^{\alpha}_{A} e^{\beta}_{B} \sum_{n=0}^{\infty} Z_n(y) Z_n(y_\pm) h^{(n)}_{\alpha\beta}.$$

  - $\mathcal{C}$ is a normalization constant (to be specified later) with dimensions of $(\text{mass})^{-4}$

Equations of motion with sources

- we make the assumptions

$$\Sigma^{\text{bulk}}_{AB} = 0 \text{ and } \Sigma^{+}_{AB} = 0 \text{ or } \Sigma^{-}_{AB} = 0$$

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  - $\mathcal{C}$ is a normalization constant (to be specified later) with dimensions of $(\text{mass})^{-4}$
- define a dimensionless brane stress-energy tensors and brane bending scalars by

$$\Theta^{\pm}_{\alpha\beta} = \mathcal{C} e^{A}_{\alpha} e^{B}_{\beta} T^{\pm}_{AB}, \quad \tilde{\xi}^{\pm} = \frac{\mathcal{C}\xi^{\pm}}{(GM)^2 \kappa_5^2}.$$

Equations of motion with sources

- the equation of motion for $h_{\alpha\beta}^{(n)}$ is

$$(GM)^2 \left[\nabla^\gamma \nabla_\gamma h_{\alpha\beta}^{(n)} + 2R_\alpha{}^\gamma{}_\beta{}^\delta h_{\gamma\delta}^{(n)}\right] - \mu_n^2 h_{\alpha\beta}^{(n)} = - 2\left(\Theta_{\alpha\beta} - \tfrac{1}{3}\Theta g_{\alpha\beta}\right) - 4(GM)^2 \nabla_\alpha \nabla_\beta \tilde{\xi}$$

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- we also have the conditions

$$\nabla^\alpha h^{(n)}_{\alpha\beta} = \nabla^\alpha \Theta_{\alpha\beta} = 0 = g^{\alpha\beta} h^{(n)}_{\alpha\beta}$$

Radiative s-wave channel

- we decompose the problem in terms of spherical harmonics:

$$\tilde{\xi} = \frac{\xi^{(s)}}{\sqrt{4\pi}} + \sum_{l=1}^{\infty} \sum_{m=-l}^{l} Y_{lm} \tilde{\xi}_{lm}$$

$$h_{\alpha\beta}^{(n)} = \frac{h_{\alpha\beta}^{(n,s)}}{\sqrt{4\pi}} + \sum_{l=1}^{\infty} \sum_{m=-l}^{l} \sum_{i=1}^{10} [Y_{lm}^{(i)}]_{\alpha\beta} \, h_i^{(nlm)}$$

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- just worry about s-wave sector from now on

Radiative s-wave channel

- write the $l = 0$ contribution to the metric perturbation as

$$h^{(n,s)}_{\alpha\beta} = \mathsf{H}_1\, t_\alpha t_\beta - 2\mathsf{H}_2\, t_{(\alpha} r_{\beta)} + \mathsf{H}_3\, r_\alpha r_\alpha + \mathsf{K}\, \gamma_{\alpha\beta},$$

where we have defined

$$t^\alpha = f^{-1/2}\partial_t, \quad r^\alpha = f^{1/2}\partial_r, \quad \gamma_{\alpha\beta} = g_{\alpha\beta} + t_\alpha t_\beta - r_\alpha r_\beta$$

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- the master variables:

$$\psi = \frac{2\rho^3}{2+\mu^2\rho^3}\left(\rho\frac{\partial \mathsf{K}}{\partial\tau} - f\mathsf{H}_2\right), \quad \varphi = \rho\frac{\partial\xi^{(s)}}{\partial\tau}.$$

Radiative s-wave channel

- $\psi = \psi(\tau, x)$ and $\varphi = \varphi(\tau, x)$ satisfy:

$$(\partial_\tau^2 - \partial_x^2 + V_\psi)\psi = \mathcal{S}_\psi + \hat{\mathcal{I}}\,\varphi$$
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- the various terms are

$$V_\psi = \frac{f}{\rho^3\,(2+\rho^3\mu^2)^2}\left[\mu^6\rho^9 + 6\,\mu^4\rho^7 - 18\,\mu^4\rho^6 - 24\,\mu^2\rho^4 + 36\,\mu^2\rho^3 + 8\right]$$
$$\mathcal{S}_\psi = \frac{2f\rho^3}{3(2+\mu^2\rho^3)^2}\left[\rho(2+\mu^2\rho^3)\partial_\tau(2\Lambda_1 + 3\Lambda_3) + 6(\mu^2\rho^3 - 4)f\Lambda_2\right]$$
$$V_\varphi = \frac{2f}{\rho^3} \qquad \mathcal{S}_\varphi = \frac{\rho f}{6}\partial_\tau\Lambda_1$$
$$\hat{\mathcal{I}} = \frac{8f}{(2+\mu^2\rho^3)^2}\left[6f\rho^2\partial_\rho + (\mu^2\rho^3 - 6\rho + 8)\right]$$

Radiative s-wave channel

- we have defined the following three scalars:

$$\Lambda_1 = -\Theta^{(s)}_{\alpha\beta} g^{\alpha\beta} \quad \Lambda_2 = -\Theta^{(s)}_{\alpha\beta} t^{\alpha} r^{\beta} \quad \Lambda_3 = +\Theta^{(s)}_{\alpha\beta} \gamma^{\alpha\beta}$$

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- also have the inversion formulae

$$\begin{aligned}
\partial_\tau \mathsf{H}_1 &= \frac{1}{\rho}\left[\left(\partial_\tau^2 + \frac{3}{\rho}\partial_\rho + \mu^2\right)\psi + \frac{4}{\mu^2}\left(\partial_\tau^2 - \frac{1}{\rho}\partial_\rho\right)\varphi\right], \\
\mathsf{H}_2 &= \frac{1}{\rho}\left[\left(\partial_\rho + \frac{2}{\rho}\right)\psi + \frac{4}{\mu^2}\left(\partial_\rho - \frac{1}{\rho}\right)\varphi\right], \\
\partial_\tau \mathsf{H}_3 &= \frac{1}{\rho}\left[\left(\partial_\tau^2 + \frac{1}{\rho}\partial_\rho\right)\psi + \frac{4}{\mu^2}\left(\partial_\tau^2 - \frac{2}{\rho}\partial_\rho\right)\varphi\right], \\
\partial_\tau \mathsf{K} &= \frac{1}{\rho}\left[\left(\frac{1}{\rho}\partial_\rho + \frac{\mu^2}{2}\right)\psi + \frac{4}{\mu^2\rho}\left(\partial_\rho - \frac{1}{\rho}\right)\varphi\right]
\end{aligned}$$

Point mass Lagrangian

- take the Lagrangian of a particle on either brane to be

$$\mathcal{L}_p^{\pm} = \frac{M_p}{2} \left\{ \int \frac{\delta^4(z^\mu - z_p^\mu)}{\sqrt{-q}} q_{\alpha\beta} \frac{dz_p^\alpha}{d\eta} \frac{dz_p^\beta}{d\eta} d\eta \right\}^{\pm} .$$

where η is an affine parameter

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- need to re-express in terms of Schwarzschild metric $g_{\alpha\beta}$
- leads to stress energy tensor

$$T_{\alpha\beta}^{\pm} = \frac{M_p}{a_\pm} \int \frac{\delta^4(z^\mu - z_p^\mu)}{\sqrt{-g}} u_\alpha u_\beta \, d\lambda, \quad u^\alpha \nabla_\alpha u^\beta = 0$$

  where λ is the affine parameter w.r.t. $g_{\alpha\beta}$

Point mass Lagrangian

- $T^{\pm}_{\alpha\beta}$ can be decomposed in spherical harmonics

$$T^{\pm}_{\alpha\beta} = \frac{f}{\mathcal{C}_{\pm} E \rho^2} u_\alpha u_\beta \delta(\rho - \rho_p) \left[\frac{1}{4\pi} + \sum_{l=1}^{\infty} \sum_{m=-l}^{l} Y_{lm}(\Omega) Y^*_{lm}(\Omega_p) \right]$$

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$$\mathcal{C}_{\pm} = \frac{(GM)^3}{M_p e^{k y_{\pm}}} \quad E = -g_{\alpha\beta} u^\alpha \xi^\beta_{(t)} \quad \xi^\alpha_{(t)} = \partial_t.$$

i.e. E is the usual energy

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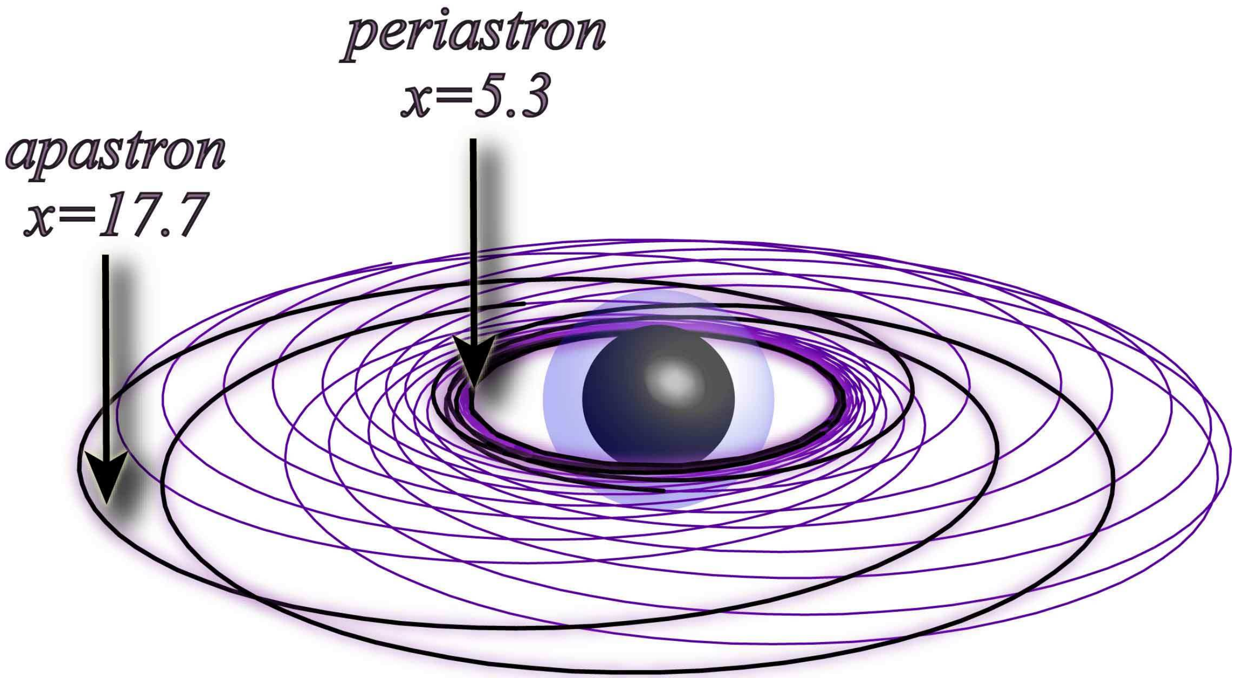
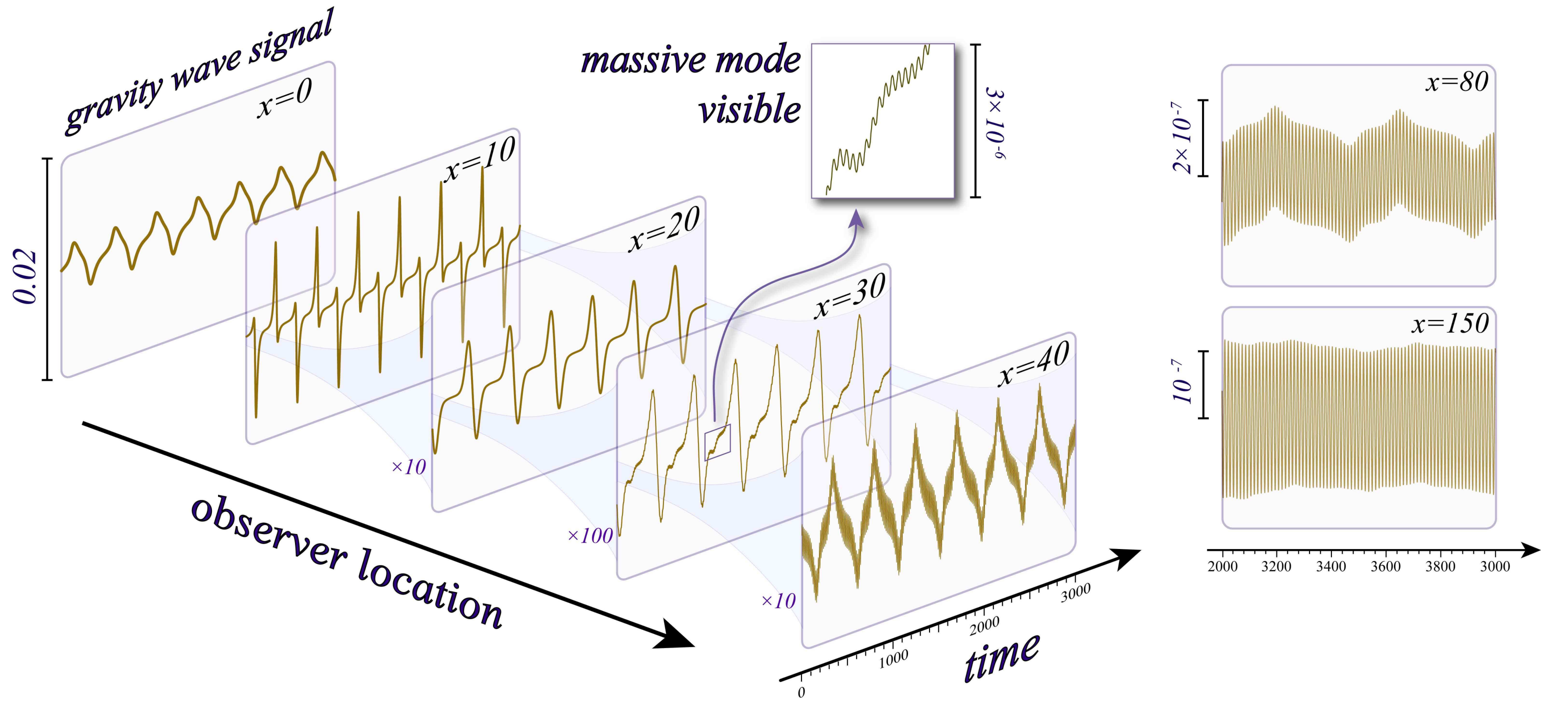
- the s-wave contribution is the first one in the square brackets

Bounded orbit



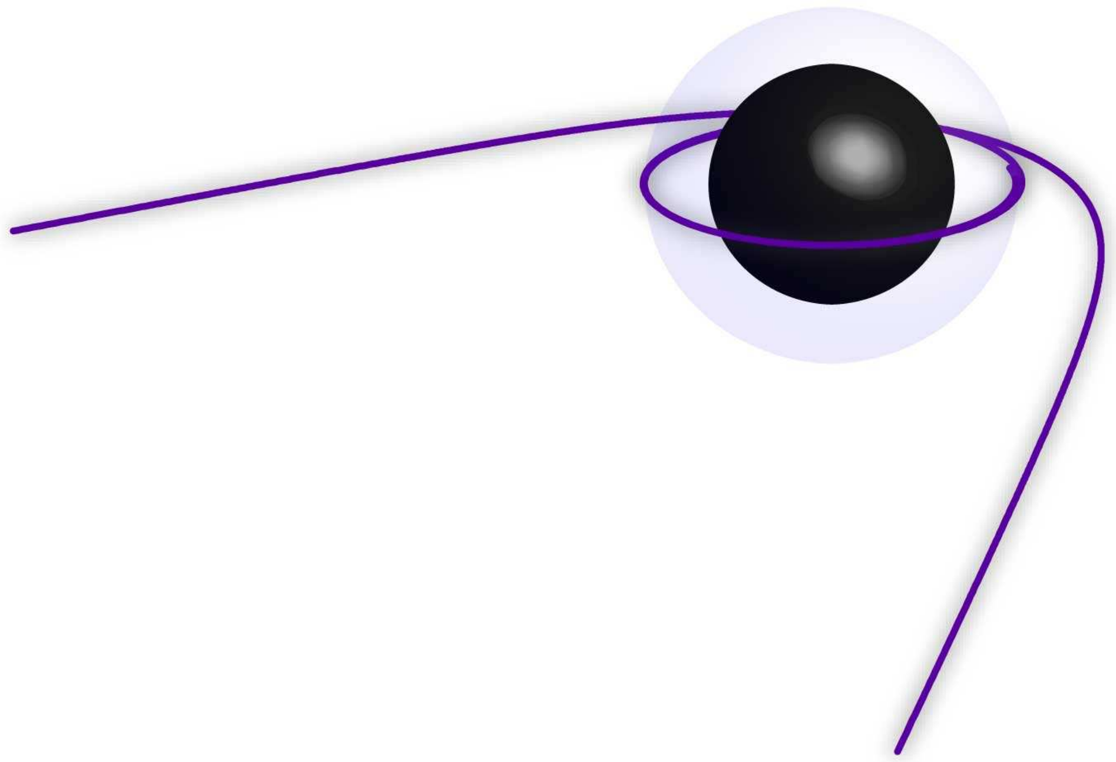
Particle in quasi-periodic orbit close to black string

Bounded orbit

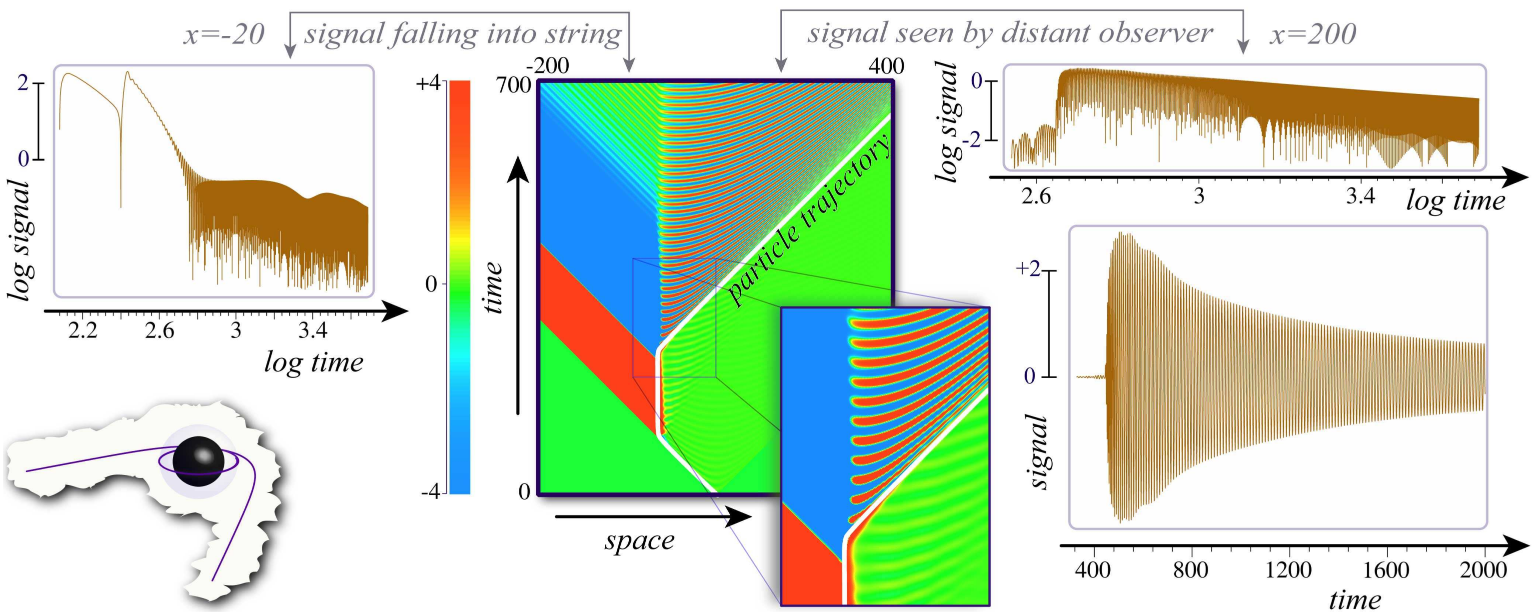


Distant observer only sees (weak) periodic signal with frequency m (no information about orbit)

Fly-by orbit



High kinetic energy particle briefly captured by black string before escaping to infinity



Much higher amplitude slowly-decaying signal observed far away due to sharp acceleration

Scaling formulae for KK amplitudes

Estimating high n amplitudes

$$(\partial_\tau^2 - \partial_x^2 + V_\psi^{(n)})\psi_n = \mathcal{S}_\psi + \hat{\mathcal{I}}\,\varphi$$

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  - $Z(y)$ eigenfunctions

Amplitudes

A detailed analysis leads to the following approximation for the perturbation far from the string:

$$h_{\alpha\beta}^{(n)} \approx h_n e^{i\omega_n t} \begin{pmatrix} 0 & & & \\ & 1 & & \\ & & -\frac{r^2}{2} & \\ & & & -\frac{r^2 \sin^2\theta}{2} \end{pmatrix} \begin{cases} 1, & \text{periodic orbits} \\ (tc^3/GM)^{-5/6}, & \text{fly-by orbits} \end{cases}$$

$$[\omega_n \sim c\ell^{-1}(n + 1/4)\pi e^{-d/\ell},\ r = \text{distance from string}]$$

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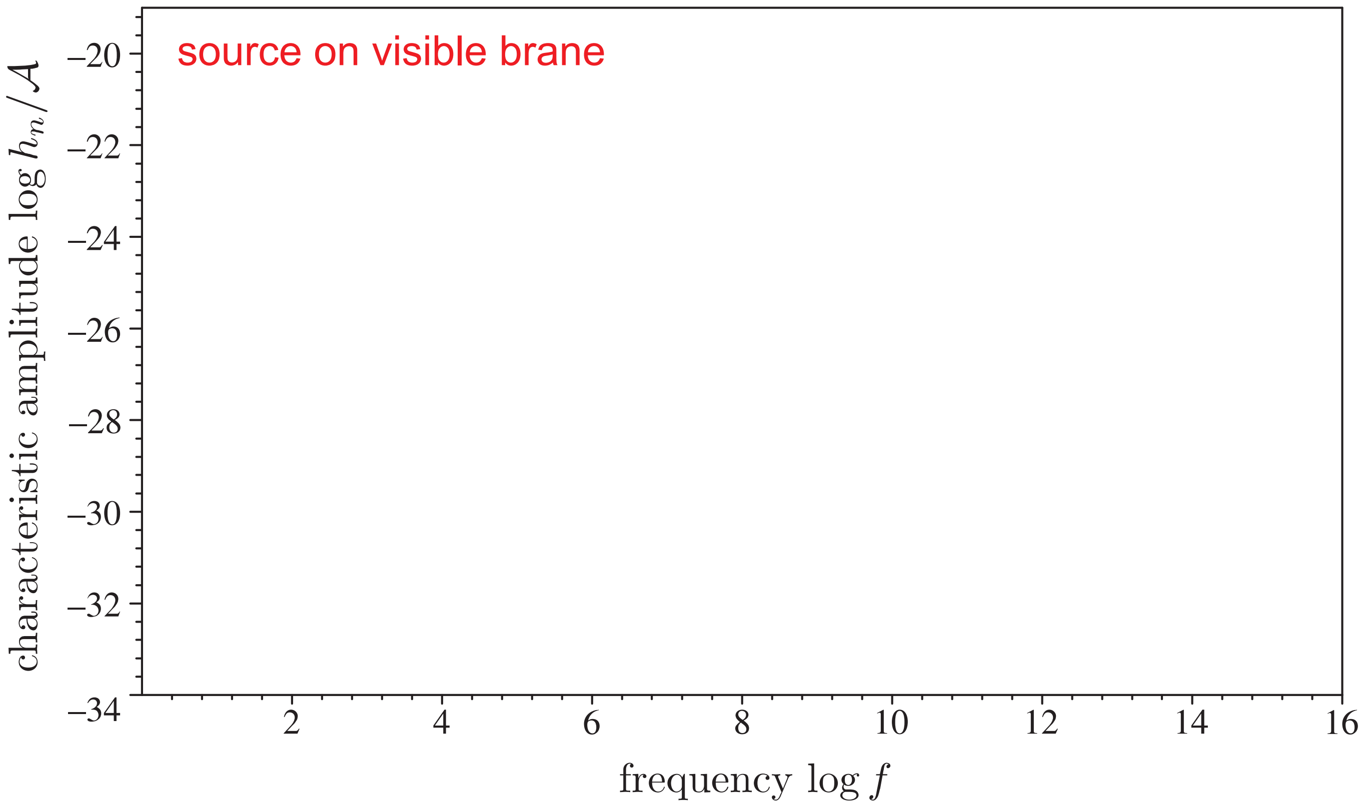
let us illustrate the properties of the characteristic amplitudes with an example:

$M = 10M_\odot$, $M_p = 1.4M_\odot$, $r = 1\,\mathrm{kpc}$, $\ell = 0.1\,\mathrm{mm}$

(M_p is the mass of the perturbing particle)

Amplitudes



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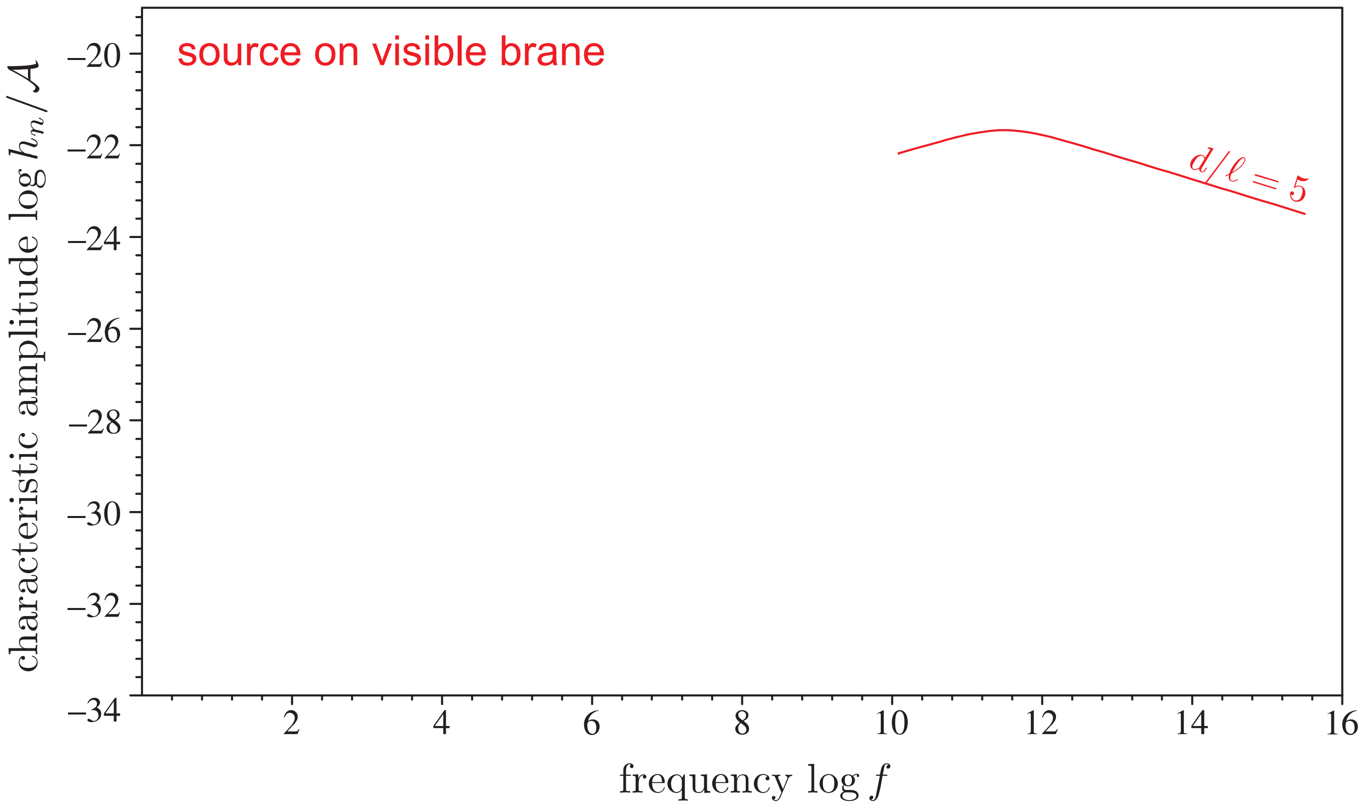

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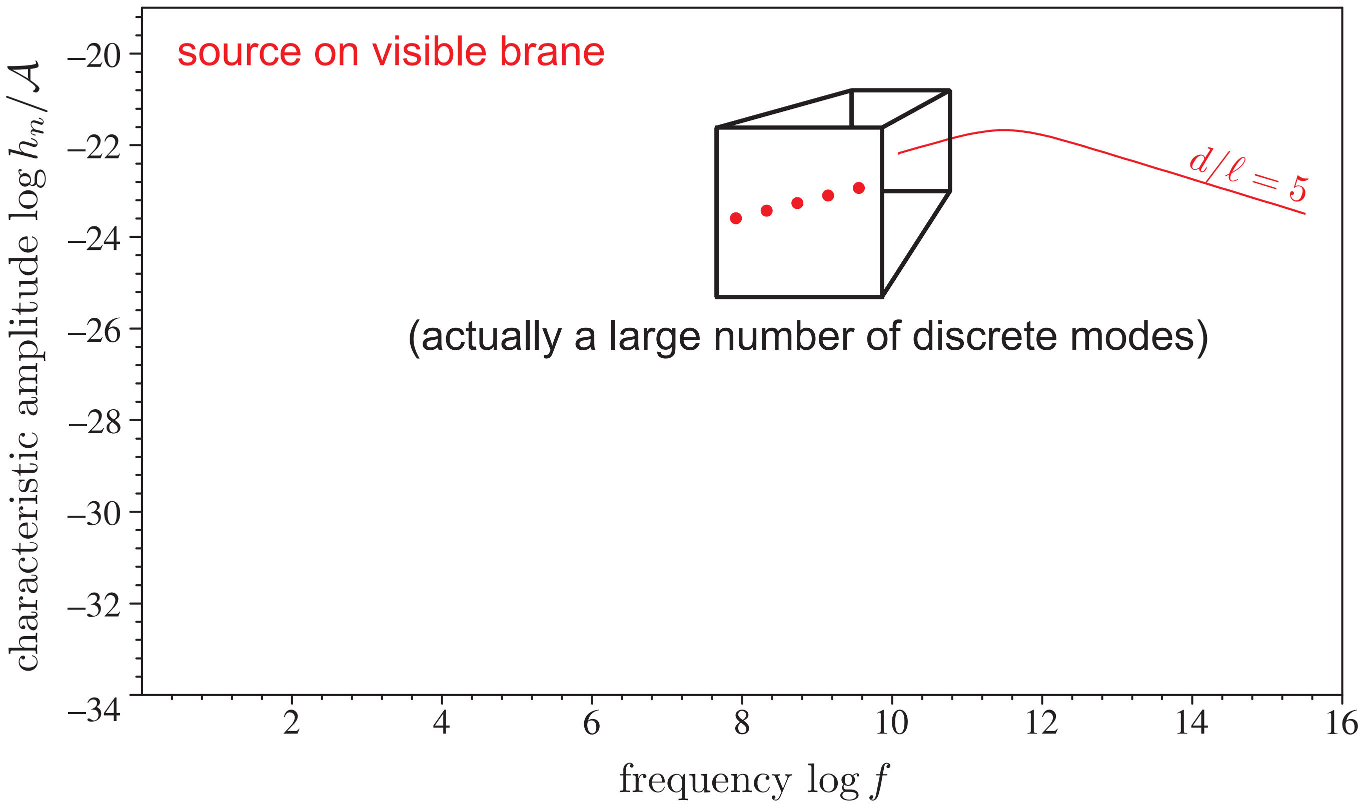

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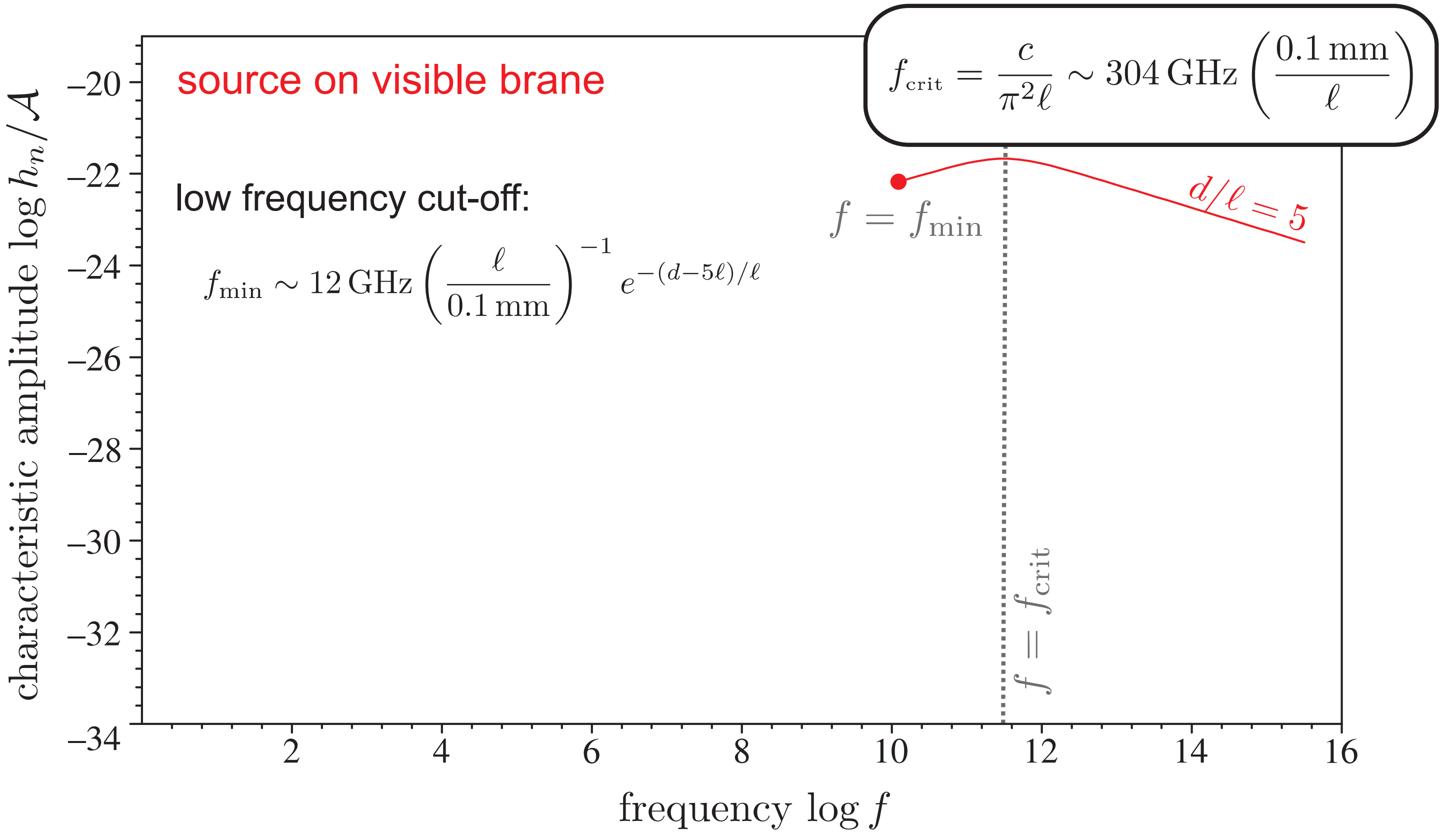

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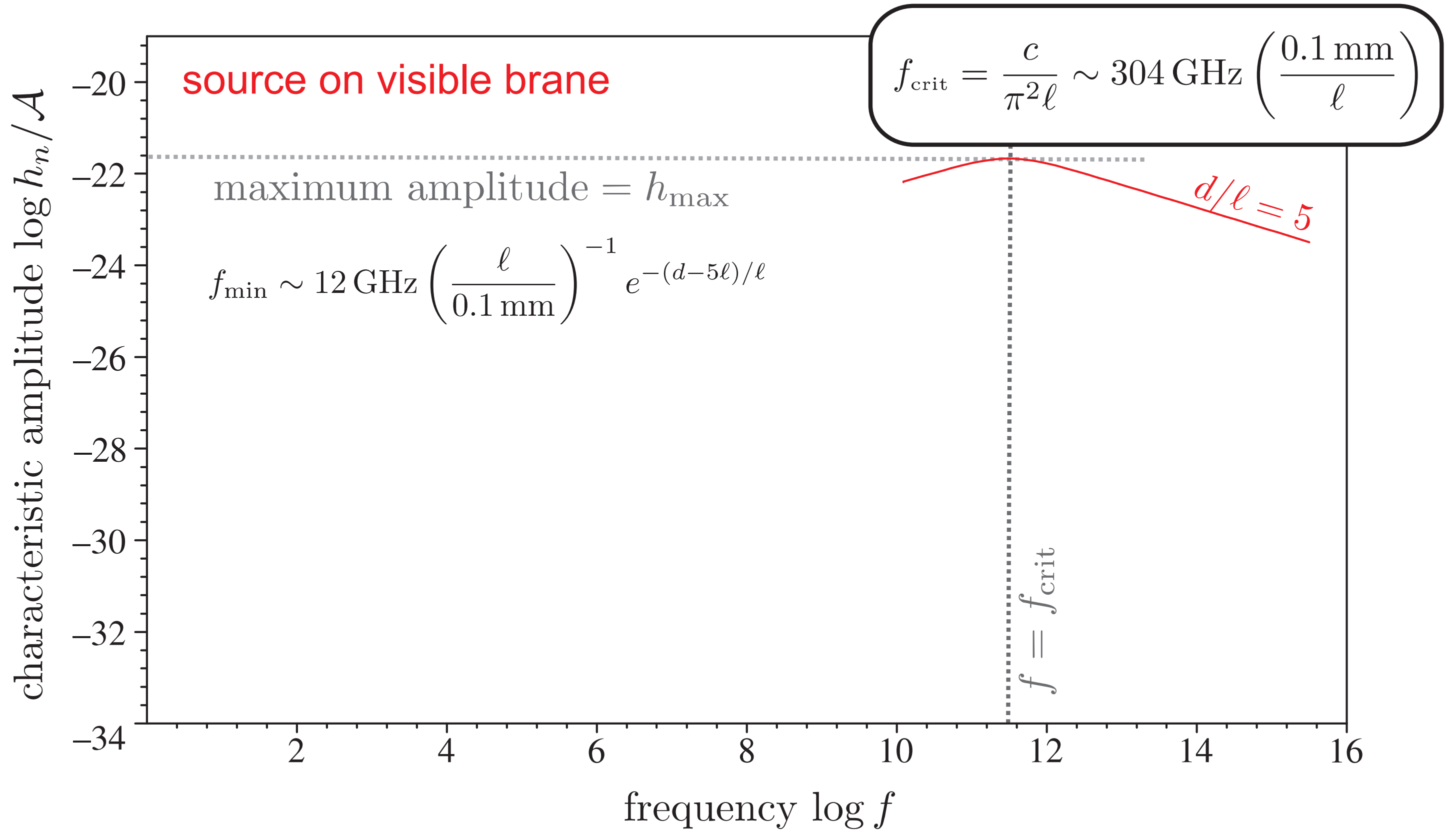

$$h_{\rm max} \sim 5.0 \times 10^{-22} \mathcal{A} \left(\frac{M_p}{M_\odot} \right) \left(\frac{r}{\rm kpc} \right)^{-1} \left(\frac{M}{M_\odot} \right)^{-1/2} \left(\frac{\ell}{0.1\,{\rm mm}} \right)^{1/2} e^{-(d-5\ell)/\ell}$$

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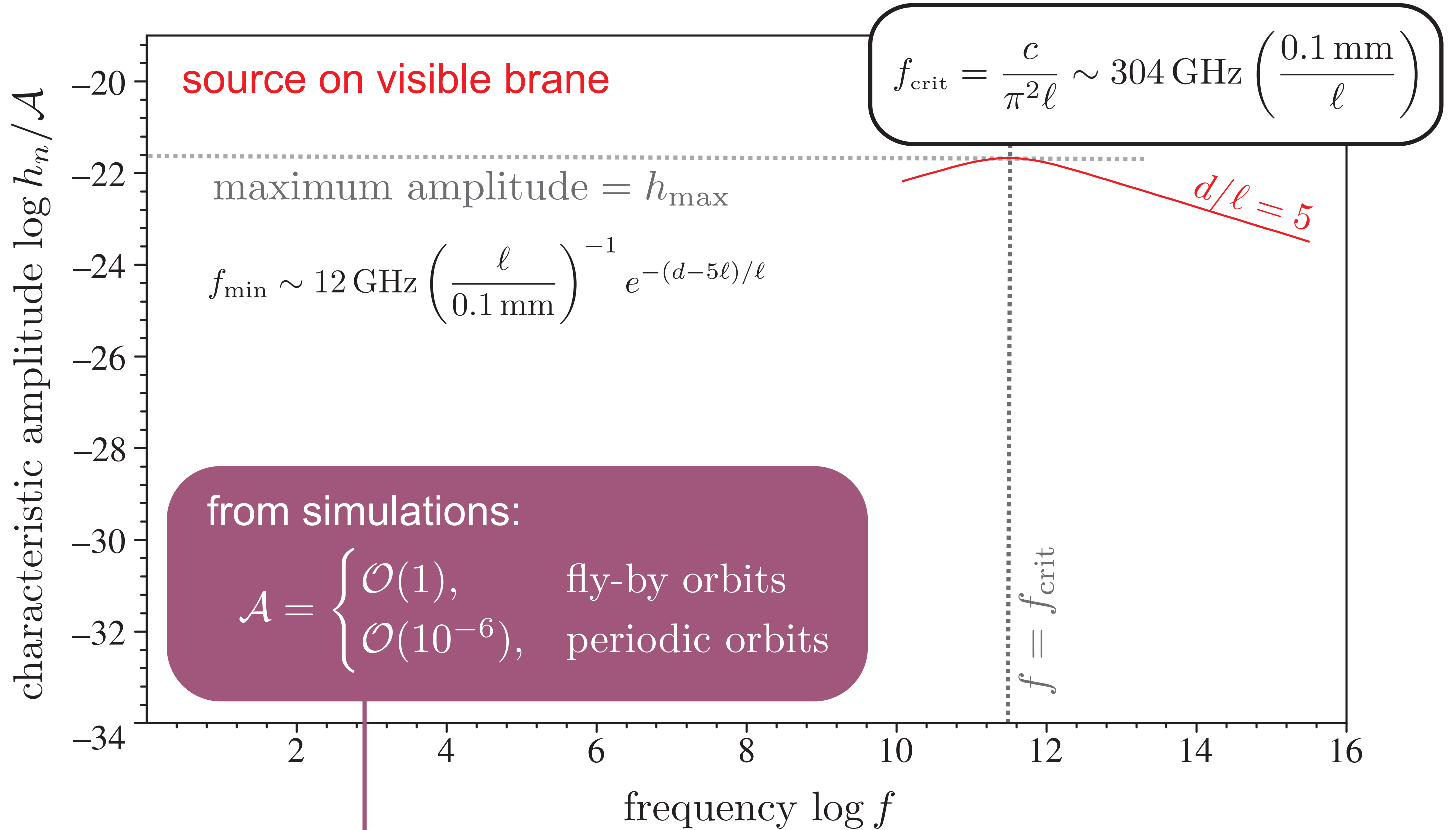
Amplitudes



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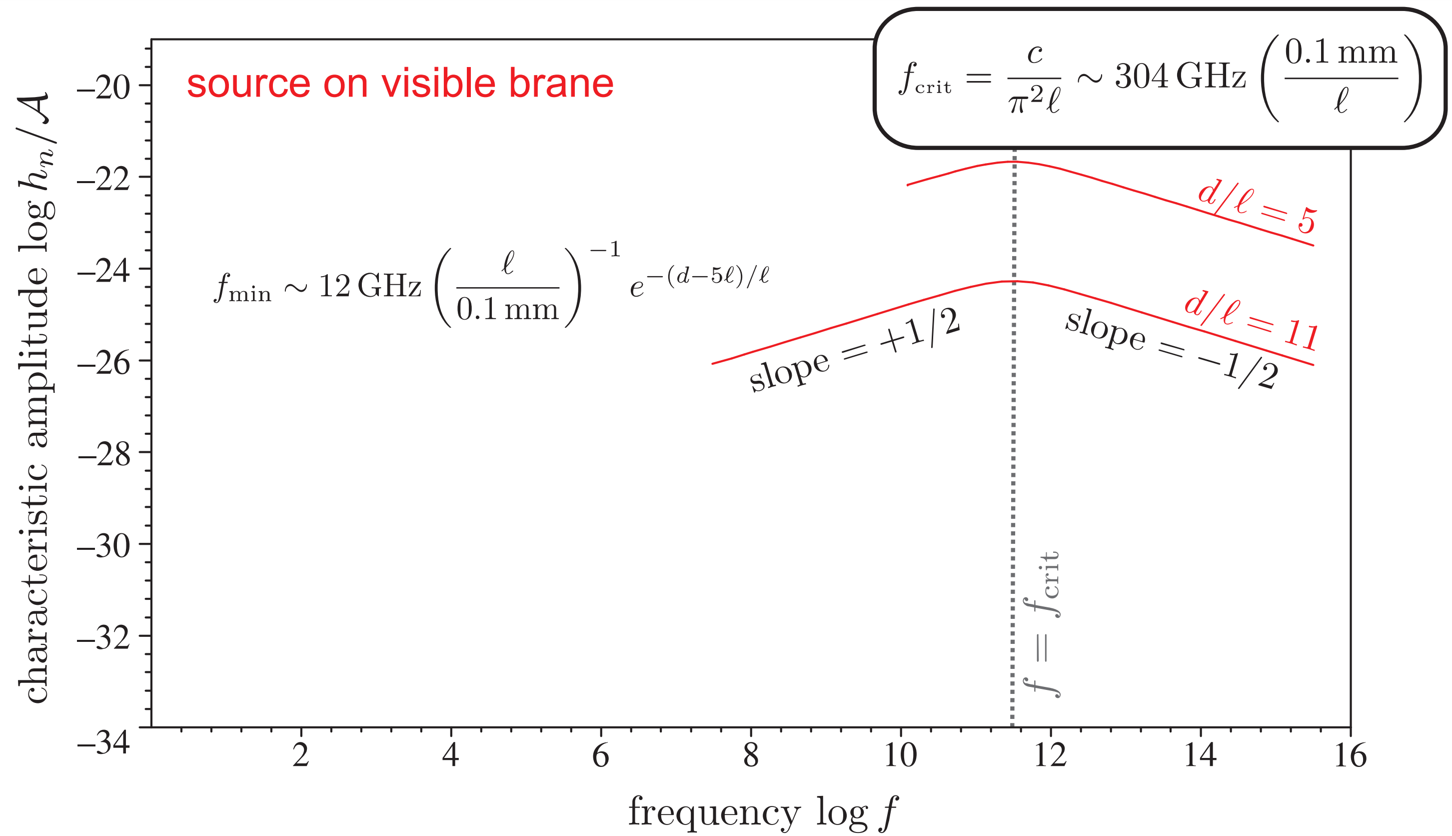

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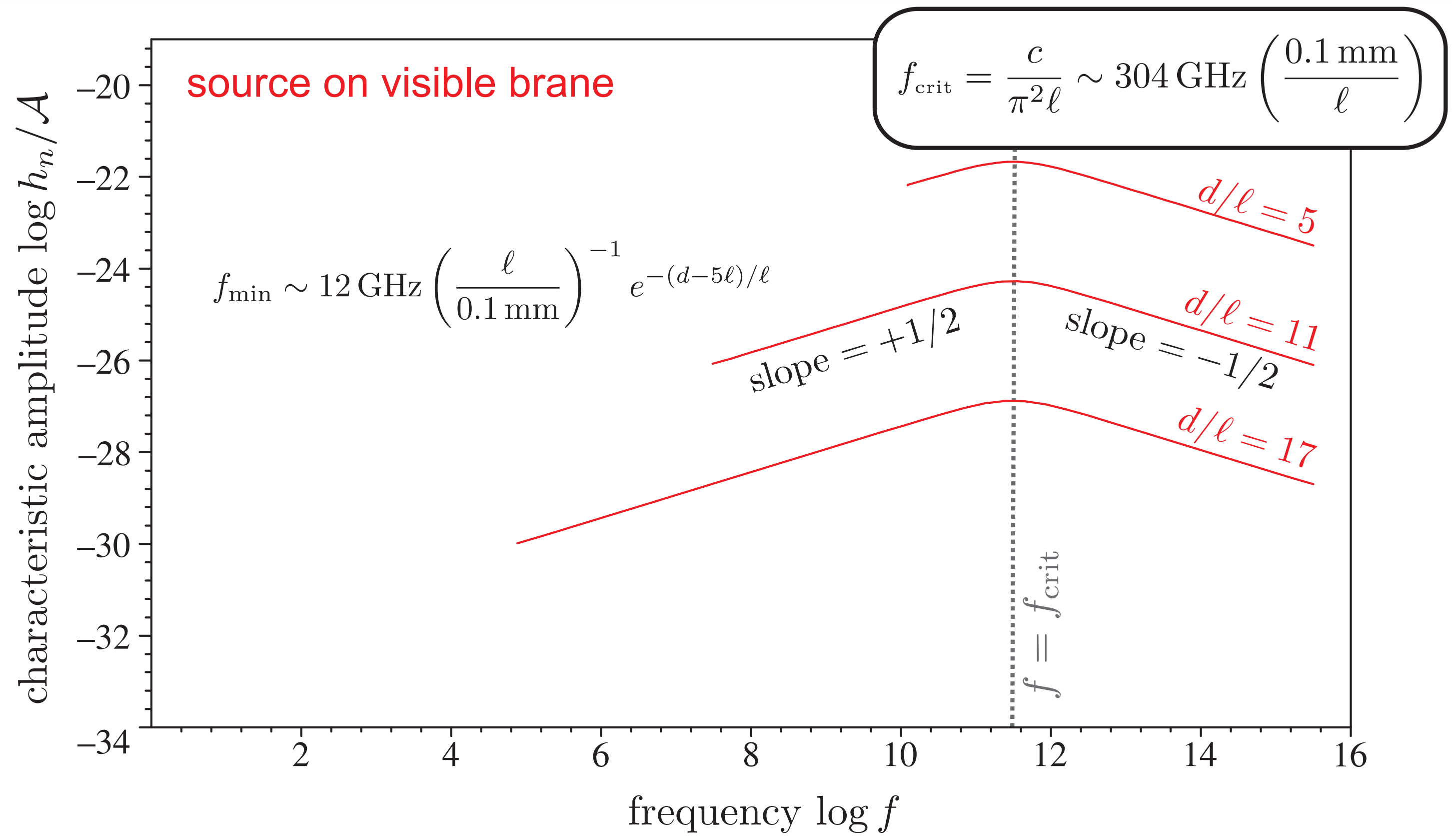

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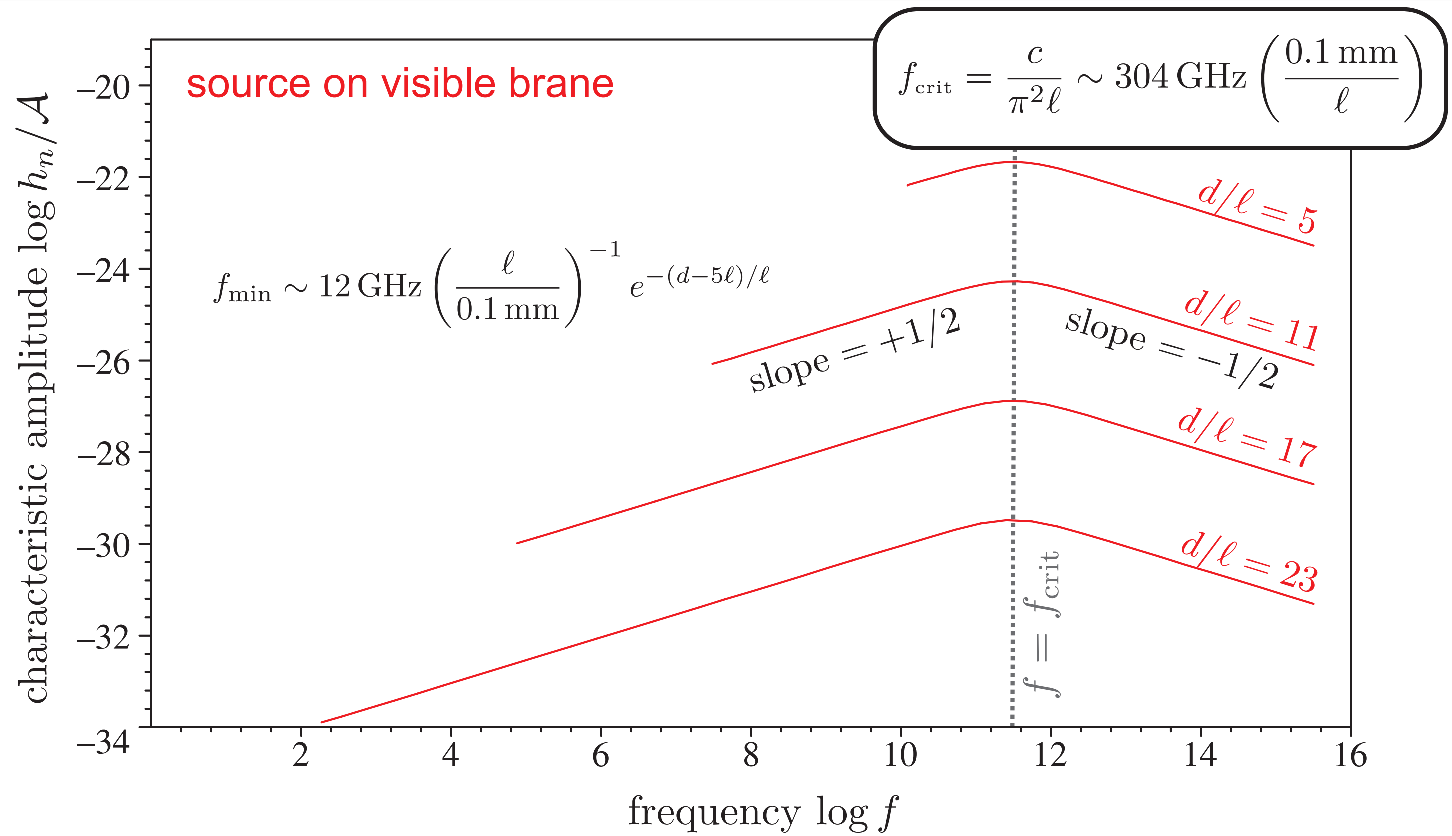

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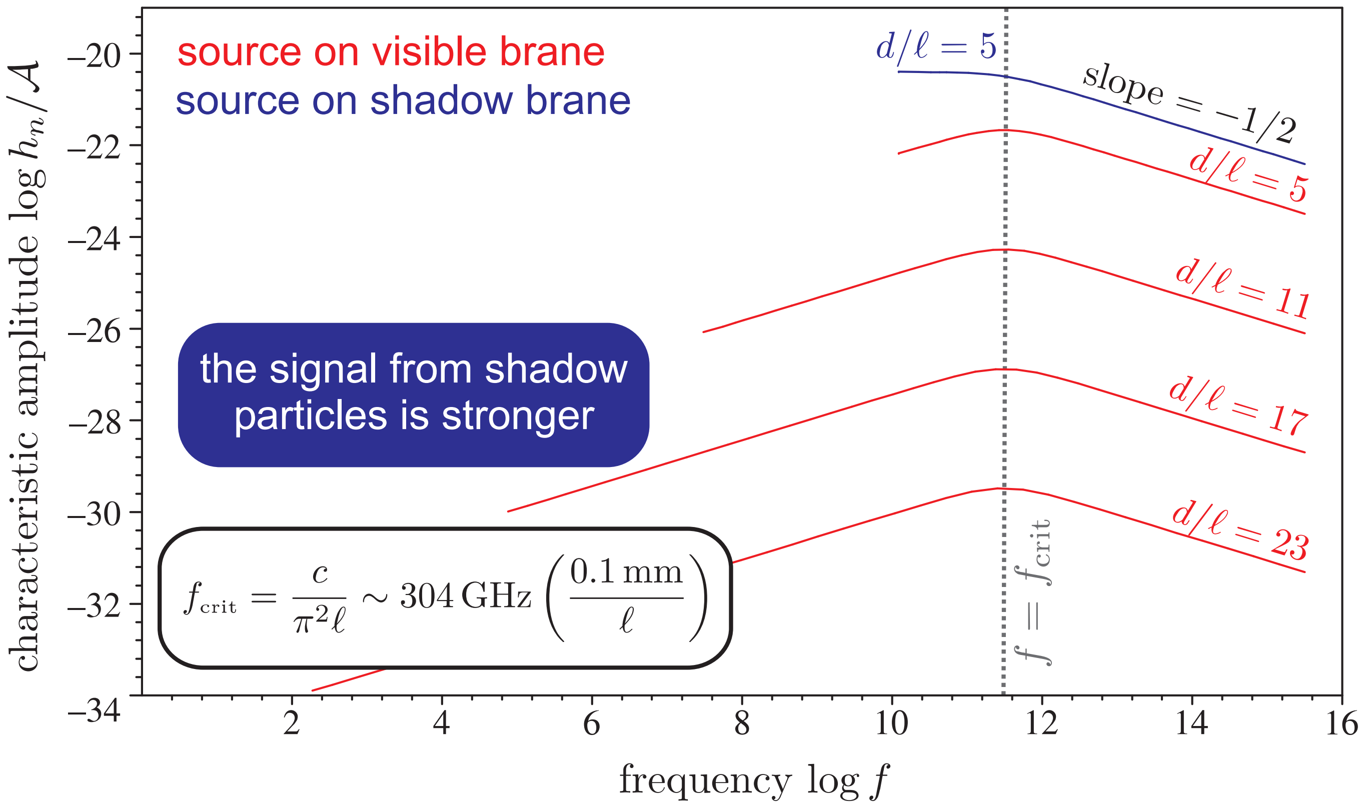

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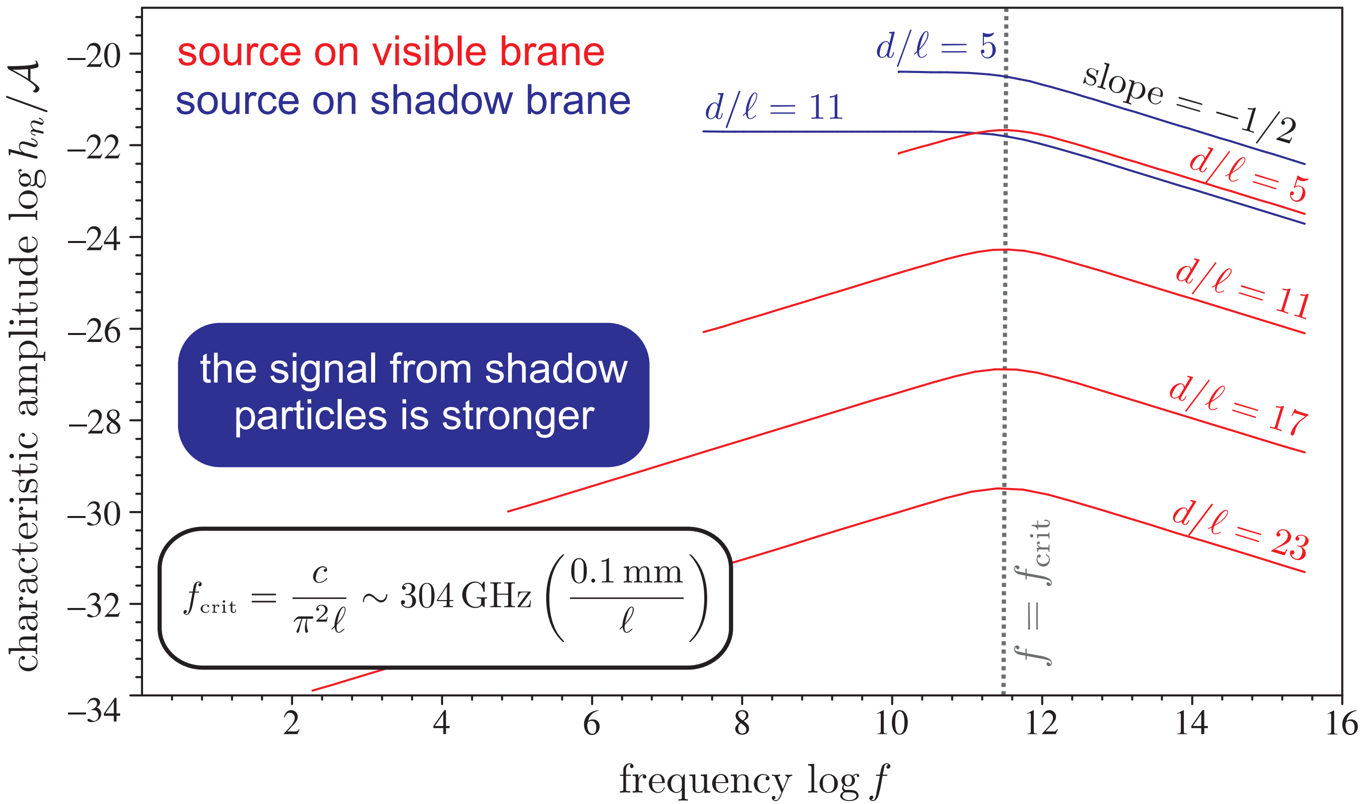

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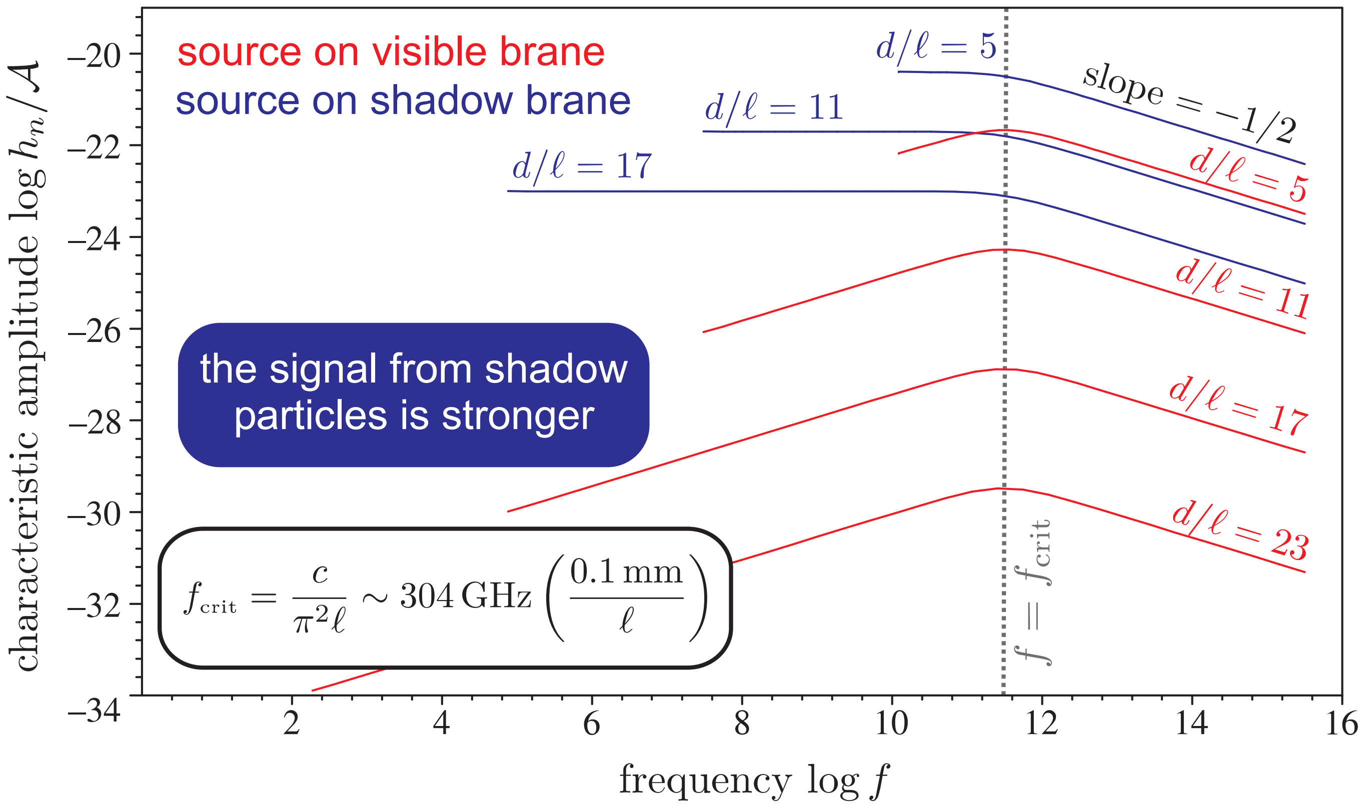

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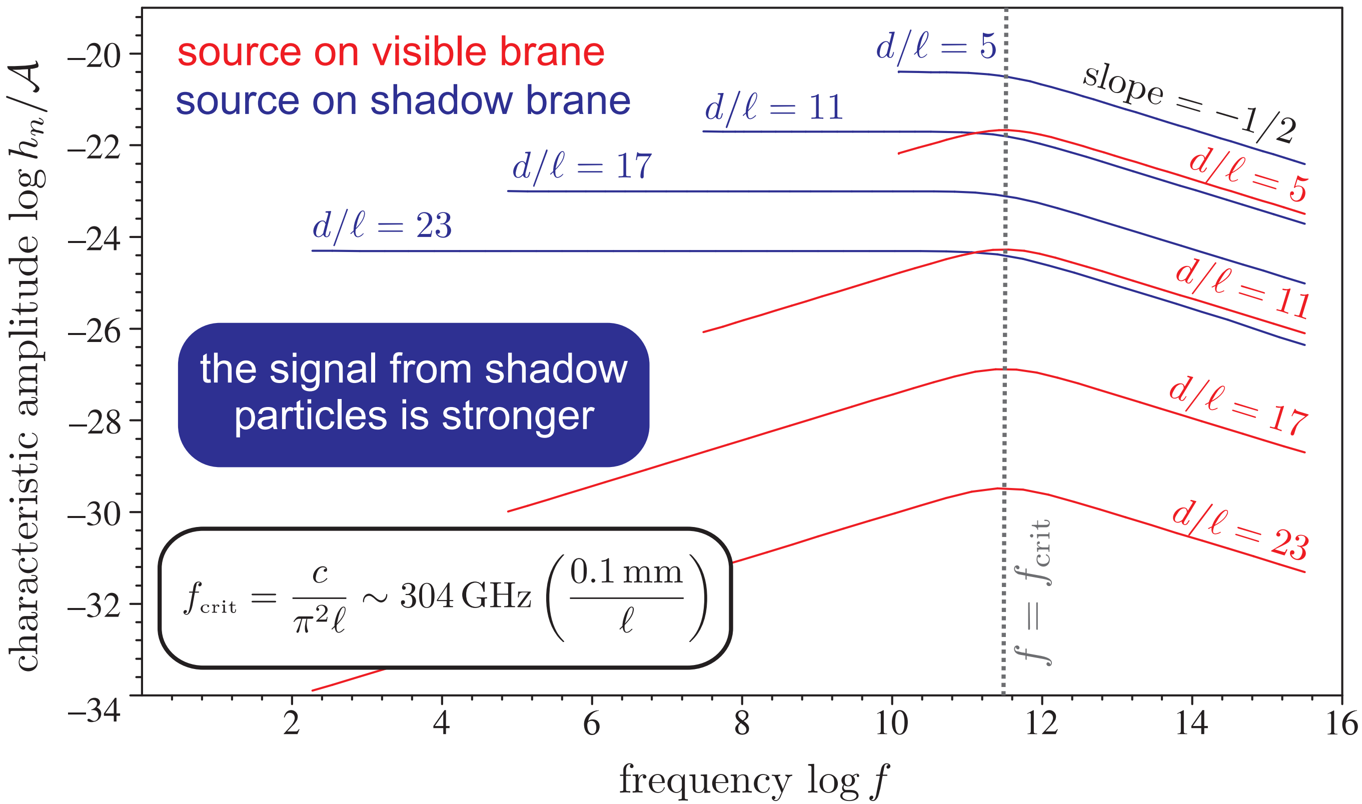

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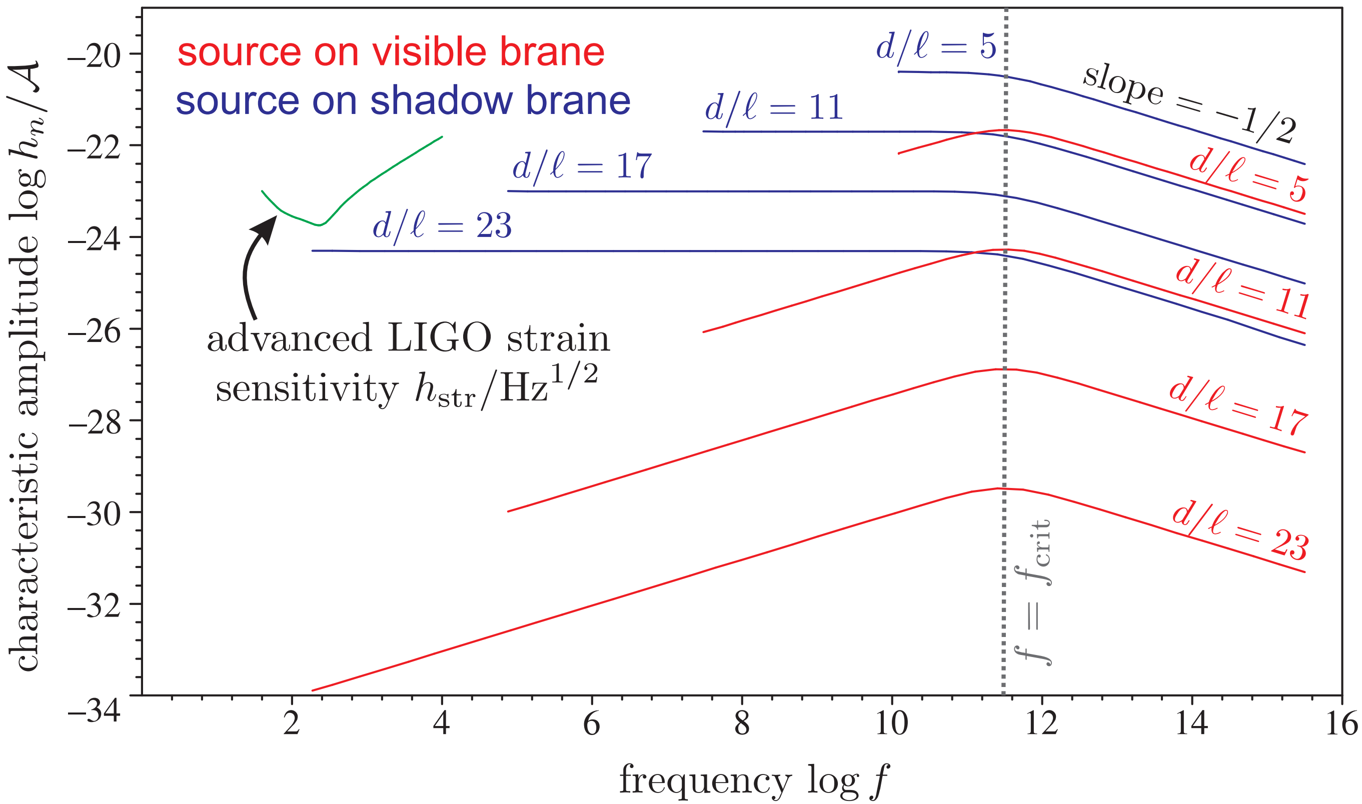

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- however, for a "low frequency" device like (A)LIGO increasing d/ℓ puts more modes in the waveband and actually increases the SNR

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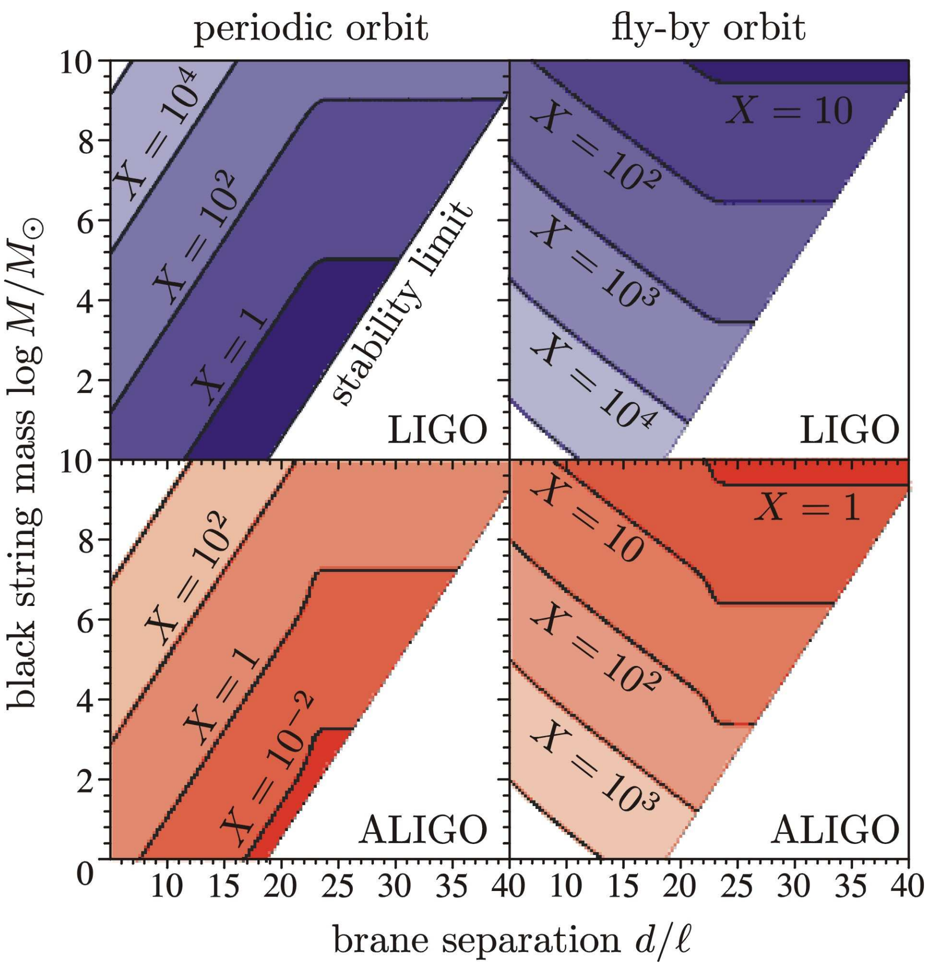

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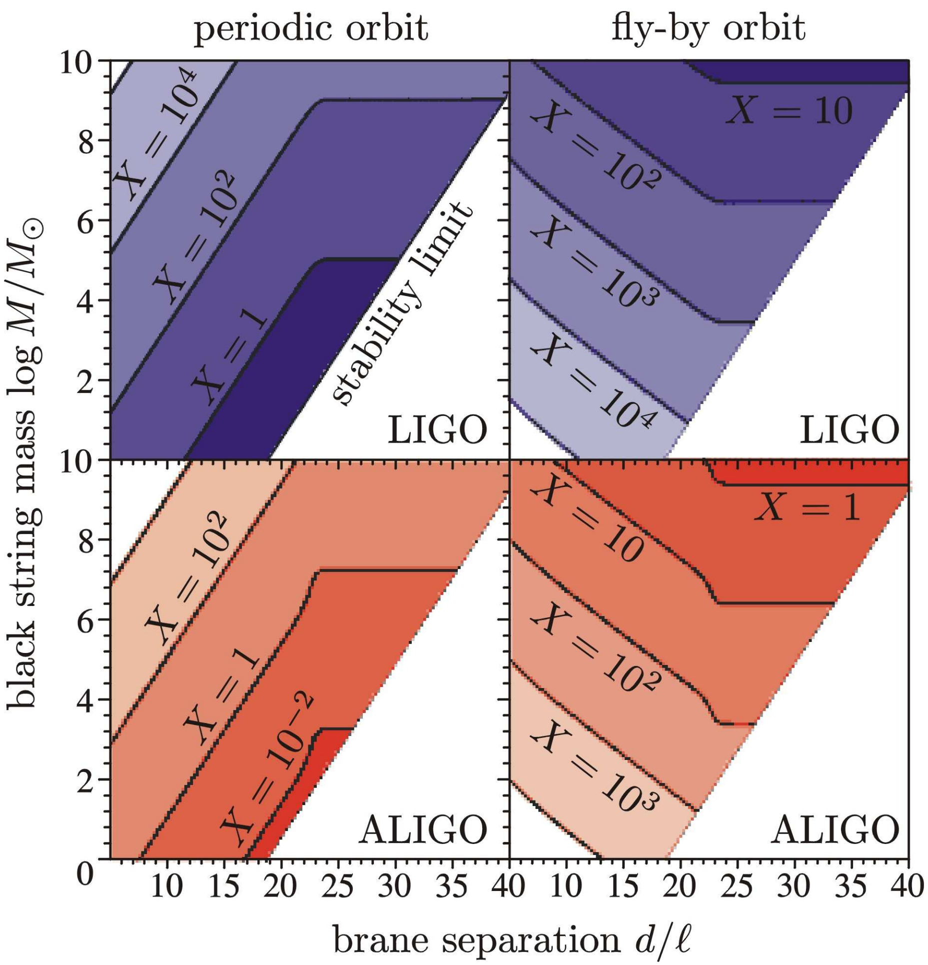

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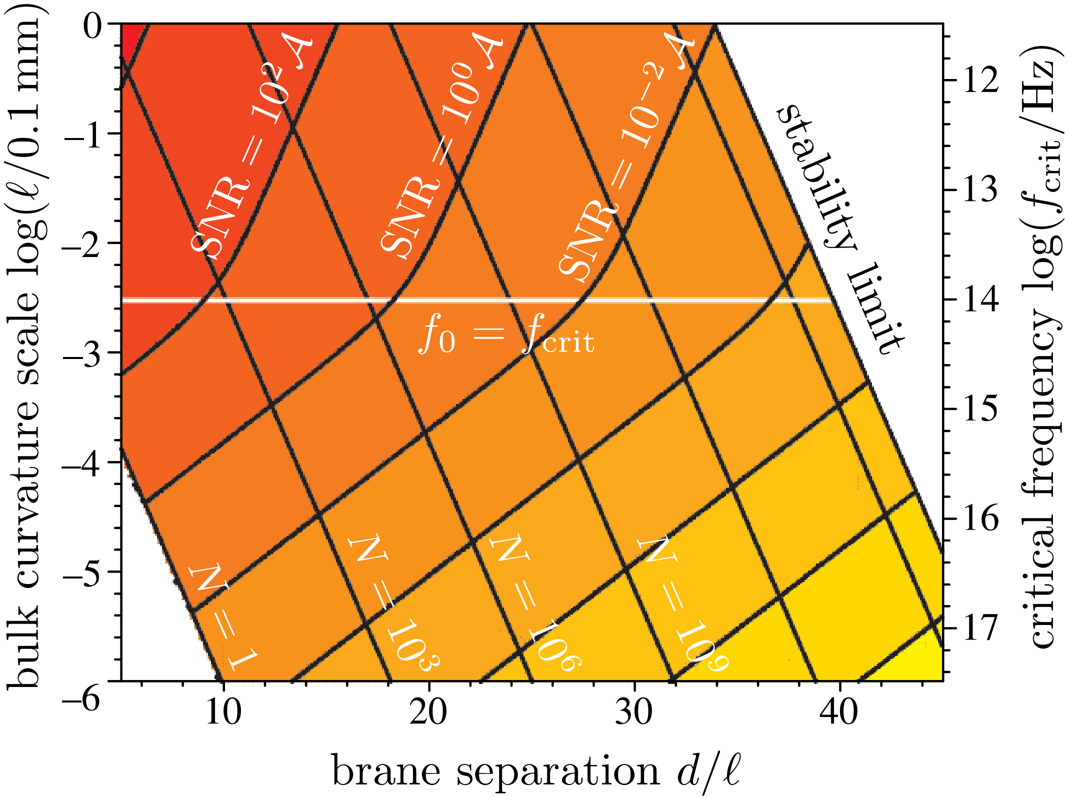
High frequency detectors

SNR induced in the detector for this event:



- N is the number of modes in the detector waveband
- measuring SNR and N fixes ℓ (assuming other parameters like M are known)

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- GWs from matter on the shadow brane are much easier to detect

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- gravitational wave background (KK masses are the same for all black strings)

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