

Higher order gravity theories

and  
their black hole solutions.

§1 Introduction - Motivations

§2 Lovelock theory basics

§3 A staticity theorem.

• Static Black holes

• Thermodynamics and geometry

§4 Applications

§5 The extended KK reduction

§6 A bunch of open problems

• Conclusions



# Introduction

Given  $\mathcal{L} = \mathcal{L}(M, g_{\mu\nu}, \nabla)$  and Einstein's postulates

Field equations are unique in  $D=4$

$$\int_M d^4x \sqrt{-g} (-2\Lambda + R) \xrightarrow{\delta g_{\mu\nu}} G_{\mu\nu} + \Lambda g_{\mu\nu} = T_{\mu\nu} \quad (8\pi G = 1)$$

Therefore  $G_{\mu\nu} + \Lambda g_{\mu\nu}$

is the unique metric dependant tensor which is

(Cartan n30)

- symmetric
  - divergence free ie,  $\nabla^\mu T_{\mu\nu} = 0$ .
  - depends up to - and is linear - to second derivatives in  $g_{\mu\nu}$
- wave type operator for the massless spin 2 graviton

$$\partial_{ab} \rightarrow \delta^2 \partial_{ab} \quad G_{ab} \rightarrow \delta G_{ab}$$



# Introduction

Given  $\mathcal{L} = \mathcal{L}(M, g_{\mu\nu}, \nabla)$  and Einstein's postulates

Field equations are unique in  $D$  dimensions (Lovelock's)

$$\int d^D x \sqrt{-g} \sum_{k=0}^{\lfloor \frac{D-1}{2} \rfloor} \alpha_k \mathcal{L}^{(k)}$$

$$k=0 \quad \mathcal{L}_0 = \Lambda$$

$$k=1 \quad \mathcal{L}_1 = R$$

$$k=2 \quad \mathcal{L}_2 = \hat{G}$$

etc.

is the unique metric dependant tensor which is

- symmetric
- divergence free ie,  $\nabla^\mu T_{\mu\nu} = 0$ .
- depends up to- and is linear- to second derivatives in  $g_{\mu\nu}$

wave type operator for the massless spin 2 graviton

$$\text{where } \mathcal{L}^{(n)} = \mathcal{R}^{A_1 B_1} \wedge \dots \wedge \mathcal{R}^{A_n B_n} \wedge \theta_{A_1 B_1}^* \wedge \dots \wedge \theta_{A_n B_n}^* \underset{\text{Sym}}{\sim} \mathcal{R}^K$$

$$\theta_{A_1 \dots A_n}^* = \frac{\epsilon_{A_1 \dots A_n D}}{(D-n)!} \theta^{A_{k+1}} \wedge \dots \wedge \theta^{AD} \quad \begin{matrix} 1 & 2 & \dots & K \\ \vdots & \vdots & \dots & \vdots \\ \vdots & \vdots & \dots & \vdots \\ K+1 & \dots & \dots & D \end{matrix}$$

$$\mathcal{R}^A{}_\beta = \frac{1}{2} R^A{}_{BCD} \theta^C \wedge \theta^D$$

therefore for arbitrary  $D$

The uniquely defined and most general classical theory satisfying Einstein's postulates is Lovelock theory

$$S = \sum_{K=0}^{\lfloor \frac{D-1}{2} \rfloor} \alpha_K \int d^D x \sqrt{-g} \mathcal{L}_K$$

$$\mathcal{L}_0 = -2\Lambda \text{ c.c.}$$

$$\mathcal{L}_1 = R \text{ Einstein-Hilbert}$$

$$\mathcal{L}_2 = \hat{G} \text{ Gauss-Barnet}$$

...

$$\mathcal{L}_K \sim R^K$$

- 2<sup>nd</sup> order field equations
- no ghost around vacuum (Zwiebach)
- Bianchi identities
- well defined junction conditions.

## §2 Lorentz theory basics

$\mathcal{L} = \mathcal{L}(M, g, \nabla)$   $M$  is a  $D$  dimensional manifold

$$g = g_{ab} dx^a dx^b, \nabla \text{ Levi-Civita connexion}$$

$\forall p \in M \quad e_A \in T_p M \quad g(e_A, e_B) = \eta_{AB}$  orthonormal basis.

$$\theta^A \in \Omega^1(TM) \quad \theta^A(e_B) = \delta_B^A \quad \text{or } g = \eta_{AB} \theta^A \otimes \theta^B$$

$$\theta^A = \theta^A_\mu dx^\mu \quad g_{\mu\nu} = \theta^A_\mu \theta^B_\nu$$

Consider now 1 forms  $\omega^A_\theta$

$$\text{Connection } d\theta^A = -\omega^A_\theta \wedge \theta^\theta, \quad \omega^A_\theta = -\omega^\theta A$$

$$\text{Curvature } \mathcal{R}^A_\theta = d\omega^A_\theta + \omega^A_\theta \wedge \omega^\theta$$

2-form

$$\mathcal{R}^A_\theta = \frac{1}{2} R^A_{\theta CD} \theta^C \wedge \theta^D$$

↑ Riemann curvature

$$\omega \in \Omega^{(k)}(TM), \text{ where } 0 \leq k \leq D \quad \omega = \omega_{A_1 \dots A_k}^{A_1 \dots A_k} \theta^{A_1} \wedge \dots \wedge \theta^{A_k}$$

Use  $\theta^A$  and the ' $\wedge$ ' product to construct higher order diff. forms



to construct  $D$ -forms out of  $\mathbb{R}^k$  need Hodge dual:

Hodge dual of  $\theta^{A_1} \wedge \dots \wedge \theta^{A_k} \in \Omega^k(TM)$

$$\theta^{A_1, \dots, A_k} = \frac{1}{(D-k)!} \varepsilon_{A_1 \dots A_k A_{k+1} \dots A_D} \theta^{A_{k+1}} \wedge \dots \wedge \theta^{A_D}$$

$\theta^{A_1, \dots, A_k}$  is a  $D-k$  form

$\theta^*$  is the volume element.

$$\text{Identity } \theta^0 \wedge \theta^{A_1, \dots, A_k} = \delta_{A_k}^B \theta^{A_1, \dots, A_{k-1}} - \delta_{A_{k-1}}^B \theta^{A_1, \dots, A_{k-2}} \wedge \theta^{A_k} + \dots + (-1)^{k-1} \delta_{A_1}^B \theta^{A_2, \dots, A_k}$$

Lorentz theory is defined by the Lagrangian density:

$$\mathcal{L} = \sum_{k=0}^{\lfloor \frac{D-1}{2} \rfloor} \alpha_k \mathcal{L}^{(k)}, \text{ where}$$

$$\mathcal{L}^{(k)} = \mathcal{R}^{A_1 B_1} \wedge \dots \wedge \mathcal{R}^{A_k B_k} \wedge \theta_{A_1 B_1, \dots, A_k B_k}^* - 0 \text{ form}$$

$$\mathcal{L}^{(0)} = \theta^* \quad \mathcal{L}^{(1)} = \mathcal{R}^{A_1 B_1} \wedge \theta_{A_1 B_1}^*$$

$$\mathcal{L}^{(2)} = \mathcal{R}^{A_1 B_1} \wedge \mathcal{R}^{A_2 B_2} \wedge \theta_{A_1 B_1, A_2 B_2}^*$$

$L_{(0)} = \Theta^*$  volume element.  $\rightarrow$  cosmological constant

Hence  $L_{(1)} = R^{A_1 B_1} \wedge \Theta^*_{A_1 B_1} = R \Theta^* \rightarrow$  Einstein-Hilbert

$L_{(2)} = R^{A_1 B_1} \wedge R^{A_2 B_2} \wedge \Theta^*_{A_1 B_1 A_2 B_2} =$

$= (R_{\mu\nu\rho\sigma} R^{\mu\nu\rho\sigma} - 4 R_{\mu\nu} R^{\mu\nu} + R^2) \Theta^*$  Gauss-Bonnet

For  $D=2n$  the  $K=n$  term is  $L_{(n)} = R_{A_1 B_1} \wedge \dots \wedge R_{A_n B_n} \Theta^*_{A_1 B_1 \dots A_n B_n}$

and  $\frac{1}{(4\pi)^n n!} \int_{\mathcal{M}} L_n = \chi(\mathcal{M})$

$\downarrow$  geometry  $\uparrow$  topology

for a manifold without boundary.

Eg. In  $D=4$   $L_{(2)} = \hat{G}$  yields no dynamics to the field equations.



the reason lies in geometry of  $M$ .

theorema egregium of Gauss 1828

Scalar Curvature of a surface depends only on 1<sup>st</sup> Fund form

Suppose  $\dim M = 2$  where  $\partial M = \emptyset$  ( $M$  orientable)

$$\int_{M_2} R \sim \chi(M_2) \quad \text{Euler number of } M_2$$

$\hat{G}$  theorem: Integral of a geometric term is a topological invariant

$$\chi = 2 - 2h$$



Remember  $g_0$  expansion

For  $\dim M = 4$  ( $\partial M = \emptyset$ )

$$\int_{M_4} \hat{G} = \chi(M_4) \quad (\text{Generalised}) \quad \text{Euler number for } M_4$$

For every  $\dim M = 2k$  (even dimensional manifold) ( $\partial M = \emptyset$ )

$$\int_{M_{2k}} L_{2k} \sim \chi(M_{2k})$$

Chern

