Higher order gravity theories and their black hole solutions.

- §1 Introduction - Motivations
- §2 Lovelock theory basics
- §3 A staticity theorem.
- Static Black holes

- Thermodynamics and geometry
- §4 2 Applications
- §5 The extended KK reduction
- §6 A bunch of open problems
  +
- Conclusions

C. Charmousis LPT Paris XI Orsay

General Relativity

Consider $S_{GR} = \frac{1}{16\pi G_4}\int d^4x \sqrt{-g}\, \mathcal{L}$ where

$$\mathcal{L} = \mathcal{L}(M, g, \nabla)$$

Field equations $G_{\mu\nu} + (\Lambda g_{\mu\nu}) = 8\pi G_4 T_{\mu\nu}$

M_{Pl} UV ... Energy ... IR $\Lambda_{acc} \sim (10^{-3} eV)^4$

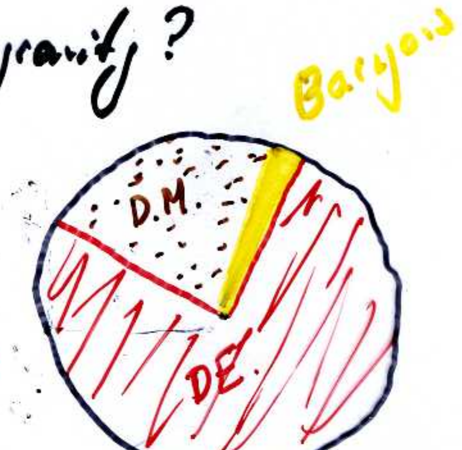
String theory
Xtra dimensions
Quantum loop Gravity

Accel. Cosmology!

Question: Is GR modified (also) at the IR?

Question: How do we modify gravity?

i) # degrees of freedom
ii) enlarge phase space.
iii) modify f.e.g.



Precision cosmology gives matter content of the Universe

Only 4%. Known matter

Introduction

Given $\mathcal{L} = \mathcal{L}(M, g_{\mu\nu}, \nabla)$ and Einstein's postulates

Field equations are unique in $D=4$

$$\int_M d^4x \sqrt{-g}\,(-2\Lambda + R) \xrightarrow[\delta g_{\mu\nu}]{} G_{\mu\nu} + \Lambda g_{\mu\nu} = T_{\mu\nu} \quad (8\pi G = 1)$$

Therefore $G_{\mu\nu} + \Lambda g_{\mu\nu}$ (Cartan ~30)
is the unique metric dependant tensor which is

- symmetric
- divergence free ie, $\nabla^\mu T_{\mu\nu} = 0$.
- depends up to - and is linear - to second derivatives in $g_{\mu\nu}$

wave type operator for the massless spin 2 graviton

$g_{ab} \to \delta g_{ab}$ $\quad G_{ab} \to \delta G_{ab}$

Introduction

Given $\mathcal{L} = \mathcal{L}(M, g_{\mu\nu}, \nabla)$ and Einstein's postulates

Field equations are unique in D dimensions (Lovelock 81)

$$\int d^D x \sqrt{-g} \sum_{K=0}^{\left[\frac{D-1}{2}\right]} \alpha_K \mathcal{L}_{(K)}$$

$K=0 \quad \mathcal{L}_0 = \Lambda$

$K=1 \quad \mathcal{L}_1 = R$

$K=2 \quad \mathcal{L}_2 = \hat{G}$

etc.

is the unique metric dependant tensor which is

- symmetric
- divergence free ie, $\nabla^\mu T_{\mu\nu} = 0$.
- depends up to- and is linear- to second derivatives in $g_{\mu\nu}$

wave type operator for the massless spin 2 graviton

where $\mathcal{L}_{(K)} = \mathcal{R}^{A_1B_1} \wedge \dots \wedge \mathcal{R}^{A_KB_K} \wedge \theta^*_{A_1B_1\dots A_KB_K} \overset{\text{sym}}{\sim} R^K$

$$\theta^*_{A_1 \dots A_K} = \frac{\varepsilon_{A_1 \dots A_D}}{(D-K)!} \theta^{A_{K+1}} \wedge \dots \wedge \theta^{A_D}$$

$1 \; 2 \; \dots K \quad K+1 \; \dots \; D$

$$\mathcal{R}^A{}_B = \frac{1}{2} R^A{}_{BCD}\, \theta^C \wedge \theta^D$$

see also B. Zumino 1986

therefore for arbitrary D

The uniquely defined and most general classical theory satisfying Einstein's postulates is Lovelock theory

$$S = \sum_{K=0}^{\left[\frac{D-1}{2}\right]} \alpha_K \int_M d^D x \sqrt{-g}\, \mathcal{L}_{(K)}$$

$\mathcal{L}_0 = -2\Lambda$ c.c.
$\mathcal{L}_1 = R$ Einstein-Hilbert
$\mathcal{L}_2 = \hat{G}$ Gauss-Bonnet
$\vdots$
$\mathcal{L}_K \sim R^K$

- 2^{nd} order field equations
- no ghost around vacuum (Zwiebach)
- Bianchi identities
- well defined junction conditions.

§2 Lovelock theory basics

$\mathcal{L} = \mathcal{L}(M, g, \nabla)$ M is a D dimensional manifold
$g = g_{ab}\, dx^a dx^b$, ∇ Levi-Civita connexion

$\forall P \in M$ $e_A \in T_P M$ $g(e_A, e_B) = \eta_{AB}$ orthonormal basis.

$\underline{\theta^A \in \Omega^1(TM)}$ $\theta^A(e_B) = \delta^A_B$ or $g = \eta_{AB}\, \theta^A \otimes \theta^B$

$\theta^A = \theta^A_\mu\, dx^\mu$ $g_{\mu\nu} = \theta^A_\mu\, \theta^B_\nu$

Consider now 1 forms $\omega^A{}_B$

Connection
1-form
$$d\theta^A = -\omega^A{}_B \wedge \theta^B \quad , \quad \omega^A{}_B = -\omega^B{}_A$$

Curvature
$$\mathcal{R}^A{}_B = d\omega^A{}_B + \omega^A{}_C \wedge \omega^C{}_B$$

2-form
$$\mathcal{R}^A{}_B = \frac{1}{2} R^A{}_{BCD}\, \theta^C \wedge \theta^D$$

Riemann curvature

$\omega \in \Omega^{(k)}(TM)$, where $0 \leq k \leq D$ $\omega = \omega_{A_1 \dots A_k}\, \theta^{A_1} \wedge \dots \wedge \theta^{A_k}$

Use θ^A and the '$\wedge$' product to construct higher order diff. forms to construct D-forms out of $\mathcal{R}^K$ need Hodge dual:

Hodge dual of $\theta^{A_1} \wedge \ldots \wedge \theta^{A_K} \in \underline{\Omega}^K(TM)$

$$\theta^*_{A_1 \ldots A_K} = \frac{1}{(D-K)!} \varepsilon_{A_1 \ldots A_K A_{K+1} \ldots A_D} \theta^{A_{K+1}} \wedge \ldots \wedge \theta^{A_D}$$

$\theta^*_{A_1 \ldots A_K}$ is a $D-K$ form

θ^* is the volume element.

Identity $\theta^B \wedge \theta^*_{A_1 \ldots A_K} = \delta^B_{A_K} \theta^*_{A_1 \ldots A_{K-1}} - \delta^B_{A_{K-1}} \theta^*_{A_1 \ldots A_{K-2} A_K} + \ldots$

$$\ldots + (-1)^{K-1} \delta^B_{A_1} \theta^*_{A_2 \ldots A_K}$$

Lovelock theory is defined by the Langrangian density:

$$\mathcal{L} = \sum_{K=0}^{\left[\frac{D-1}{2}\right]} \alpha_K \mathcal{L}_{(K)} \quad , \text{ where}$$

$$\mathcal{L}_{(K)} = \underbrace{\mathcal{R}^{A_1 B_1} \wedge \ldots \wedge \mathcal{R}^{A_K B_K}}_{K} \wedge \theta^*_{A_1 B_1 \ldots A_K B_K} \text{ - } D \text{ form}$$

$$\mathcal{L}_{(0)} = \theta^* \qquad \mathcal{L}_{(1)} = \mathcal{R}^{A_1 B_1} \wedge \theta^*_{A_1 B_1}$$

$$\mathcal{L}_{(2)} = \mathcal{R}^{A_1 B_1} \wedge \mathcal{R}^{A_2 B_2} \wedge \theta^*_{A_1 B_1 A_2 B_2}$$

$$\mathcal{L}_{(0)} = \theta^* \quad \text{volume element.} \rightarrow \text{cosmological constant}$$

Hence

$$\mathcal{L}_{(1)} = \mathcal{R}^{A_1B_1} \wedge \theta^*_{A_1B_1} = R\,\theta^* \rightarrow \text{Einstein-Hilbert}$$

$$\mathcal{L}_{(2)} = \mathcal{R}^{A_1B_1} \wedge \mathcal{R}^{A_2B_2} \wedge \theta^*_{A_1B_1A_2B_2} =$$

$$= \left(R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} - 4R_{\mu\nu}R^{\mu\nu} + R^2\right)\theta^* \quad \text{Gauss-Bonnet}$$

For $D=2n$ the $K=n$ term is $\mathcal{L}_{(n)} = \mathcal{R}_{A_1B_1} \wedge \cdots \wedge \mathcal{R}_{A_nB_n}\, \varepsilon_{A_1B_1\ldots A_nB_n}$

and

$$\frac{1}{(4\pi)^n n!}\int_M \underset{\substack{\downarrow\\ \text{geometry}}}{\mathcal{L}_n} = \underset{\substack{\uparrow\\ \text{topology}}}{\chi(M)}$$

for a manifold without boundary.

Eg. In $D=4$ $\mathcal{L}_{(2)} = \hat{G}$ yields no dynamics to the field equations.

the reason lies in geometry of M.

theorema Egregium of Gauss 1828
Scalar Curvature of a surface depends only on $1^{\underline{st}}$ fund form

Suppose $\dim M = 2$ where $\partial M = \varphi$ (M orientable)

$$\int_{M_2} R \sim \chi(M_2) \quad \text{Euler number of } M_2$$

$\hat{C}$ **theorem**: Integral of a geometric term is a topological invariant

$\chi = 2 - 2h$

Remember g_s expansion

For $\dim M = 4$ $(\partial M = \varphi)$

$$\int_{M_4} \hat{G} = \chi(M_4) \quad \text{(Generalised) Euler number for } M_4$$

For every $\dim M = 2K$ (even dimensional manifold) $(\partial M = \varphi)$

Chern $\displaystyle\int_{M_{2K}} L(x) \sim \chi(M_{2K})$

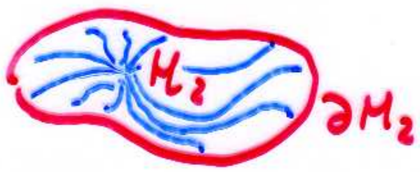
the reason lies in geometry of M.

theorema Egregium of Gauss 1828
Scalar Curvature of a surface depends only on
$1^{\underline{st}}$ Fund form

Suppose $\dim M = 2$ with boundary

$$\int_{\partial M_2} K + \int_{M_2} R \sim \chi(M_2)$$ Euler number of M_2

$\hat{C}$ theorem : Integral of a geometric term is a topological invariant



For $\dim M = 4$

$$\int_{\partial M_4} K(R+K^2) + \int_{M_4} \hat{G} = \chi(M_4)$$ (Generalised) Euler number for M_4

For every $\dim M = 2K$ (even dimensional manifold)

Chern $$\int_{\partial M_{2K}} \partial \mathcal{L} + \int_{M_{2K}} \mathcal{L}(x) \sim \chi(M_{2K})$$

Boundary term for EGB Myers 1987.

Lovelock theory is special because its Lagrangian densities are dim. continuation of Chern-Euler forms

The Chern forms give well-defined junction conditions. for manifolds with boundary.

therefore given $\int_{\mathcal{M}} \sum_{k=0}^{\left[\frac{D-1}{2}\right]} \alpha_k \mathcal{L}_{(k)}$ where $\mathcal{L}_{(k)} \sim \mathcal{R}^k$

variation with respect to θ^A gives Lovelock's equations

$$\sum_{k=0}^{\left[\frac{D-1}{2}\right]} \alpha_k \mathcal{E}_{(k)A} = -2 T_{AB} \theta^{*B}$$

with $\mathcal{E}_{(k)A} = \mathcal{R}^{A_1B_1} \wedge \dots \wedge \mathcal{R}^{A_kB_k} \wedge \theta^{*}_{AA_1B_1\dots A_kB_k}$

Example: $-\frac{1}{2} \mathcal{R}^{A_1B_1} \wedge \theta^{*}_{CA_1B_1} = G^A_C \theta^{*}_A$

Introduction

Einstein-Gauss-Bonnet

Given $\mathcal{L} = \mathcal{L}(M, g_{\mu\nu}, \nabla)$ and Einstein's postulates

Field equations are unique in $D=5$ or 6

$$\int d^{(5,6)}x \sqrt{-g}\, (-2\Lambda + R + \alpha \hat{G}) \qquad \text{where} \quad \hat{G} = R^2 - 4R_{\mu\nu}^2 + R_{\mu\nu\rho\sigma}^2$$

Gauss Bonnet term

Field eqns: $\mathcal{L}_{\mu\nu} =$

$$= G_{\mu\nu} + \Lambda g_{\mu\nu} - \alpha\left[\frac{g_{\mu\nu}}{2}\hat{G} - 2RR_{\mu\nu} + 4R_{\mu\gamma}R^{\gamma}{}_{\nu} + 4R_{\gamma\delta}R^{\gamma}{}_{\mu}{}^{\delta}{}_{\nu} - 2R_{\mu\gamma\delta\lambda}R_{\nu}{}^{\gamma\delta\lambda}\right] = 0$$

is the unique metric dependant tensor which is

- symmetric (Lanczos 1938)
- divergence free ie, $\nabla^{\mu}T_{\mu\nu} = 0$.
- depends up to - and is linear - to second derivatives in $g_{\mu\nu}$

wave type operator for the massless spin 2 graviton

- α has dimensions $[\text{length}]^2$ ... like string tension α'

$O(\alpha')$ correction for the heterotic string effective action around the vacuum.

$$\delta\mathcal{L}_{\mu\nu} = \pm\sqrt{1 + \frac{4\kappa\Lambda}{3}}\, \delta G_{\mu\nu}$$

Gross & Sloan 87, Metsaev & Tseytlin

Zwiebach 85

§3 A Staticity theorem.

(Y) Consider spacetime with a $D-2$ maximally symmetric subspace. We seek the general solution to the Lovelock field equations (in cosmological constant and EM vector background)

Max. symmetric space in $D-2$ dimensions. $\quad \dfrac{d\chi^2}{1-\kappa\chi^2} + \chi^2\, d\Omega^2_{D-3}$, $\kappa = 0, 1, -1$.

$$\text{where } d\Omega^2_{D-3} = d\theta^2 + \sin^2\theta\, d\Omega^2_{D-4}$$

$$\dots$$

$$d\Omega_1^2 = d\varphi^2$$

Note that a "Wick rotation" can give us a maximally symmetric spacetime adS, Mink or dS spacetime.

Example: For $\kappa = 1$ $\quad \theta \to \frac{\pi}{2} - it$ gives $\quad$ de Sitter spacetime

$$d\theta^2 + \sin^2\theta\, d\Omega^2 \longrightarrow -dt^2 + \cosh^2 t\, d\Omega^2$$

Spacetime metric $\quad ds^2 = e^{2\nu}\left(-dt^2 + dz^2\right) B^{-\frac{D-3}{D-2}} + B^{\frac{2}{D-2}}\left(\dfrac{d\chi^2}{1-\kappa\chi^2} + \chi^2 d\Omega^2_{D-3}\right)$

→ Class I

Theorem Under hypothesis (Y) the general solution admits an extra local timelike Killing vector $(\partial/\partial t)$

(Wiltshire, CC & Dufaux, R. Zegers)

Sketch of proof in $D=5$ and $Q=0$ (no charge)

The action reads: $S = \frac{1}{16\pi G_5} \int d^5x \sqrt{-g} \left(R - 2\Lambda + \hat{\alpha}\hat{G}\right)$

$$ds^2 = e^{2\nu(u,v)} B^{-2/3}_{(u,v)} (-4\,du\,dv) + B^{2/3}(u,v) \left(\frac{d\chi^2}{1-\kappa\chi^2} + \chi^2 d\Omega^2_{D-3}\right)$$

Field Eq^{ons} give:

$$\mathcal{I}\,(B_{uu} - 2\nu_u B_u) = 0$$
$$\mathcal{I}\,(B_{vv} - 2\nu_v B_v) = 0$$

where $\mathcal{I} = 9B^{4/3}e^{2\nu} + 36\alpha\kappa B^{2/3}e^{2\nu} + 4\alpha B_u B_v$

Two cases: Either $\mathcal{I}_1 = 0$ class 1 solutions.

We get $$ds^2 = -\frac{A(R)^2}{\kappa + \frac{R^2}{4\alpha}} dt^2 + \frac{dR^2}{\kappa + \frac{R^2}{4\alpha}} + R^2\left(\frac{d\chi^2}{1-\kappa\chi^2} + \chi^2 d\Omega^2\right)$$
with $A(R)$ undetermined and

$8\alpha K^2 = 1$ Fine tuning condition between α and the cosmo. const.

otherwise $B = B(U+V)$ $e^{2\nu} = B'U'V'$

$\begin{cases} u \to U(u) \\ v \to V(v) \end{cases}$ gauge symmetries

Apart from $8\kappa K^2 = 1$ (pathological case) we have a unique solution

$$ds^2 = -V(r)dt^2 + \frac{dr^2}{V(r)} + r^2\left(\frac{d\chi^2}{1-\kappa\chi^2} + \chi^2 d\Omega^2_{D-3}\right)$$ (Myers and Simon)

and $$F = \frac{q^2}{4\pi r^{2(D-2)}}\, dt \wedge dr$$

and $V(r) = \kappa - r^2 f(r)$ with $P(f) = \sum_{l=0}^{K} \hat{c}_l f^l = \frac{\mu}{r^{D-1}} - \frac{Q^2}{r^{2D-4}}$

ie, $f(r)$ is a root of a K-th order polynomial

$K = \left[\frac{D-1}{2}\right]$.

μ is an integration parameter associated to mass.
Q charge
and $P(f) = 0$ gives the flat, positive and negative vacua of the theory.

e.g.: $D=4$ $\quad P(f) = \hat{c}_0 + \hat{c}_1 f = \frac{\mu}{r^3} - \frac{Q^2}{r^4}$

Hence $V(r) = \kappa - r^2(\ldots)$

Stick to $D=5$ or $D=6$ where we have

Cosmological constant + Einstein + Gauss-Bonnet

$\mathcal{L}_0 \sim \Lambda$ $\qquad \mathcal{L}_1 \sim R$ $\qquad \mathcal{L}_2 \sim R^2 - 4R_{ab}^2 + R_{abcd}^2$

Summary of 1st lecture.

(Classical) modifications of GR conceivable both at the UV and the IR

In $D=4$ $\quad G_{\mu\nu} + 2\Lambda g_{\mu\nu} = 8\pi G T_{\mu\nu}$ are the unique metric field equations with $\nabla^\mu T_{\mu\nu} = 0$ and which are 2nd order PDE's.

In D dimensions the most general metric theory is Lovelock's theory.

$$S = \int d^D x \sum_{K=0}^{\left[\frac{D-1}{2}\right]} \mathcal{L}_{(K)}$$

$\mathcal{L}_0 \sim \Lambda \theta^*$ CC

$\mathcal{L}_1 \sim R \theta^*$ E-H

$\mathcal{L}_2 \sim \hat{G} \theta^*$ G-B

...

for $D=2n$ $\quad \int \mathcal{L}_{(n)} d^n x \sim \chi(\mathcal{M})$

$$\mathcal{L}_{(n)} = \underbrace{\mathcal{R}^{A_1 B_1} \wedge \ldots \wedge \mathcal{R}^{A_n B_n} \varepsilon_{A_1 \ldots B_n}}_{2n\text{-form}} \xrightarrow{d>n} \mathcal{L}_{(n)} = \underbrace{\mathcal{R}^{A_1 B_1} \wedge \ldots \wedge \mathcal{R}^{A_n B_n} \wedge \theta_{A_1 \ldots B_n}}_{2d\text{-form}}$$

Assuming $(D-2)$ maximally sym. subspaces, solutions to Lovelock's eq$^{\underline{ns}}$ have an extra local timelike Killing vector

$$ds^2 = A^2(t,r)\,(-4\,du\,dv) + B^2(t,r)\left(\frac{d\chi^2}{1-\kappa\chi^2} + \chi^2 d\Omega^2_{(D-3)}\right)$$

$u \longrightarrow U(u)$

$v \longrightarrow V(v)$

Brief reminder

Static Einstein black holes.

$$ds^2 = -h(r)\, dt^2 + \frac{dr^2}{h(r)} + r^2 \left(\frac{d\chi^2}{1-\kappa\chi^2} + \chi^2 d\Omega^2_{D-3} \right)$$

$$h(r) = \kappa - \frac{2\Lambda}{(D-1)(D-2)} r^2 - \frac{\mu}{r^{D-3}} + \frac{Q^2}{r^{2(D-3)}}$$

- For $\mu = Q^2 = 0$ we pick up vacua parametrised by Λ expressed in different coords parametrised by κ
  For $\Lambda < 0$, $\kappa = 1$ global patch of AdS
  $\Lambda > 0$, $\kappa = 1$ static patch of dS.

- For $\kappa = 0$ and $\Lambda < 0$ we have a planar black hole
  The horizon is toroidal or of more complicated topology.

- $\kappa = 1$ $\Lambda > 0$, $Q^2 = 0$, de-Sitter black hole
  Event horizon and a dSitter horizon

Boulware & Deser
Wiltshire
Myers & Simon
Cai

Einstein Gauss-Bonnet static
black holes.

$$I = \frac{1}{16\pi G}\int d^Dx \sqrt{-g}\,(R - 2\Lambda + \hat{\alpha}\hat{G}) - \frac{1}{4}\int d^Dx \sqrt{-g}\, \mathcal{F}^2$$

$$ds^2 = -V(r)\,dt^2 + \frac{dr^2}{V(r)} + r^2\left(\frac{d\chi^2}{1-\kappa\chi^2} + \chi^2 d\Omega^2_{D-3}\right)$$

$$\mathcal{F} = \frac{q^2}{4\pi r^{2(D-2)}}\; dt \wedge dr$$

$$V(r) = \kappa + \frac{r^2}{2\alpha}\left[1 + \varepsilon\sqrt{1 + 4\alpha\left(-K^2 + \frac{\mu}{r^{D-1}} - \frac{Q^2}{r^{2(D-2)}}\right)}\,\right]$$

$\alpha = (D-3)(D-4)\hat{\alpha}$ Gauss-Bonnet coupling $[\alpha] \sim \ell^2$

$2\Lambda = -(D-1)(D-2)K^2$ cosmological constant (<0)

$\mu = \dfrac{16\pi G M}{(D-2)\Sigma_K}$ mass M.

double branch!

$Q^2 = \dfrac{q^2}{2\pi(D-2)(D-3)}$ charge $\varepsilon = \pm 1$

- Vacua of the theory: Set $\mu = Q = K^2 = 0$ $\kappa = 1$.

Need $1 - 4\alpha K^2 \geq 0$.

For $\varepsilon = -1$ flat vacuum (Einstein branch)

$\varepsilon = 1$ Cosm. const vacuum $\sim \dfrac{1}{\alpha}$ (Gauss-Bonnet branch)

Both stable (Deser & Tekin)

Asymptotics:

Indeed for large r we get asymptotically Einstein solutions

for $\varepsilon=-1$, $V(r) \sim 1+K^2r^2-\frac{\mu}{r^{D-3}}-\frac{Q^2}{r^{2(D-3)}}$ ΛRN.

For $\varepsilon=1$ there is no Einstein limit (although the solutions have dS or adS asymptotics)

Zanelli

* Special case $4\alpha K^2=1$ single vacuum. (Chern-Simons)

Naked or curvature singularities. Two possible singularities

$$\left(r_0=0\right) \text{ or } \left\{ \begin{array}{l} r_1 \text{ such that} \\ \left(1+4\alpha\left(-K^2+\frac{\mu}{r_1^{D-1}}-\frac{Q^2}{r_1^{2(D-2)}}\right)=0\right) \end{array} \right\}$$

Horizons We have a black hole solution iff there exists r_H such that $V(r_H)=0$ and $r_H > (r_1, r_0)$

Indeed $r=r_H$ is a coord. singularity since (as usual)

$$dv_\pm = dt \pm \frac{dr}{V(r)} \quad \text{Kruskal extension}$$

(v_+, r) & (v_-, r) regular chart across future and past horizons.

read → Myers & Simon.

Proposition: $r = r_H$ is an event horizon iff $r_H > r_1, r_0$

and satisfies the following conditions:

$$\varepsilon\,(2\alpha\kappa + r_H^2) \leq 0 \quad (1)$$

and

(2) $r = r_H$ is <u>root of $\mathbb{P}_\alpha(x) = k^2 x^{2D-4} + \kappa x^{2D-6} - \mu x^{D-3} + Q^2 + \alpha k^2 x^{2D-8}$</u>

Therefore * $(1) \overset{\varepsilon=1}{\Longrightarrow} r_H^2 \leq -2\alpha\kappa$. bounded from above

Examples $\kappa = 0$ or $\kappa = 1$

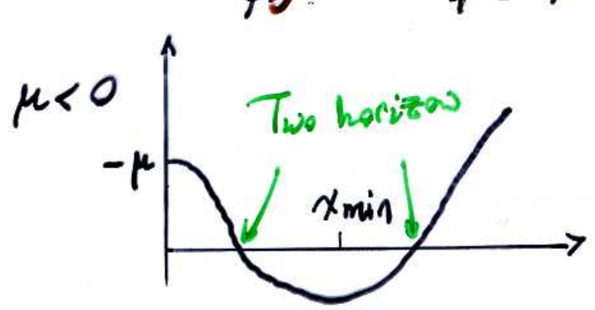
** $\mathbb{P}_{\alpha=0}(x)$ is the relevant ΛRN polynomial.

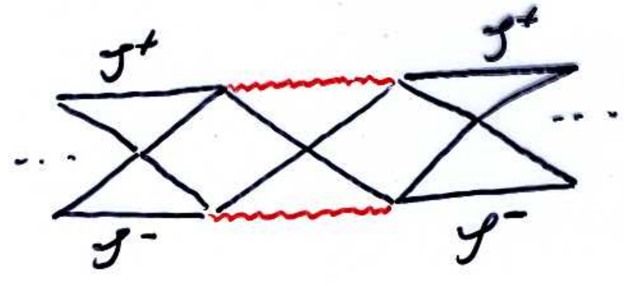
Example: For $\kappa = 0$ $r = r_H$ are the same as in the Einstein case!!

The position of the horizon only picks up corrections from the horizon curvature!

Example: Take $\underline{\varepsilon = 1}$ and $\underline{k = 0}$, $\underline{\kappa = 1}$ $(1) \Rightarrow r_H^2 \leq -2\alpha$ hence $\alpha = -\beta < 0$.

Need $1 - 4\beta \dfrac{\mu}{r^5} + 4\beta \dfrac{Q^2}{r^8} > 0$.





Myers & Simon
Clunan, Ross & Smith.

Thermodynamics and geometry. $Q=0$

Mass : $M = \frac{(D-2)\Sigma_k}{16\pi G}\left(\kappa + r_H^2 K^2 + \frac{\alpha}{r_H^2}\right) r_H^{D-3}$

For temperature we will rotate $t \to i\tau$ and ask for the resulting Euclidean manifold to be smooth ie τ is an angular coordinate with no conical singularity. $r = r_H$

Define $R \underset{r\to r_H}{=} \frac{2\sqrt{V}}{V_H'}$

$$ds^2 \underset{r\to r_H}{\sim} \left(\frac{V_H' R}{2}\right)^2 d\tau^2 + \left(\frac{V_H'}{V'}\right)^2 dR^2 + \ldots$$

compare with $dR^2 + R^2 d\theta^2$ flat space $r \to r_H$

Identify $\frac{V_H'}{2}\tau \sim \frac{V_H'}{2}\tau + 2\pi \Rightarrow$ Period $\beta = \frac{4\pi}{V_H'}$

and the temperature is its inverse $T = \frac{V_H'}{4\pi}$

$$T = \frac{k^2(D-1) r_H^4 + (D-3)\kappa r_H^2 + (D-5)\alpha\kappa^2}{4\pi r_H (r_H^2 + 2\alpha\kappa)}$$

Gibbons & Perry.

For entropy : $dM = T dS \Rightarrow S = \int T^{-1} dM = \int_{r_{min}}^{r_H} T^{-1} \frac{\partial M}{\partial \tilde{r}_H} d\tilde{r}_H$

$$S = \frac{r_H^{D-2}\Sigma_\kappa}{4G}\left(1 + \frac{2\alpha\kappa(D-2)}{(D-4) r_H^2}\right)$$

Entropy Area + $\hat{G}$ correction of the horizon surface.

Iyer & Wald

$S = \frac{1}{4G}\int d^{D-2}x \sqrt{\tilde{h}_{\mu\nu}}\,(1 + 2\alpha \tilde{R}^{ind})$, $\tilde{R}$ Ricci scalar of the horizon !! Extended Euler density of the horizon surface!

2 applications of the staticity theorem.

Consider a 3-brane with the energy-momentum tensor of a perfect fluid in $D=5$

$$T_\mu{}^\nu = \mathrm{diag}\left(-\rho(t), p(t), p(t), p(t)\right) \frac{\delta(z)}{\sqrt{g_{zz}}}$$

at $z=0$.

Hom. & Isot. Space.

$$ds^2 = \left(-dt^2 + dz^2\right) A(t,z) + B(t,z)\left(\frac{d\chi^2}{1-k\chi^2} + \chi^2 d\Omega_{II}^2\right)$$

| Corresponding metric symmetries.

time xtra dimension

$-4\,du\,dv$

2 dim. conformal symmetries $u \to U(u)$, $v \to V(v)$ in the bulk.

In the presence of the brane $U=V$

In the presence of a brane one of the two bulk gauge symmetries survives and the remaining physical degree of freedom is the brane trajectory $r = R(t)$ now evolving in a static background (EGB black hole)

$$\left(\frac{\rho}{16\alpha}\right)^2 = \left(H^2 + \frac{V(R)}{R^2}\right)^3 - \varepsilon C\left(H^2 + \frac{V}{R^2}\right)^2 + \frac{1}{4}C^2\left(H^2 + \frac{V}{R^2}\right)$$

with $C = \dfrac{3}{4\alpha}\left(1 - 8\alpha k^2 + \dfrac{8\alpha\mu}{R^4}\right)$

Codimension 2 braneworlds

Consider a double Wick rotation $\qquad D=6$

$$ds^2 = -V(r)\,dt^2 + \frac{dr^2}{V(r)} + r^2\left(\frac{d\chi^2}{1-\kappa\chi^2} + \chi^2 d\Omega_{II}^2\right) \quad \text{max sym space}$$

$$\downarrow \qquad\qquad\qquad\qquad \downarrow$$

$$ds^2 = V(r)\,d\theta^2 + \frac{dr^2}{V(r)} + r^2 h_{\mu\nu}\,dx^\mu dx^\nu \qquad \text{max sym s-t.}$$

where $h_{\mu\nu}$ is flat de-Sitter or adSitter space-time in 4 dimensions! (see Gregory & Padilla)

Codimension 2 braneworlds are characterised by conical singularities like cosmic strings in 4 dimensions.

The staticity theorem provides ALL the possible codimension 2 braneworld solutions. $\partial/\partial\theta$ of axial symmetry comes for free!

Proposition. * The possible positions of the brane are the zeros of the potential in other words $r = r_H$

** The junction equations will be given by

$$2\pi(1-\beta)\left[\delta^\mu_\nu + 4\alpha\, G^{\mu\,(ind)}_\nu\right] = S^\mu_\nu$$

$$\text{and} \quad T^{brane}_{\mu\nu} = S_{\mu\nu}\,\delta^{(2)}_{(y)} = S_{\mu\nu}\,\frac{\delta(r)}{2\pi r}.$$

*** In the presence of two horizons space will be dynamically compactified and we expect a localised graviton zero-mode ie effective 4 dimensional gravity on the brane.

Therefore candidate backgrounds are de-Sitter Schw. or RN since they have two horizons for their static patch.

Assume then $r_- < r < r_+$ with $V(r_\pm) = 0$.

$r = r_\pm$ defines the two brane positions

$$V \underset{r \sim r_\pm}{\sim} V(r_\pm) + \frac{1}{2}(r - r_\pm)\, V'(r_\pm)$$

$$ds^2 \underset{r \sim r_\pm}{\sim} \frac{1}{2}(r - r_\pm)\, V'(r_\pm)\, d\theta^2 + \frac{dr^2}{\frac{1}{2}(r - r_\pm) V'(r_\pm)} + \dots$$

$$\hookrightarrow \rho_\pm = \sqrt{\frac{2(r - r_\pm)}{V'_\pm}}$$

$$= \left(\frac{1}{4} V_\pm'^2\right) \rho_\pm^2\, d\theta^2 + d\rho_\pm^2 + \dots$$

flat space with deficit angle $\beta_\pm$

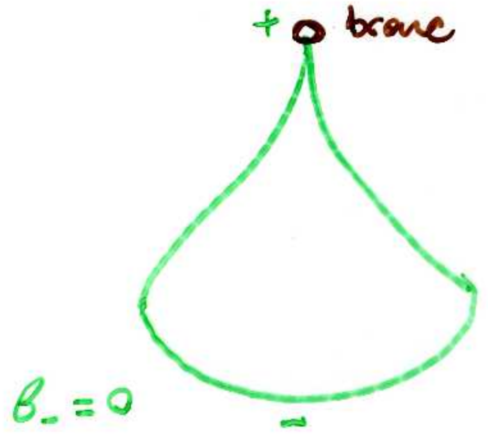
$$\frac{1}{2} V_\pm' \,\theta \overset{\text{ident}}{\sim} \frac{1}{2} V_\pm' \,\theta + (2\pi - \beta_\pm)$$

$$\hookrightarrow \quad \theta \sim \theta + \frac{2\pi - \beta_\pm}{\frac{1}{2} V_\pm'}$$

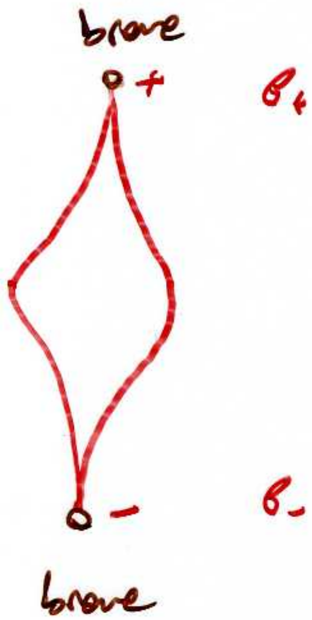
The period is the same at $\varphi_\pm$. In other words

$$\frac{2\pi - \delta_+}{\frac{1}{2} V_+} = \frac{2\pi - \delta_-}{\frac{1}{2} V_-}$$

This allows us to get rid of one conical deficit and keep 1 brane.



or



For Einstein case
check out Mukhoyama et.al.
Papazoglou et al.
Sorbo et.al.
Kobayashi et.al.

§5 The extended Kaluza-Klein reduction

Here we stick to $D=5$ and reduce to $d=4$.

For KK reduction we need that the $D=5$ metric has a Killing vector in the 5^{th} dimension.

Start with $$S = \frac{1}{16\pi G}\int d^5x\, \sqrt{-g^{(5)}}\,(R^{(5)} - 2\Lambda + \alpha\, \hat{G}^{(5)})$$

metric $$ds^2 = (g^{(5)}_{ab})\, dx^a dx^b =$$

$$= (g_{\mu\nu} + e^{-4\varphi} A_\mu A_\nu)\, dx^\mu dx^\nu + e^{-2\varphi}\, 2A_\mu dx^\mu dy + e^{-4\varphi} dy^2$$

Now integrate out the y-direction with periodic boundary conditions.

We obtain:

$$S_{eff} = \int d^4x\, \sqrt{-g}\; e^{-2\varphi} \Big\{ \overbrace{R - (\nabla\varphi)^2 - 2\Lambda e^{\gamma\varphi} - \tfrac{1}{4}F^2}^{\text{usual KK}} + \alpha\,\hat{G} + \frac{3\alpha}{16} e^{-8\varphi}\big[(F_{\mu\nu}F^{\mu\nu})^2 - 2F^\mu{}_\nu F^\nu{}_\lambda F^\lambda{}_\kappa F^\kappa{}_\mu\big]\Big\}$$

$$-\frac{1}{2}\int d^4x\, \sqrt{-g}\; e^{-6\varphi}\left(F_{\mu\nu}F^{\kappa\lambda}R^{\mu\nu}{}_{\kappa\lambda} - 4F_{\mu\kappa}F^{\nu\kappa}R^\mu{}_\nu - F_{\mu\nu}F^{\mu\nu}R\right)$$

Interaction term!

The reduced KK theory is the unique gauge field scalar tensor theory which is higher order (in power) and (only) yields second order field equations

Starting from a geometric 5 dimensional theory we have picked up the general (and hard to find) 4 dim theory with matter.

Freeze the dilaton: $\Phi = \text{const.}$

$$S_{eff} = \int d^4x \sqrt{-g} \quad R - 2\Lambda + \frac{3\kappa}{16}\left((\tilde{F}_{\mu\nu}F^{\mu\nu})^2 - 2\tilde{F}F\tilde{F}F \right) - \frac{1}{4}F^2$$

$$- \frac{1}{2}\int d^4x \sqrt{-g} \quad \left(F F R^{\mu\nu}{}_{\kappa\lambda} - 4 \tilde{F} \tilde{F} R^{\mu}{}_{\nu} - F \tilde{F} R \right)$$

Freeze geometry: $g_{\mu\nu} = \eta_{\mu\nu}$

$$S_{eff} = \int d^4x \sqrt{-g} \quad \left(-\frac{1}{4}F^2 + \frac{3\kappa}{16}\left[(\tilde{F}_{\mu\nu}F^{\mu\nu})^2 - 2 F \tilde{F} \tilde{F} F \right] \right)$$

generalised Maxwell theory.

Set $\alpha = \frac{3\gamma}{16e^2}$ (Kerner)

Field equations:

$$\partial_\lambda \left[F^{\lambda\rho} - \frac{3\gamma}{2e^2} F_{\mu\nu} F^{\mu\nu} F^{\lambda\rho} \right] + \frac{3\gamma}{e^2} \partial_\lambda \left[F_{\mu\nu} F^{\lambda\mu} F^{\rho\nu} \right] = 0$$

and $\partial_{[\mu} F_{\lambda\rho]} = 0$ as usual.

Also introducing $\vec{E}$ & $\vec{B}$.

$$\operatorname{div} \vec{E} = -\frac{3\gamma}{e^2} B \cdot \vec{\operatorname{grad}} (\vec{E} \cdot \vec{B})$$

$$\vec{\operatorname{rot}}\, \vec{B} = \frac{\partial \vec{E}}{\partial t} + \frac{3\gamma}{e^2} \left[\vec{B} \frac{\partial (\vec{E} \cdot \vec{B})}{\partial t} - \vec{E} \wedge \vec{\operatorname{grad}} (\vec{E} \cdot \vec{B}) \right]$$

$$\operatorname{div} \vec{B} = 0$$

$$\vec{\operatorname{rot}}\, \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

If $\vec{E} \cdot \vec{B} = 0$ ie $\vec{E} \perp \vec{B}$ higher order terms drop out

Hence EM wave solutions remain the same!

Induced charge density $\rho_{ind} = -\frac{3\gamma}{e^2} \vec{B} \cdot \vec{\operatorname{grad}} (\vec{E} \cdot \vec{B})$

Induced current $j_{ind} = \frac{3\gamma}{e^2} \left[\vec{B} \frac{\partial (\vec{E} \cdot \vec{B})}{\partial t} - \vec{E} \wedge \vec{\operatorname{grad}} (\vec{E} \cdot \vec{B}) \right]$

from EM fields themselves! with $\frac{\partial \rho_{ind}}{\partial t} + \operatorname{div} (\vec{j}_{ind}) = 0$

Freeze gauge field $\mathcal{F}=0$.

$$S_{eff} = \int d^4x \sqrt{-g}\, e^{-2\varphi} \left\{ R - (\nabla\varphi)^2 - 2\Lambda e^{\gamma\varphi} + \alpha \hat{G} \right\}$$

by a conformal rescaling we can go to the Einstein frame.

$$\overset{*}{S_{eff}} = \int d^4x \sqrt{-g} \left(R - \lambda (\nabla\varphi)^2 - 2\Lambda e^{\tilde{\gamma}\varphi} + \alpha\, e^{\delta\varphi} \hat{G} \right)$$

So if there is a scalar field in the action $\hat{G}$ term is no longer non-dynamical and will affect the field equations.

For example in d.E. models it appears as the leading order interaction between gravity and the quint. field φ. This can give solar system constraints for the d.e. model.

* Numerical solutions for b.h. by Kanti et.al.

A bunch of Open problems.

Stationary black holes.
Kerr-Myers Perry.

The Black string. (Kobayashi & Tanaka)

Perturbation theory

KK solutions

Cod 2 Braneworld cosmology etc

Conclusions.

Lovelock theory is well motivated but hard to handle beyond a given symmetry.

Should we use rather differential forms?

A technical breakthrough is needed in order to find solutions of axial symmetry.

Intriguing junction conditions and relation to horizon entropy formula!

Perturbation theory & stability