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Analytical calculation of quasi-normal modes of black holes

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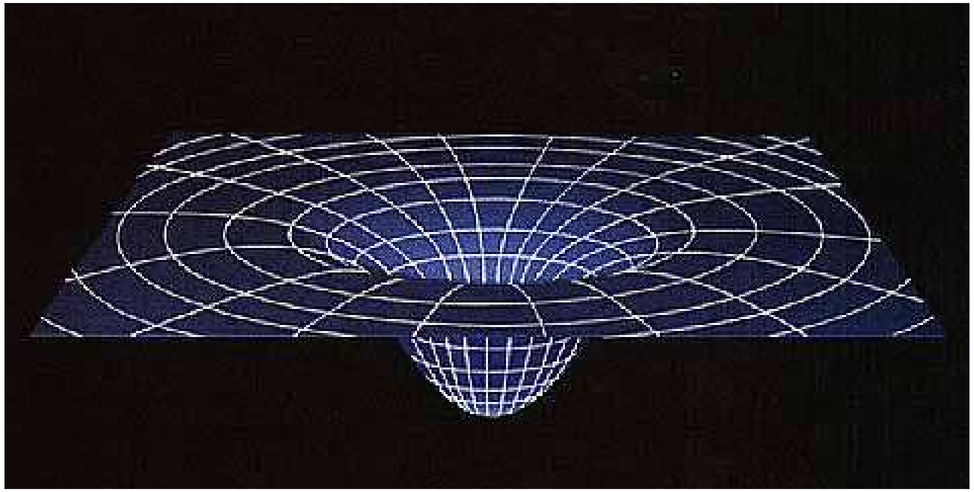
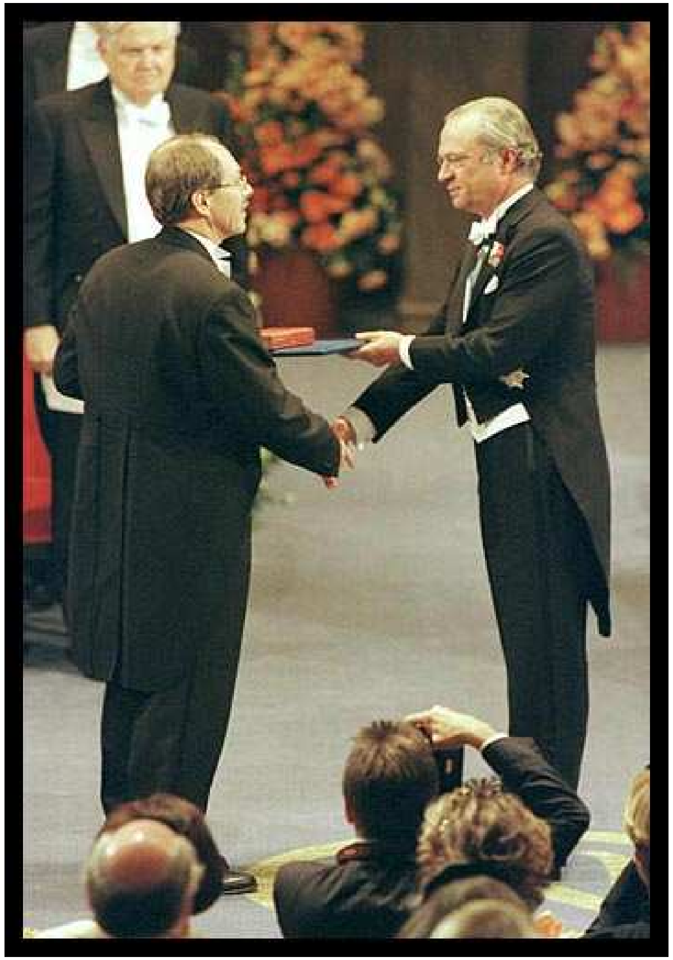
OUTLINE

- Introduction
- Flat spacetime
- AdS spacetime
- Unitarity
- Stability
- Hydrodynamics
- Conclusions





"To many practitioners of quantum gravity the black hole plays the role of a soliton, a non-perturbative field configuration that is added to the spectrum of particle-like objects only after the basic equations of their theory have been put down, much like what is done in gauge theories of elementary particles, where Yang-Mills equations with small coupling constants determine the small-distance structure, and solitons and instantons govern the large-distance behavior.

Such an attitude however is probably not correct in quantum gravity. The coupling constant increases with decreasing distance scale which implies that the smaller the distance scale, the stronger the influences of "solitons". At the Planck scale it may well be impossible to disentangle black holes from elementary particles."

– G. 't Hooft

Quasi-normal modes (QNMs) describe small perturbations of a black hole.

- A black hole is a thermodynamical system whose (Hawking) temperature and entropy are given in terms of its global characteristics (total mass, charge and angular momentum).

QNMs obtained by solving a wave equation for small fluctuations subject to the conditions that the flux be

- ingoing at the horizon and
- outgoing at asymptotic infinity.

$\Rightarrow$ discrete spectrum of complex frequencies.

- imaginary part determines the decay time of the small fluctuations

$$\Im\omega = \frac{1}{\tau}$$

Flat spacetime

Schwarzschild black holes

► study QNMs in asymptotically flat space-times

Metric:

$$ds^2 = -f(r)\,dt^2 + \frac{dr^2}{f(r)} + r^2 d\Omega^2 \ , \ \ f(r) = 1 - \frac{2GM}{r}$$

Hawking temperature:

$$T_H = \frac{1}{8\pi GM} = \frac{1}{4\pi r_0}$$

$r_0 = 2GM$: radius of horizon.

A spin-j perturbation of frequency ω is governed by the radial equation

$$-f(r)\frac{d}{dr}\left(f(r)\frac{d\Psi}{dr}\right) + V(r)\Psi = \omega^2\Psi$$

where $V(r)$ is the "Regge-Wheeler" potential

$$V(r) = f(r)\left(\frac{\ell(\ell+1)}{r^2} + \frac{(1-j^2)r_0}{r^3}\right)$$

j	Wave
0	scalar
1	electromagnetic
2	gravitational

avoid integer values of j throughout the discussion and only take the limit

$$j \to \text{ integer}$$

at the end of the calculation.

“tortoise coordinate”

$$r_* = \int \frac{dr}{f(r)} = r + r_0 \ln\left(\frac{r}{r_0} - 1\right)$$

Wave equation:

$$-\frac{d^2\Psi}{dr_*^2} + V(r(r_*))\Psi = \omega^2\Psi$$

to be solved along the entire real axis.

At both ends the potential vanishes

$$V \to 0 \ \text{ as } \ r_* \to \pm\infty$$

∴ solutions behave as

$$\Psi \sim e^{\pm i\omega r_*}$$

For QNMs, demand

$$\Psi \sim e^{\mp i\omega r_*} \quad , \quad r_* \to \pm\infty$$

assuming $\Re\omega > 0$.

Limit $\ell \to \infty$

[Ferrari, Mashhoon]

expand around the maximum of the potential: $V_0'(r_{max}) = 0$,

$$r_{max} = \frac{3}{2} r_0 + \mathcal{O}(1/\ell) .$$

$$V_0[r(r_*)] \approx \alpha^2 - \beta^2 (r_* - r_*(r_{max}))^2 ,$$

where

$$\alpha^2 = \frac{4}{27} \left(\ell + \tfrac{1}{2}\right) r_0^2 + \mathcal{O}(1/\ell)$$

$$\beta^2 = \frac{16}{729} \left(\ell + \tfrac{1}{2}\right) + \mathcal{O}(1/\ell) .$$

solutions

$$\Psi_n = H_n(\sqrt{i\beta}x)e^{i\beta x^2/2} \ , \ \ n = 0, 1, 2, \dots$$

where H_n are Hermite polynomials.
corresponding eigenvalues

$$\omega_n = \frac{2}{3\sqrt{3}\, r_0} \left\{ \ell + \tfrac{1}{2} + i(n + \tfrac{1}{2}) \right\} + \mathcal{O}(1/\ell)$$

in agreement with standard WKB approach

Limit $n \to \infty$

Asymptotic form of QNMs:

$$\frac{\omega_n}{T_H} = (2n+1)\pi i + \ln 3$$

- derived numerically
  [Chandrasekhar and Detweiler; Leaver; Nollert; Andersson; Bachelot and Motet-Bachelot]
- subsequently confirmed analytically
  [Motl and Neitzke]

$\Im\omega_n$ is large

$\Rightarrow$ numerical analysis cumbersome

$\Rightarrow$ easy to understand $\because$ spacing of frequencies is $2\pi i T_H$

- same as spacing of poles of a thermal Green function on the Schwarzschild black hole background.

$\Re\omega_n$ is small

- Analytical value proposed by Hod.

Number of microstates is related to Bekenstein-Hawking entropy

$$g_n = e^{S_{BH}} \sim k^n \quad , \quad k = 2, 3, \dots$$

[Mukhanov and Bekenstein]

spacing of eigenvalues

$$e^{\delta S_{BH}} = \frac{g_{n+1}}{g_n} \quad \Rightarrow \quad \delta S_{BH} = \ln k$$

Area spectrum of black holes

$$\delta A = 4G \ln k \quad , \quad k = 2, 3, \dots$$

since

$$S_{BH} = \frac{1}{4G} A$$

Bohr's correspondence principle

$$\delta M = \hbar \Re\omega$$

and first law of black hole mechanics

$$\delta M = T_H \delta S_{BH}$$

imply

$$\delta S = \frac{\Re \omega}{T_H} = \ln 3$$

$\therefore k = 3$

Intriguing value from LQG point of view:

⇒ gauge group should be $SO(3)$ rather than $SU(2)$
$\because k = 3$ instead of $k = 2$

⇒ The study of QNMs may lead to a deeper understanding of black holes and quantum gravity.

Analytical derivation of asymptotic form of QNMs by Motl and Neitzke offered a new surprise

$\because$ it heavily relied on the black hole singularity.

It is intriguing that the unobservable region beyond the horizon influences the behavior of physical quantities.

GOAL

- Calculate asymptotic formula for QNMs incl. first-order correction
  - by solving wave equation perturbatively for arbitrary spin of the wave.

We shall obtain agreement with results from

- numerical analysis for gravitational and scalar waves

  [Nollert; Berti, Kokkotas]

- a WKB analysis for gravitational waves.

  [Maassen van den Brink]

Let

$$\Psi = e^{-i\omega r_*} f(r_*)$$

$\therefore$

$$f(r_*) \sim 1 \;\; \text{as} \;\; r_* \to +\infty$$

and near the horizon

$$f(r_*) \sim e^{2i\omega r_*} \;\; \text{as} \;\; r_* \to -\infty$$

continue r analytically into the complex plane and define the boundary condition at the horizon in terms of the monodromy of $f(r_*(r))$ around the singular point $r = r_0$,

$$\mathcal{M}(r_0) = e^{-4\pi\omega r_0}$$

along a contour running counterclockwise.

Deform contour in complex r-plane so that it either lies

- beyond the horizon ($\mathrm{Re} r < r_0$) or
- at infinity ($r \to \infty$).

$\Rightarrow$ monodromy only gets a contribution from the segment lying beyond the horizon.

Change variables to

$$z = \omega(r_* - i\pi r_0) = \omega(r + r_0 \ln(1 - r/r_0))$$

(choose branch s.t. $z \to 0$ as $r \to 0$.)

The potential can be written as a series in $\sqrt{z}$,

$$V(z) = -\frac{\omega^2}{4z^2}\left(1 - j^2 + \frac{3\ell(\ell+1) + 1 - j^2}{3}\sqrt{-\frac{2z}{\omega r_0}} + \dots\right)$$

(formal expansion in $1/\sqrt{\omega}$).

Deform contour so that it gets mapped onto the real axis in the z-plane.

Near the singularity $z = 0$,

$$z \approx -\frac{\omega}{2r_0} r^2$$

Choose contour in r-plane so that near $r = 0$, positive (negative) real axis in z-plane are mapped onto

$$\arg r = \pi - \frac{\arg \omega}{2} \quad , \quad \arg r = \frac{3\pi}{2} - \frac{\arg \omega}{2}$$

in the r-plane, respectively.

Segments form a $\pi/2$ angle (independent of $\arg \omega$).

Avoid the $r = 0$ singularity: go around an arc of angle $3\pi/2$

$\Rightarrow$ angle of 3π around $z = 0$ in the z-plane.

Considering black hole singularity ($r = 0$)

$\Rightarrow$ two solutions

$$f_\pm(r) = r^{1\pm j} Z_\pm(r)$$

$Z_\pm$: analytic functions of r.

Go around an arc of angle of $3\pi/2$,

$$f_\pm(e^{3\pi i/2} r) = e^{3\pi(1\pm j)i/2} f_\pm(r)$$

Exact result!

To proceed further, relate

- behavior near the black hole singularity to
- behavior at large r in the complex r-plane.

Solve the wave equation perturbatively.

$\Rightarrow$ write wavefunction as a perturbation series in $1/\sqrt{\omega}$.

Zeroth order:

$$\frac{d^2\Psi^{(0)}}{dz^2} + \left(\frac{1-j^2}{4z^2} + 1\right)\Psi^{(0)} = 0$$

Solutions:

$$f_\pm^{(0)}(z) = e^{iz}\,\Psi_\pm^{(0)} = e^{iz}\sqrt{\frac{\pi z}{2}}J_{\pm j/2}(z)$$

behavior at infinity ($z \to \infty$)

$$f_\pm^{(0)}(z) \sim e^{iz}\cos(z - \pi(1 \pm j)/4)$$

B.C. $\Rightarrow$

$$f(z) \sim \text{const. as } z \to \infty$$

along the positive real axis in the z-plane.

$\therefore$ adopt linear combination

$$f^{(0)} = f_+^{(0)} - e^{-\pi j i/2}\,f_-^{(0)} \sim e^{iz}\sqrt{z}\,H_{j/2}^{(1)}(z)$$

As desired,

$$f^{(0)}(z) \sim -e^{-\pi(1+j)i/4}\sin(\pi j/2)$$

Go along the 3π arc around $z = 0$ in the z-plane

$$f^{(0)}(e^{3\pi i}z) = e^{3\pi(1+j)i/2}\left(f_+^{(0)}(z) - e^{-7\pi ji/2}\, f_-^{(0)}(z)\right)$$

as $z \to \infty$,

$$\begin{aligned} f^{(0)}(z) \sim \; & e^{-\pi(1+j)i/4}\sin(3\pi j/2) \\ & + e^{\pi(1-j)i/4}\sin(2\pi j)e^{2iz} \end{aligned}$$

Monodromy to zeroth order

$$\mathcal{M}(r_0) = -\frac{\sin(3\pi j/2)}{\sin(\pi j/2)} = -(1 + 2\cos(\pi j))$$

$\Rightarrow$ discrete set of complex frequencies (QNMs)

$$\boxed{\frac{\omega_n}{T_H} = (2n+1)\pi i + \ln(1 + 2\cos(\pi j)) + o(1/\sqrt{n})}$$

[Motl and Neitzke]

First order

expand wavefunction in $1/\sqrt{\omega}$

$$\Psi = \Psi^{(0)} + \frac{1}{\sqrt{-\omega r_0}} \Psi^{(1)} + o(1/\omega)$$

First-order correction obeys

$$\frac{d^2\Psi^{(1)}}{dz^2} + \left(\frac{1-j^2}{4z^2} + 1 \right) \Psi^{(1)} = \sqrt{-\omega r_0}\, \delta V \Psi^{(0)}$$

$$\delta V(z) = \frac{1-j^2}{4z^2} + \frac{1}{\omega^2} V(r(z))$$

Solutions:

$$\begin{aligned} \Psi_\pm^{(1)}(z) &= \mathcal{C}\Psi_+^{(0)}(z) \int_0^z \Psi_-^{(0)} \delta V \Psi_\pm^{(0)} \\ &\quad - \mathcal{C}\Psi_-^{(0)}(z) \int_0^z \Psi_+^{(0)} \delta V \Psi_\pm^{(0)} \end{aligned}$$

$$\mathcal{C} = \frac{\sqrt{-\omega r_0}}{\sin(\pi j/2)}$$

integral along positive real axis on z-plane
($z > 0$).

Large-z behavior:

$$\begin{aligned}\Psi_{\pm}^{(1)}(z) \;\sim\; & c_{-\pm}\cos(z - \pi(1+j)/4) \\ & -c_{+\pm}\cos(z - \pi(1-j)/4)\end{aligned}$$

$$c_{\pm\pm} = \mathcal{C}\int_0^\infty \Psi_{\pm}^{(0)}\delta V \Psi_{\pm}^{(0)}$$

For small-z behavior, expand

$$\delta V(z) = -\frac{3\ell(\ell+1)+1-j^2}{6\sqrt{-2\omega r_0}}\, z^{-3/2} + o(1/\omega)$$

It follows that

$$\Psi_{\pm}^{(1)} = z^{1\pm j/2} G_{\pm}(z) + o(1/\omega)$$

$G_{\pm}$ even analytic functions of z.

For desired behavior as $z \to \infty$, define

$$\Psi = \Psi_+^{(0)} + \frac{1}{\sqrt{-\omega r_0}} \left\{ \Psi_+^{(1)} - e^{-\pi j i/2} \Psi_-^{(1)} + e^{-\pi j i/2} \xi \Psi_-^{(0)} \right\} + \dots$$

- $\xi \sim \mathcal{O}(1)$
- dots represent terms of order higher than $\mathcal{O}(1/\sqrt{\omega})$.

Demanding

$$\Psi \sim e^{-iz} \text{ as } z \to +\infty$$

fixes

$$\xi = \xi_+ + \xi_- \ , \ \xi_+ = c_{++} e^{\pi j i/2} - c_{+-}$$

$$\xi_- = c_{--} e^{-\pi j i/2} - c_{+-}$$

Then $f = e^{iz}\Psi \sim$ const. as $z \to \infty$,

$$f(z) \sim -e^{-\pi(1+j)i/4} \sin(\pi j/2) \left\{ 1 - \frac{\xi_-}{\sqrt{-\omega r_0}} \right\}$$

In the neighborhood of the black hole singularity (around $z = 0$), go around a 3π arc,

$$\Psi_{\pm}^{(1)}(e^{3\pi i}z) = e^{3\pi(2\pm j)i/2}\Psi_{\pm}^{(1)}(z)$$

$\therefore$

$$\Psi(e^{3\pi i}z) = \Psi^{(0)}(e^{3\pi i}z)$$

$$-ie^{3\pi(1+j)i/2}\frac{1}{\sqrt{-\omega r_0}}\left\{\Psi_{+}^{(1)}(z)\right.$$

$$\left.-e^{-7\pi ji/2}(\Psi_{-}^{(1)}(z) - i\xi\Psi_{-}^{(0)}(z))\right\}$$

As $z \to \infty$ along the real axis,

$$\begin{aligned} f(z) \;\sim\; & e^{-\pi(1+j)i/4}\sin(3\pi j/2)\left\{1 - \frac{1}{\sqrt{-\omega r_0}}A\right\} \\ & + e^{\pi(1-j)i/4}\sin(2\pi j)\left\{1 - \frac{1}{\sqrt{-\omega r_0}}B\right\}e^{2iz} \end{aligned}$$

where

$$A = \frac{i-1}{2}\, e^{\pi j i/2}\, \left(\xi_+ + i\xi_- - \xi\, \cot(3\pi j/2)\right)$$

and B is not needed.

monodromy to this order:

$$\mathcal{M}(r_0) = -\frac{\sin(3\pi j/2)}{\sin(\pi j/2)}$$

$$\times \left\{ 1 + \frac{i-1}{2\sqrt{-\omega r_0}}\, e^{\pi j i/2}\, \left(\xi_- - \xi_+ + \xi \cot(3\pi j/2)\right) \right\}$$

∴ QNM frequencies

$$\boxed{\frac{\omega_n}{T_H} = (2n+1)\pi i + \ln(1 + 2\cos(\pi j)) + \frac{e^{\pi j i/2}}{\sqrt{n+1/2}} (\xi_- - \xi_+ + \xi \cot(3\pi j/2)) + \mathcal{O}(1/n)}$$

(includes $\mathcal{O}(1/\sqrt{n})$ correction to original $\mathcal{O}(1)$ asymptotic formula)

For explicit expression, use

$$\mathcal{J}(\nu,\mu) \equiv \int_0^\infty dz\, z^{-1/2} J_\nu(z) J_\mu(z) = \frac{\sqrt{\pi/2}\,\Gamma(\frac{\nu+\mu+1/2}{2})}{\Gamma(\frac{-\nu+\mu+3/2}{2})\Gamma(\frac{\nu+\mu+3/2}{2})\Gamma(\frac{\nu-\mu+3/2}{2})}$$

We obtain

$$c_{\pm\pm} = \pi \frac{3\ell(\ell+1)+1-j^2}{12\sqrt{2}\,\sin(\pi j/2)} \mathcal{J}(\pm j/2, \pm j/2)$$

$\therefore$

$$\xi_- - \xi_+ + \xi \cot(3\pi j/2) = (1-i)\,\frac{3\ell(\ell+1)+1-j^2}{24\sqrt{2}\pi^{3/2}}\,\frac{\sin(2\pi j)}{\sin(3\pi j/2)}\,\Gamma^2(1/4)\,\Gamma(1/4+j/2)\,\Gamma(1/4-j/2)$$

[Musiri, Siopsis]

where we used the identity

$$\Gamma(y)\Gamma(1-y) = \frac{\pi}{\sin(\pi y)}$$

$\Rightarrow$ well-defined finite limit as $j \to$ integer.

Scalar waves

$\boxed{j \to 0^+}$

$$\frac{\omega_n}{T_H} = (2n+1)\pi i + \ln 3 + \frac{1-i}{\sqrt{n+1/2}} \frac{\ell(\ell+1)+1/3}{6\sqrt{2}\pi^{3/2}} \Gamma^4(1/4) + \mathcal{O}(1/n)$$

in agreement with numerical results

[Berti and Kokkotas]

Gravitational waves

$\boxed{j \to 2}$

$$\frac{\omega_n}{T_H} = (2n+1)\pi i + \ln 3 + \frac{1-i}{\sqrt{n+1/2}} \frac{\ell(\ell+1)-1}{18\sqrt{2}\pi^{3/2}} \Gamma^4(1/4) + \mathcal{O}(1/n)$$

in agreement with the results from

- a WKB analysis

[Maassen van den Brink]

- numerical analysis

[Nollert]

Kerr black holes

Extend above to rotating (Kerr) black holes

- NOT straightforward!

Bohr's correspondence principle

$$\delta M = \hbar \Re\omega$$

and first law of black hole mechanics

$$\delta M = T_H \delta S_{BH} + \Omega \delta J$$

$\Rightarrow$ asymptotic expression

[Hod]

$$\Re\omega = T_H \ln 3 + m\Omega$$

m: azimuthal eigenvalue of wave
Ω: angular velocity of horizon.

Some numerical results $\Rightarrow$

[Berti, Cardoso, Kokkotas, Onozawa]

$$\Re\omega = m\Omega$$

CONFLICT!

GOAL

Analytic solution to the wave (Teukolsky) equation

- valid for asymptotic modes bounded from above by $1/a$

$$a = \frac{J}{M}$$

  J: angular momentum, M: mass of Kerr black hole.

Calculation valid for

$$a \ll 1$$

includes Schwarzschild case ($a = 0$).

Results

[Musiri, Siopsis]

- confirm Hod's expression
- do not necessarily contradict numerical results (may be valid in asymptotic regime $1/a \lesssim \omega$)
- In Schwarzschild limit ($a = 0$)
  - range of frequencies extends to infinity
  - our expression reduces to the expected form

Metric

$$
\begin{aligned}
ds^2 = & -\left(1-\frac{2Mr}{\Sigma}\right)dt^2+\frac{4Mar\sin^2\theta}{\Sigma}\,dtd\phi+\frac{\Sigma}{\Delta}\,dr^2 \\
& + \Sigma d\theta^2+\sin^2\theta\left(r^2+a^2+\frac{2Ma^2r\sin^2\theta}{\Sigma}\right)\,d\phi^2
\end{aligned}
$$

$$
\Sigma = r^2+a^2\cos^2\theta \ , \ \ \Delta = r^2-2Mr+a^2 = (r-r_-)(r-r_+)
$$

M: mass of black hole
Newton's constant $G = 1$.

Angular velocity

$$
\Omega = \frac{a}{2Mr_+}
$$

Hawking temperature

$$
T_H = \frac{1-r_-/r_+}{8\pi M}
$$

Small perturbations are governed by the Teukolsky wave equation

$$\left(\frac{(r^2+a^2)^2}{\Delta}-a^2\sin^2\theta\right)\frac{\partial^2\Psi}{\partial t^2}+\frac{4Mar}{\Delta}\frac{\partial^2\Psi}{\partial t\partial\phi}+\left(\frac{a^2}{\Delta}-\frac{1}{\sin^2\theta}\right)\frac{\partial^2\Psi}{\partial\phi^2}$$

$$-\frac{1}{\Delta^s}\frac{\partial}{\partial r}\left(\Delta^{s+1}\frac{\partial\Psi}{\partial r}\right)-2s\left(\frac{M(r^2-a^2)}{\Delta}-r-ia\cos\theta\right)\frac{\partial\Psi}{\partial t}$$

$$-\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\Psi}{\partial\theta}\right)-2s\left(\frac{a(r-M)}{\Delta}+\frac{i\cos\theta}{\sin^2\theta}\right)\frac{\partial\Psi}{\partial\phi}+(s^2\cot^2\theta-s)\Psi=0$$

s	Wave
0	scalar
-1	electromagnetic
-2	gravitational

Solution

$$\Psi=e^{-i\omega t}e^{im\phi}S(\theta)f(r)$$

Angular equation:

$$\frac{1}{\sin\theta}(\sin\theta\, S')' + \left(a^2\omega^2\cos^2\theta - \frac{m^2}{\sin^2\theta} - 2a\omega s\cos\theta - \frac{2ms\cos\theta}{\sin^2\theta} - s^2\cot^2\theta\right) S = -(A+s)S$$

A: separation constant (eigenvalue)

Radial equation:

$$\frac{1}{\Delta^s}(\Delta^{s+1} f')' + V(r) f = (A + a^2\omega^2) f$$

where

$$\Delta V(r) = (r^2+a^2)^2\omega^2 - 4aMr\omega m + a^2m^2 + 2ia(r-M)ms - 2iM(r^2-a^2)\omega s + 2ir\omega s\Delta$$

simplify by placing horizon at $r = 1$

$$2M = 1 + a^2 \ , \quad r_- = a^2 \ , \ r_+ = 1$$

Solve the two wave equations by expanding in a

- keep terms up to $o(a)$
- assume ω is large but bounded from above by $1/a$,

$$1 \lesssim \omega \lesssim 1/a$$

  ω is in an intermediate range
  (asymptotic in Schwarzschild limit $a \to 0$)

Solutions to angular equation to lowest order: spin-weighted spherical harmonics, and

$$A = \ell(\ell+1) - s(s+1) + o(a\omega)$$

Near the horizon ($r \to 1$)

$$f(r) \sim (r-1)^{\lambda} \ , \ \ \lambda = i(\omega - am) + o(1/\omega)$$

At infinity ($r \to \infty$)

$$f(r) \sim e^{i\omega r}$$

Introduce "tortoise coordinate"

$$z = \omega r + (\omega - am)\ln(r-1)$$

$\therefore$ boundary conditions

$$f(z) \sim e^{\pm iz} \ , \ \ z \to \pm\infty$$

Define B.C. at horizon. . .

- Observe

$$\mathcal{F}(z) \equiv e^{iz} f(z) \sim \text{const. as } z \to \infty$$

Monodromy of $\mathcal{F}$ around the singular point $r = 1$ to this order ($o(a)$)

$$\mathcal{M}(1) = e^{4\pi(\omega - am)}$$

To express the radial equation in terms of the tortoise coordinate, define

$$f(r) = \Delta_0^{-s/2} \frac{R(r)}{\sqrt{r(\omega r - am)}}$$

$\Delta_0 = r(r-1)$ (NB: $\Delta = \Delta_0 + o(a^2)$).

Inverting $z = z(r)$,

$$r = \sqrt{-\frac{2z}{\omega}} + o(1/\omega)$$

$\Rightarrow$ radial equation to lowest order in $1/\sqrt{\omega}$ in terms of R,

$$\frac{d^2R}{dz^2} + \left\{ 1 + \frac{3is}{2z} + \frac{4 - s^2 - 4iams}{16z^2} \right\} R = 0$$

to be solved along the entire real axis.

Whittaker's equation!

Solutions (set $x = 2iz$)

$$M_{\kappa,\pm\mu}(x) = e^{-x/2} x^{\pm\mu+1/2} M(\tfrac{1}{2} \pm \mu - \kappa, 1 \pm 2\mu, x)$$

$$\kappa = \frac{3s}{4} \quad , \quad \mu^2 = \frac{s(s + 4iam)}{16}$$

M: Kummer's function (also called Φ).

Need Whittaker's function:

$$W_{\kappa,\mu}(x) = \frac{\Gamma(-2\mu)}{\Gamma(\frac{1}{2}-\mu-\kappa)}\, M_{\kappa,\mu}(x) + \frac{\Gamma(2\mu)}{\Gamma(\frac{1}{2}+\mu-\kappa)}\, M_{\kappa,-\mu}(x)$$

due to its clean asymptotic behavior,

$$W_{\kappa,\mu}(x) \sim e^{-x/2}\, x^{\kappa}\, (1 + o(1/x))$$

as $|x| \to \infty$.

Compute monodromy by deforming contour as before.

Go around an arc of angle 3π

$$M_{\kappa,\pm\mu}(e^{3\pi i}x) = -ie^{\pm 3\pi i\mu} M_{-\kappa,\pm\mu}(x)$$

where we used

$$M(a,b,-x) = e^{-x} M(b-a,b,x)$$

$\therefore$

$$W_{\kappa,\mu}(e^{3\pi i}x) = -ie^{3\pi i\mu}\, \frac{\Gamma(-2\mu)}{\Gamma(\frac{1}{2}-\mu-\kappa)}\, M_{-\kappa,\mu}(x) - ie^{-3\pi i\mu}\, \frac{\Gamma(2\mu)}{\Gamma(\frac{1}{2}+\mu-\kappa)}\, M_{-\kappa,-\mu}(x)$$

For asymptotic behavior, we need

$$M_{-\kappa,\mu}(x) = \frac{\Gamma(1+2\mu)}{\Gamma(\frac{1}{2}+\mu+\kappa)}\, e^{-i\pi\kappa} W_{\kappa,\mu}(e^{i\pi}x) + \frac{\Gamma(1+2\mu)}{\Gamma(\frac{1}{2}+\mu-\kappa)}\, e^{-i\pi(\frac{1}{2}+\mu+\kappa)} W_{-\kappa,\mu}(x)$$

As $|x| \to \infty$,

$$M_{-\kappa,\mu}(x) \sim \frac{\Gamma(1+2\mu)}{\Gamma(\frac{1}{2}+\mu+\kappa)}\, e^{-i\pi\kappa}\, e^{x/2}\,(-x)^{\kappa} + \frac{\Gamma(1+2\mu)}{\Gamma(\frac{1}{2}+\mu-\kappa)}\, e^{-i\pi(\frac{1}{2}+\mu+\kappa)} e^{-x/2}\, x^{-\kappa}$$

$\therefore$

$$W_{\kappa,\mu}(e^{3\pi i}x) \sim \mathcal{A} e^{x/2} x^{\kappa} + \mathcal{B} e^{-x/2} x^{-\kappa}$$

$$\mathcal{A} = -ie^{3\pi i\mu}\, \frac{\Gamma(-2\mu)}{\Gamma(\frac{1}{2}-\mu-\kappa)}\, \frac{\Gamma(1+2\mu)}{\Gamma(\frac{1}{2}+\mu+\kappa)}\, e^{-\pi i\kappa} + (\mu \to -\mu)$$

$$\mathcal{B} = -ie^{3\pi i\mu}\, \frac{\Gamma(-2\mu)}{\Gamma(\frac{1}{2}-\mu-\kappa)}\, \frac{\Gamma(1+2\mu)}{\Gamma(\frac{1}{2}+\mu-\kappa)}\, e^{-i\pi(\frac{1}{2}+\mu+\kappa)} + (\mu \to -\mu)$$

After some algebra,

$$\mathcal{A} = -(1+2\cos\pi s) + o(a^2)$$

where we used the identities

$$\Gamma(1-x)\Gamma(x) = \frac{\pi}{\sin\pi x}\ ,\quad \Gamma(\tfrac{1}{2}+x)\Gamma(\tfrac{1}{2}-x) = \frac{\pi}{\cos\pi x}$$

- correct Schwarzschild limit
- no $\mathcal{O}(a)$ corrections.

Monodromy around $r = 1$

$$\mathcal{M}(1) = e^{4\pi(\omega - ma)} = \mathcal{A}$$

$\therefore$

$$\boxed{\Re\omega = \frac{1}{4\pi}\ln(1 + 2\cos\pi s) + ma + o(a^2)}$$

[Musiri, Siopsis]

in agreement with Hod's formula

- for gravitational waves ($s = -2$)
- in the small-a limit

$$\Omega \approx a \ , \ \ T_H \approx \frac{1}{4\pi}$$

NB: QNMs bounded from above by $1/a$.

Half-integer spin

Need Teukolsky equation

[Khriplovich, Ruban]

Set $r_0 = 1$. Potential

$$V(r) = f(r)\left(\frac{\ell(\ell+1)}{r^2} + \frac{1}{r^3}\right) + \frac{2i\omega j}{r} - \frac{3i\omega j}{r^2} + \frac{j^2}{4r^4}$$

for a spin-j field (e.g., $j = 1/2$ for Dirac fermion).

Expand around singularity $z = \omega r_* = 0$,

$$\frac{1}{\omega^2}V(z) = \frac{3ij}{2z} - \frac{4-j^2}{16z^2} + \frac{\mathcal{A}}{\omega^{1/2}z^{3/2}} + \mathcal{O}(1/\omega) \ , \quad \mathcal{A} = \frac{\ell(\ell+1) + \frac{1-j^2}{3}}{2\sqrt{2}}$$

Zeroth-order wave equation

$$\frac{d^2\Psi}{dz^2} + \left[1 - \frac{3ij}{2z} - \frac{4-j^2}{16z^2}\right]\Psi = 0$$

solutions are Whittaker functions

$$\Psi_\pm^{(0)}(z) = M_{\lambda,\pm\mu}(-2iz) \ , \quad \lambda = \frac{3j}{4} \ , \ \mu = \frac{j}{4}$$

Calculation of monodromy $\Rightarrow$

$$\frac{\omega_n}{T_H} = -(2n+1)\pi i + \ln(1 + 2\cos\pi j) + \mathcal{O}(1/\sqrt{n})$$

in agreement with integer spin (Regge-Wheeler equation)!

For Dirac fermion,

$$\frac{\omega_n}{T_H} = -(2n+1)\pi i + \mathcal{O}(1/\sqrt{n}) \ , \quad j + \frac{1}{2} \in \mathbb{N}$$

$\Rightarrow$ asymptotically, real part vanishes.

First-order correction

[Musiri, Siopsis]

Result:

$$\frac{\omega_n}{T_H} = -(2n+1)\pi i + \ln(1 + 2\cos \pi j)$$

$$-\frac{2i}{\sqrt{-in/2}} \sin 4\pi\mu \, \frac{\bar{b}_+ A_- B_- + \bar{b}_- A_+ B_+}{e^{-4\pi i\mu} A_+ B_- - e^{4\pi i\mu} A_- B_+} + \mathcal{O}(1/n)$$

where

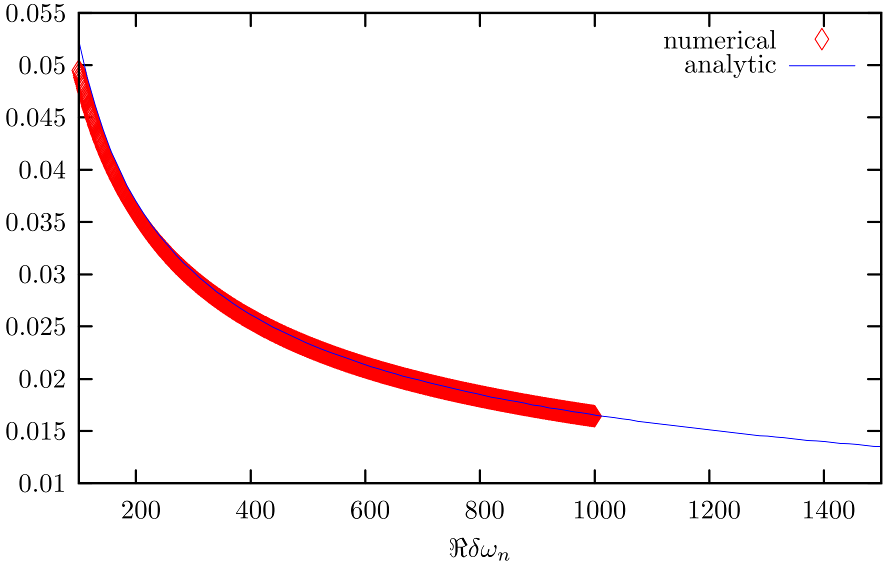
$$\bar{b}_\pm = \frac{\mathcal{A}}{4\mu} \int_0^\infty \frac{dz}{z^{3/2}} M_{\lambda,\pm\mu}(-2iz) M_{\lambda,\pm\mu}(-2iz)$$

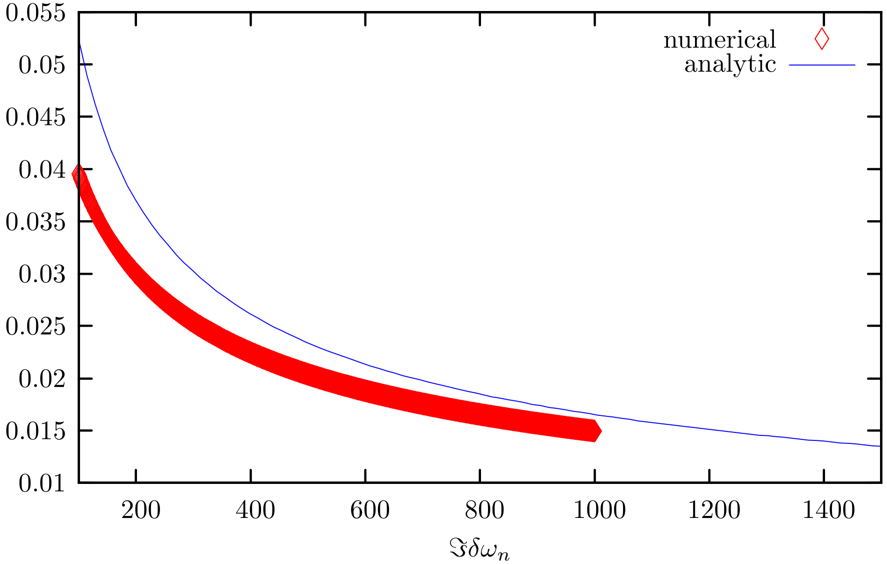
$$A_\pm = \frac{\Gamma(1 \pm 2\mu)}{\Gamma(\frac{1}{2} \pm \mu + \lambda)} e^{i\pi(\frac{1}{2} \pm \mu - \lambda)} \quad , \qquad B_\pm = \frac{\Gamma(1 \pm 2\mu)}{\Gamma(\frac{1}{2} \pm \mu - \lambda)} e^{-i\pi\lambda}$$

looks complicated, but look at cases. . .

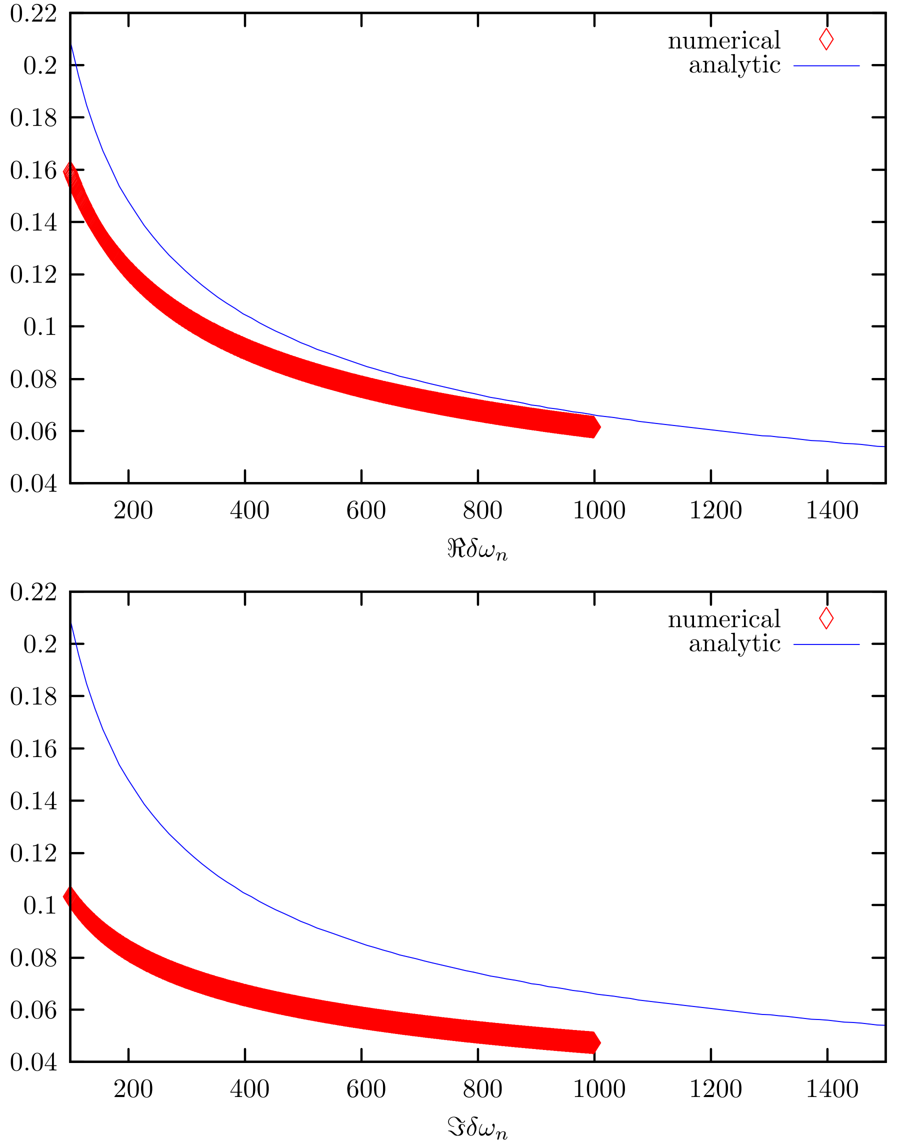
$j = 1/2$ (Dirac)

$$\frac{\omega_n}{T_H} = -(2n+1)\pi i + \frac{1+i}{2\sqrt{n}} \left(\ell + \frac{1}{2}\right)^2 \Gamma^2\left(\frac{1}{4}\right) + \mathcal{O}(1/n)$$





High overtones ($n \geq 100$) of massless Dirac fermions for $\ell + j = 1$. Numerical data by Konoplya.



High overtones ($n \geq 100$) of massless Dirac fermions for $\ell + j = 2$. Numerical data by Konoplya.

$j = 3/2$

$$\frac{\omega_n}{T_H} = -(2n+1)\pi i + \mathcal{O}(1/n)$$

no first-order corrections to the spectrum!

$j = 5/2$

$$\frac{\omega_n}{T_H} = -(2n+1)\pi i + \frac{1+i}{\sqrt{2n}}\,\mathcal{A}\,\Gamma^2\left(\frac{1}{4}\right) + \mathcal{O}(1/n)$$

All above spectra agree with spectrum from Regge-Wheeler equation!

- including integer spin
- no general proof

What is the relation between the Regge-Wheeler and Teukolsky equations?

AdS

AdS Black Holes

AdS/CFT correspondence:

$\Rightarrow$ QNMs for AdS b.h. expected to correspond to perturbations of dual CFT.

establishment of correspondence hindered by difficulties in solving wave eq.

- In 3d: Hypergeometric equation $\therefore$ solvable

  [Cardoso, Lemos; Birmingham, Sachs, Solodukhin]

- In 5d: Heun equation $\therefore$ unsolvable.
- Numerical results in 4d, 5d and 7d

  [Horowitz, Hubeny; Starinets; Konoplya]

Asymptotic form of QNMs of AdS black holes

Approximation to the wave equation valid in the high frequency regime.

- In 3d: exact equation.
- In 5d: Heun eq. $\rightarrow$ Hypergeometric eq., as in low frequency regime.
  - analytical expression for asymptotic form of QNM frequencies
  - in agreement with numerical results.

AdS_3

wave equation

$$\frac{1}{R^2\, r}\partial_r\left(r^3\left(1-\frac{r_h^2}{r^2}\right)\partial_r\Phi\right)-\frac{R^2}{r^2-r_h^2}\partial_t^2\Phi+\frac{1}{r^2}\partial_x^2\Phi=m^2\Phi$$

Solution:

$$\Phi=e^{i(\omega t-px)}\Psi(y),\qquad y=\frac{r_h^2}{r^2}$$

where Ψ satisfies

$$y^2(y-1)\left((y-1)\Psi'\right)'+\hat{\omega}^2\,y\Psi+\hat{p}^2\,y(y-1)\Psi+\tfrac{1}{4}\hat{m}^2\,(y-1)\Psi=0$$

in the interval $0<y<1$, and

$$\hat{\omega}=\frac{\omega R^2}{2r_h}=\frac{\omega}{4\pi T_H},\quad \hat{p}=\frac{pR}{2r_h}=\frac{p}{4\pi R T_H}\,,\quad \hat{m}=mR$$

Two independent solutions obtained by examining the behavior near the horizon ($y \to 1$),

$$\Psi_\pm \sim (1-y)^{\pm i\hat{\omega}}$$

Ψ_+ : outgoing; Ψ_- : ingoing.

Different set obtained by studying behavior at large r ($y \to 0$).

$$\Psi \sim y^{h_\pm} \quad , \quad h_\pm = \frac{1}{2} \pm \frac{1}{2}\sqrt{1+\hat{m}^2}$$

In massless case ($m = 0$): $h_+ = 1$ and $h_- = 0$

$\therefore$ one of the solutions contains logarithms.

For QNMs, we are interested in the analytic solution

$$\Psi(y) = y(1-y)^{i\hat{\omega}}{}_2F_1(1+i(\hat{\omega}+\hat{p}), 1+i(\hat{\omega}-\hat{p}); 2; y)$$

Near the horizon ($y \to 1$): mixture of ingoing and outgoing waves
[∵ standard Hypergeometric function identities]

$$\Psi \sim A(1-y)^{-i\hat{\omega}} + B(1-y)^{i\hat{\omega}}$$

$$A = \frac{\Gamma(2i\hat{\omega})}{\Gamma(1+i(\hat{\omega}+\hat{p}))\Gamma(1+i(\hat{\omega}-\hat{p}))}$$
$$B = \frac{\Gamma(-2i\hat{\omega})}{\Gamma(1-i(\hat{\omega}+\hat{p}))\Gamma(1-i(\hat{\omega}-\hat{p}))}$$

Ψ linear combination of Ψ_+ and Ψ_- ∴

$$\Psi = A\Psi_- + B\Psi_+$$

For QNMs: Ψ purely ingoing at horizon, so set

$$B = 0$$

Solutions (QNM frequencies):

$$\boxed{\hat{\omega} = \pm\hat{p} - in} \quad , \quad n = 1, 2, \dots$$

discrete set of complex frequencies with $\Im\hat{\omega} < 0$.
NB: we obtained two sets of frequencies, with opposite $\Re\hat{\omega}$.

AdS$_5$

For a large black hole, scalar wave equation with $m = 0$

$$\frac{1}{r^3}\partial_r(r^5\, f(r)\,\partial_r\Phi) - \frac{R^4}{r^2\, f(r)}\partial_t^2\Phi - \frac{R^2}{r^2}\,\vec{\nabla}^2\Phi = 0$$

$$\hat{f}(r) = 1 - \frac{r_h^4}{r^4}$$

Solution:

$$\Phi = e^{i(\omega t - \vec{p}\cdot\vec{x})}\Psi(r)$$

change coordinate r to y,

$$y = \frac{r^2}{r_h^2}$$

Wave equation:

$$(y^2-1)\left(y(y^2-1)\Psi'\right)' + \left(\frac{\hat{\omega}^2}{4}y^2 - \frac{\hat{p}^2}{4}(y^2-1)\right)\Psi = 0$$

Two solutions by examining behavior near the horizon ($y \to 1$),

$$\Psi_\pm \sim (y-1)^{\pm i\hat{\omega}/4}$$

Different set by studying behavior at large r
($y \to \infty$)

$$\Psi \sim y^{h_\pm} \quad , \quad h_\pm = 0, -2$$

so one of the solutions contains logarithms.

For QNMs, we are interested in analytic solution

$$\Psi \sim y^{-2} \text{ as } y \to \infty$$

By considering the other (unphysical) singularity at $y = -1$,
$\Rightarrow$ another set of solutions

$$\Psi \sim (y+1)^{\pm\hat{\omega}/4} \text{ near } y = -1$$

Write wavefunction as

$$\Psi(y) = (y-1)^{-i\hat{\omega}/4}(y+1)^{\pm\hat{\omega}/4}F(y)$$

$\Rightarrow$ Two sets of modes with same $\Im\hat{\omega}$, but opposite $\Re\hat{\omega}$.

$F(y)$ satisfies the Heun equation

$$y(y^2-1)F'' + \left\{ \left(3 - \frac{i \pm 1}{2}\hat{\omega} \right) y^2 - \frac{i \pm 1}{2}\hat{\omega} y - 1 \right\} F'$$

$$+ \left\{ \frac{\hat{\omega}}{2} \left(\pm \frac{i\hat{\omega}}{4} \mp 1 - i \right) y - (i \mp 1)\frac{\hat{\omega}}{4} - \frac{\hat{p}^2}{4} \right\} F = 0$$

Solve in a region in the complex y-plane containing $|y| \geq 1$
(includes physical regime $r > r_h$)

For large $\hat{\omega}$: constant terms in Polynomial coefficients of F' and F small compared with other terms

$\therefore$ they may be dropped.

$\therefore$ wave eq. may be approximated by Hypergeometric equation

$$(y^2-1)F'' + \left\{ \left(3 - \frac{i \pm 1}{2}\hat{\omega} \right) y - \frac{i \pm 1}{2}\hat{\omega} \right\} F' + \frac{\hat{\omega}}{2} \left(\pm \frac{i\hat{\omega}}{4} \mp 1 - i \right) F = 0$$

in asymptotic limit of large frequencies $\hat{\omega}$.

Analytic solution:

$$F_0(x) = {}_2F_1(a_+, a_-; c; (y+1)/2) \ , \quad a_\pm = 1 - \tfrac{i \pm 1}{4}\hat{\omega} \pm 1 \quad , \quad c = \tfrac{3}{2} \pm \tfrac{1}{2}\hat{\omega}$$

For proper behavior at $y \to \infty$, demand that F be a *Polynomial*.

$\therefore$

$$a_+ = -n \ , \ n = 1, 2, \dots$$

$\therefore$ F is a Polynomial of order n, so as $y \to \infty$,

$$F \sim y^n \sim y^{-a_+}$$

$$\Psi \sim y^{-i\hat{\omega}/4} y^{\pm\hat{\omega}/4} y^{-a_+} \sim y^{-2}$$

as expected.

$\therefore$ QNM frequencies

$$\boxed{\hat{\omega} = \frac{\omega}{4\pi T_H} = 2n(\pm 1 - i)}$$

[Musiri, Siopsis]

in agreement with numerical results.

Monodromy argument

If the function has no singularities other than $y = \pm 1$, the contour around $y = +1$ may be unobstructedly deformed into the contour around $y = -1$,

$$\mathcal{M}(1)\mathcal{M}(-1) = 1$$

Since

$$\mathcal{M}(1) = e^{\pi\widehat{\omega}/2} \ , \ \mathcal{M}(-1) = e^{\mp i\pi\widehat{\omega}/2}$$

and using $\Im\widehat{\omega} < 0$, we deduce

$$\widehat{\omega} = 2n(\pm 1 - i)$$

same as before.

Gravitational perturbations

AdS Schwarzschild black holes with metric in d dimensions

$$ds^2 = -f(r)dt^2 + \frac{dr^2}{f(r)} + r^2 d\Omega_{d-2}^2 \ , \quad f(r) = \frac{r^2}{R^2} + 1 - \frac{2\mu}{r^{d-3}} \ .$$

- ► derive analytical expressions including first-order corrections.
- ► results in good agreement with results of numerical analysis.

radial wave equation

$$-\frac{d^2\Psi}{dr_*^2} + V[r(r_*)]\Psi = \omega^2\Psi \ ,$$

in terms of the tortoise coordinate defined by

$$\frac{dr_*}{dr} = \frac{1}{f(r)} \ .$$

potential V from Master Equation *[Ishibashi and Kodama]*

For tensor, vector and scalar perturbations, we obtain, respectively,

[Natário and Schiappa]

$$V_{\mathsf{T}}(r) = f(r)\left\{\frac{\ell(\ell+d-3)}{r^2} + \frac{(d-2)(d-4)f(r)}{4r^2} + \frac{(d-2)f'(r)}{2r}\right\}$$

$$V_{\mathsf{V}}(r) = f(r)\left\{\frac{\ell(\ell+d-3)}{r^2} + \frac{(d-2)(d-4)f(r)}{4r^2} - \frac{rf'''(r)}{2(d-3)}\right\}$$

$$\begin{aligned}
V_{\mathsf{S}}(r) \quad = \quad & \frac{f(r)}{4r^2}\left[\ell(\ell+d-3) - (d-2) + \frac{(d-1)(d-2)\mu}{r^{d-3}}\right]^{-2} \\
\times \quad & \left\{\frac{d(d-1)^2(d-2)^3\mu^2}{R^2 r^{2d-8}} - \frac{6(d-1)(d-2)^2(d-4)[\ell(\ell+d-3)-(d-2)]\mu}{R^2 r^{d-5}}\right. \\
& + \frac{(d-4)(d-6)[\ell(\ell+d-3)-(d-2)]^2 r^2}{R^2} + \frac{2(d-1)^2(d-2)^4\mu^3}{r^{3d-9}} \\
& + \frac{4(d-1)(d-2)(2d^2-11d+18)[\ell(\ell+d-3)-(d-2)]\mu^2}{r^{2d-6}} \\
& + \frac{(d-1)^2(d-2)^2(d-4)(d-6)\mu^2}{r^{2d-6}} - \frac{6(d-2)(d-6)[\ell(\ell+d-3)-(d-2)]^2\mu}{r^{d-3}} \\
& - \frac{6(d-1)(d-2)^2(d-4)[\ell(\ell+d-3)-(d-2)]\mu}{r^{d-3}} \\
& \left. + 4[\ell(\ell+d-3)-(d-2)]^3 + d(d-2)[\ell(\ell+d-3)-(d-2)]^2\right\}
\end{aligned}$$

Near the black hole singularity ($r \sim 0$),

$$V_{\mathsf{T}} = -\frac{1}{4r_*^2} + \frac{\mathcal{A}_{\mathsf{T}}}{[-2(d-2)\mu]^{\frac{1}{d-2}}} r_*^{-\frac{d-1}{d-2}} + \dots\,, \quad \mathcal{A}_{\mathsf{T}} = \frac{(d-3)^2}{2(2d-5)} + \frac{\ell(\ell+d-3)}{d-2},$$

$$V_{\mathsf{V}} = \frac{3}{4r_*^2} + \frac{\mathcal{A}_{\mathsf{V}}}{[-2(d-2)\mu]^{\frac{1}{d-2}}} r_*^{-\frac{d-1}{d-2}} + \dots\ , \quad \mathcal{A}_{\mathsf{V}} = \frac{d^2-8d+13}{2(2d-15)} + \frac{\ell(\ell+d-3)}{d-2}$$

and

$$V_{\mathsf{S}} = -\frac{1}{4r_*^2} + \frac{\mathcal{A}_{\mathsf{S}}}{[-2(d-2)\mu]^{\frac{1}{d-2}}} r_*^{-\frac{d-1}{d-2}} + \dots\,,$$

where

$$\mathcal{A}_{\mathsf{S}} = \frac{(2d^3-24d^2+94d-116)}{4(2d-5)(d-2)} + \frac{(d^2-7d+14)[\ell(\ell+d-3)-(d-2)]}{(d-1)(d-2)^2}$$

We may summarize the behavior of the potential near the origin by

$$V = \frac{j^2-1}{4r_*^2} + \mathcal{A}\, r_*^{-\frac{d-1}{d-2}} + \dots$$

where $j = 0$ (2) for scalar and tensor (vector) perturbations.

for large r,

$$V = \frac{j_\infty^2 - 1}{4(r_* - \bar{r}_*)^2} + \dots \quad , \qquad \bar{r}_* = \int_0^\infty \frac{dr}{f(r)}$$

where $j_\infty = d-1$, $d-3$ and $d-5$ for tensor, vector and scalar perturbations, respectively.

After rescaling the tortoise coordinate $(z = \omega r_*)$, wave equation

$$\left(\mathcal{H}_0 + \omega^{-\frac{d-3}{d-2}} \mathcal{H}_1 \right) \Psi = 0,$$

where

$$\mathcal{H}_0 = \frac{d^2}{dz^2} - \left[\frac{j^2 - 1}{4z^2} - 1 \right] \ , \ \mathcal{H}_1 = -\mathcal{A}\ z^{-\frac{d-1}{d-2}}.$$

By treating $\mathcal{H}_1$ as a perturbation, we may expand the wave function

$$\Psi(z) = \Psi_0(z) + \omega^{-\frac{d-3}{d-2}} \Psi_1(z) + \dots$$

and solve wave eq. perturbatively.

The zeroth-order wave equation,

$$\mathcal{H}_0 \Psi_0(z) = 0,$$

may be solved in terms of Bessel functions,

$$\Psi_0(z) = A_1 \sqrt{z}\, J_{\frac{j}{2}}(z) + A_2 \sqrt{z}\, N_{\frac{j}{2}}(z).$$

For large z, it behaves as

$$\begin{aligned} \Psi_0(z) &\sim \sqrt{\frac{2}{\pi}} \left[A_1 \cos(z - \alpha_+) + A_2 \sin(z - \alpha_+) \right], \\ &= \frac{1}{\sqrt{2\pi}} (A_1 - iA_2) e^{-i\alpha_+} e^{iz} + \frac{1}{\sqrt{2\pi}} (A_1 + iA_2) e^{+i\alpha_+} e^{-iz} \end{aligned}$$

where $\alpha_\pm = \frac{\pi}{4}(1 \pm j)$.

large z ($r \longrightarrow \infty$)

wavefunction ought to vanish $\therefore$ acceptable solution

$$\Psi(r_*) = B\sqrt{\omega(r_* - \bar{r}_*)}\; J_{\frac{j_\infty}{2}}(\omega(r_* - \bar{r}_*))$$

NB: $\Psi \longrightarrow 0$ as $r_* \longrightarrow \bar{r}_*$, as desired.

Asymptotically, it behaves as

$$\Psi(r_*) \sim \sqrt{\frac{2}{\pi}}\, B \cos\left[\omega(r_* - \bar{r}_*) + \beta\right] \ , \quad \beta = \frac{\pi}{4}(1 + j_\infty)$$

match this to asymptotic behavior in the vicinity of the black-hole singularity along the Stokes line $\Im z = \Im(\omega r_*) = 0$

$\Rightarrow$ constraint on the coefficients $A_1,\ A_2$,

$$A_1 \tan(\omega \bar{r}_* - \beta - \alpha_+) - A_2 = 0.$$

impose boundary condition at the horizon

$$\Psi(z) \sim e^{iz} \ , \quad z \to -\infty \ ,$$

$\Rightarrow$ second constraint

analytically continue wavefunction near the origin to negative values of z.

- rotation of z by $-\pi$ corresponds to a rotation by $-\frac{\pi}{d-2}$ near the origin in the complex r-plane.

using

$$J_\nu(e^{-i\pi}z) = e^{-i\pi\nu}J_\nu(z)\ , \quad N_\nu(e^{-i\pi}z) = e^{i\pi\nu}N_\nu - 2i\cos\pi\nu\, J_\nu(z)$$

for $z < 0$, the wavefunction changes to

$$\Psi_0(z) = e^{-i\pi(j+1)/2}\sqrt{-z}\left\{\left[A_1 - i(1+e^{i\pi j})A_2\right] J_{\frac{j}{2}}(-z) + A_2 e^{i\pi j}\, N_{\frac{j}{2}}(-z)\right\}$$

whose asymptotic behavior is given by

$$\Psi \sim \frac{e^{-i\pi(j+1)/2}}{\sqrt{2\pi}}\left[A_1 - i(1+2e^{j\pi i})A_2\right] e^{-iz} + \frac{e^{-i\pi(j+1)/2}}{\sqrt{2\pi}}\left[A_1 - iA_2\right] e^{iz}$$

$\Rightarrow$ second constraint

$$A_1 - i(1+2e^{j\pi i})A_2 = 0$$

constraints compatible provided

$$\begin{vmatrix} 1 & -i(1+2e^{j\pi i}) \\ \tan(\omega\bar{r}_* - \beta - \alpha_+) & -1 \end{vmatrix} = 0$$

∴ quasi-normal frequencies

$$\boxed{\omega \bar{r}_* = \frac{\pi}{4}(2 + j + j_\infty) - \tan^{-1} \frac{i}{1 + 2e^{j\pi i}} + n\pi}$$

[Natário and Schiappa]

First-order corrections

[Musiri, Ness and Siopsis]

To first order, the wave equation becomes

$$\mathcal{H}_0\Psi_1 + \mathcal{H}_1\Psi_0 = 0$$

The solution is

$$\Psi_1(z) = \sqrt{z}\, N_{\frac{j}{2}}(z) \int_0^z dz' \frac{\sqrt{z'}\, J_{\frac{j}{2}}(z')\mathcal{H}_1\Psi_0(z')}{\mathcal{W}} - \sqrt{z}\, J_{\frac{j}{2}}(z) \int_0^z dz' \frac{\sqrt{z'}\, N_{\frac{j}{2}}(z')\mathcal{H}_1\Psi_0(z')}{\mathcal{W}}$$

$\mathcal{W} = 2/\pi$ is the Wronskian.

$\therefore$ wavefunction up to first order

$$\Psi(z) = \{A_1[1 - b(z)] - A_2 a_2(z)\}\sqrt{z} J_{\frac{j}{2}}(z) + \{A_2[1 + b(z)] + A_1 a_1(z)\}\sqrt{z} N_{\frac{j}{2}}(z)$$

where

$$\begin{aligned} a_1(z) &= \frac{\pi\mathcal{A}}{2}\omega^{-\frac{d-3}{d-2}} \int_0^z dz'\, z'^{-\frac{1}{d-2}} J_{\frac{j}{2}}(z') J_{\frac{j}{2}}(z') \\ a_2(z) &= \frac{\pi\mathcal{A}}{2}\omega^{-\frac{d-3}{d-2}} \int_0^z dz'\, z'^{-\frac{1}{d-2}} N_{\frac{j}{2}}(z') N_{\frac{j}{2}}(z') \\ b(z) &= \frac{\pi\mathcal{A}}{2}\omega^{-\frac{d-3}{d-2}} \int_0^z dz'\, z'^{-\frac{1}{d-2}} J_{\frac{j}{2}}(z') N_{\frac{j}{2}}(z') \end{aligned}$$

$\mathcal{A}$ depends on the type of perturbation.

asymptotically

$$\Psi(z) \sim \sqrt{\frac{2}{\pi}}\,[A_1' \cos(z-\alpha_+) + A_2' \sin(z-\alpha_+)] \;,$$

where

$$A_1' = [1-\bar{b}]A_1 - \bar{a}_2 A_2 \;\;,\quad A_2' = [1+\bar{b}]A_2 + \bar{a}_1 A_1$$

and we introduced the notation

$$\bar{a}_1 = a_1(\infty) \;\;,\quad \bar{a}_2 = a_2(\infty) \;\;,\quad \bar{b} = b(\infty) \;.$$

First constraint modified to

$$A_1' \tan(\omega \bar{r}_* - \beta - \alpha_+) - A_2' = 0$$

$\therefore$

$$[(1-\bar{b})\tan(\omega\bar{r}_* - \beta - \alpha_+) - \bar{a}_1]A_1 - [1+\bar{b}+\bar{a}_2 \tan(\omega\bar{r}_* - \beta - \alpha_+)]A_2 = 0$$

For second constraint,

$\hookrightarrow$ approach the horizon

$\hookrightarrow$ rotate by $-\pi$ in the z-plane

$$
\begin{aligned}
a_1(e^{-i\pi}z) &= e^{-i\pi\frac{d-3}{d-2}}e^{-i\pi j}a_1(z)\ , \\
a_2(e^{-i\pi}z) &= e^{-i\pi\frac{d-3}{d-2}}\left[e^{i\pi j}a_2(z) - 4\cos^2\frac{\pi j}{2}a_1(z) - 2i(1+e^{i\pi j})b(z)\right]\ , \\
b(e^{-i\pi}z) &= e^{-i\pi\frac{d-3}{d-2}}\left[b(z) - i(1+e^{-i\pi j})a_1(z)\right]
\end{aligned}
$$

$\therefore$ in the limit $z \to -\infty$,

$$
\Psi(z) \sim -ie^{-ij\pi/2}B_1\cos(-z-\alpha_+) - ie^{ij\pi/2}B_2\sin(-z-\alpha_+)
$$

where

$$
\begin{aligned}
B_1 &= A_1 - A_1 e^{-i\pi\frac{d-3}{d-2}}[\bar{b} - i(1+e^{-i\pi j})\bar{a}_1] \\
&\quad -A_2 e^{-i\pi\frac{d-3}{d-2}}\left[e^{+i\pi j}\bar{a}_2 - 4\cos^2\frac{\pi j}{2}\bar{a}_1 - 2i(1+e^{+i\pi j})\bar{b}\right] \\
&\quad -i(1+e^{i\pi j})\left[A_2 + A_2 e^{-i\pi\frac{d-3}{d-2}}[\bar{b} - i(1+e^{-i\pi j})\bar{a}_1] + A_1 e^{-i\pi\frac{d-3}{d-2}}e^{-i\pi j}\bar{a}_1\right] \\
B_2 &= A_2 + A_2 e^{-i\pi\frac{d-3}{d-2}}[\bar{b} - i(1+e^{-i\pi j})\bar{a}_1] + A_1 e^{-i\pi\frac{d-3}{d-2}}e^{-i\pi j}\bar{a}_1
\end{aligned}
$$

$\therefore$ second constraint

$$
[1 - e^{-i\pi\frac{d-3}{d-2}}(i\bar{a}_1 + \bar{b})]A_1 - [i(1+2e^{i\pi j}) + e^{-i\pi\frac{d-3}{d-2}}((1+e^{i\pi j})\bar{a}_1 + e^{i\pi j}\bar{a}_2 - i\bar{b})]A_2 = 0
$$

compatibility of the two first-order constraints,

$$\left| \begin{array}{cc} 1+\bar{b}+\bar{a}_2\tan(\omega\bar{r}_*-\beta-\alpha_+) & i(1+2e^{i\pi j})+e^{-i\pi\frac{d-3}{d-2}}((1+e^{i\pi j})\bar{a}_1+e^{i\pi j}\bar{a}_2-i\bar{b}) \\ (1-\bar{b})\tan(\omega\bar{r}_*-\beta-\alpha_+)-\bar{a}_1 & 1-e^{-i\pi\frac{d-3}{d-2}}(i\bar{a}_1+\bar{b}) \end{array} \right| = 0$$

$\Rightarrow$ first-order expression for quasi-normal frequencies,

$$\begin{aligned} \omega\bar{r}_* &= \frac{\pi}{4}(2+j+j_\infty)+\frac{1}{2i}\ln 2+n\pi \\ &\quad -\frac{1}{8}\left\{6i\bar{b}-2ie^{-i\pi\frac{d-3}{d-2}}\bar{b}-9\bar{a}_1+e^{-i\pi\frac{d-3}{d-2}}\bar{a}_1+\bar{a}_2-e^{-i\pi\frac{d-3}{d-2}}\bar{a}_2\right\} \end{aligned}$$

where

$$\begin{aligned} \bar{a}_1 &= \frac{\pi\mathcal{A}}{4}\left(\frac{n\pi}{2\bar{r}_*}\right)^{-\frac{d-3}{d-2}}\frac{\Gamma(\frac{1}{d-2})\Gamma(\frac{j}{2}+\frac{d-3}{2(d-2)})}{\Gamma^2(\frac{d-1}{2(d-2)})\Gamma(\frac{j}{2}+\frac{d-1}{2(d-2)})} \\ \bar{a}_2 &= \left[1+2\cot\frac{\pi(d-3)}{2(d-2)}\cot\frac{\pi}{2}\left(-j+\frac{d-3}{d-2}\right)\right]\bar{a}_1 \\ \bar{b} &= -\cot\frac{\pi(d-3)}{2(d-2)}\,\bar{a}_1 \end{aligned}$$

▶ first-order correction is $\sim O(n^{-\frac{d-3}{d-2}})$.

4d

compare with numerical results *[Cardoso, Konoplya and Lemos]*

set the AdS radius $R = 1$: radius of horizon r_H related to black hole mass μ by

$$2\mu = r_H^3 + r_H$$

$f(r)$ has two more (complex) roots, r_- and its complex conjugate, where

$$r_- = e^{i\pi/3}\left(\sqrt{\mu^2 + \frac{1}{27}} - \mu\right)^{1/3} - e^{-i\pi/3}\left(\sqrt{\mu^2 + \frac{1}{27}} + \mu\right)^{1/3}$$

The integration constant in the tortoise coordinate is

$$\bar{r}_* = \int_0^\infty \frac{dr}{f(r)} = -\frac{r_-}{3r_-^2 + 1}\ln\frac{r_-}{r_H} - \frac{r_-^*}{3r_-^{*2} + 1}\ln\frac{r_-^*}{r_H}$$

Scalar perturbations

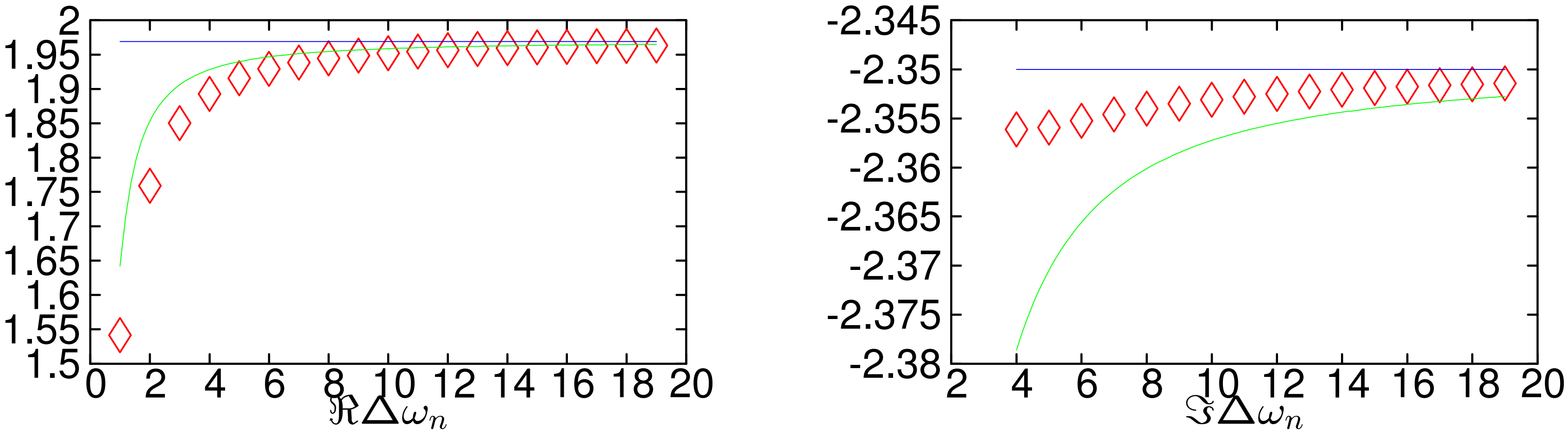


Fig. 1: $r_H = 1$ and $\ell = 2$: zeroth (horizontal line) and first order (curved line) analytical compared with numerical data (diamonds).

$$\omega_n \bar{r}_* = \left(n + \frac{1}{4}\right)\pi + \frac{i}{2}\ln 2 + e^{i\pi/4}\frac{\mathcal{A}_S \Gamma^4(\frac{1}{4})}{16\pi^2}\sqrt{\frac{\bar{r}_*}{2\mu n}}\,, \quad \mathcal{A}_S = \frac{\ell(\ell+1)-1}{6}$$

only the first-order correction is ℓ-dependent.

In the limit of large horizon radius ($r_H \approx (2\mu)^{1/3} \gg 1$),

$$\bar{r}_* \approx \frac{\pi(1+i\sqrt{3})}{3\sqrt{3}r_H}$$

Numerically for $\ell = 2$,

$$\frac{\omega_n}{r_H} = (1.299 - 2.250i)n + 0.573 - 0.419i + \frac{0.508 + 0.293i}{r_H^2\sqrt{n}}$$

which compares well with the result of numerical analysis,

$$\left(\frac{\omega_n}{r_H}\right)_{\text{numerical}} \approx (1.299 - 2.25i)n + 0.581 - 0.41i$$

including both leading order and offset.

For an intermediate black hole, $r_H = 1$, we obtain

$$\omega_n = (1.969 - 2.350i)n + 0.752 - 0.370i + \frac{0.654 + 0.458i}{\sqrt{n}}$$

In Fig. 1 we compare with data from numerical analysis. We plot the gap

$$\Delta\omega_n = \omega_n - \omega_{n-1}$$

because the offset does not always agree with numerical results.

▶ numerical estimates of the offset ought to be improved.

For a small black hole, $r_H = 0.2$, we obtain

$$\omega_n = (1.695 - 0.571i)n + 0.487 - 0.0441i + \frac{1.093 + 0.561i}{\sqrt{n}}$$

to be compared with the result of numerical analysis,

$$(\omega_n)_{\text{numerical}} \approx (1.61 - 0.6i)n + 2.7 - 0.37i$$

The two estimates of the offset disagree with each other.

Tensor perturbations

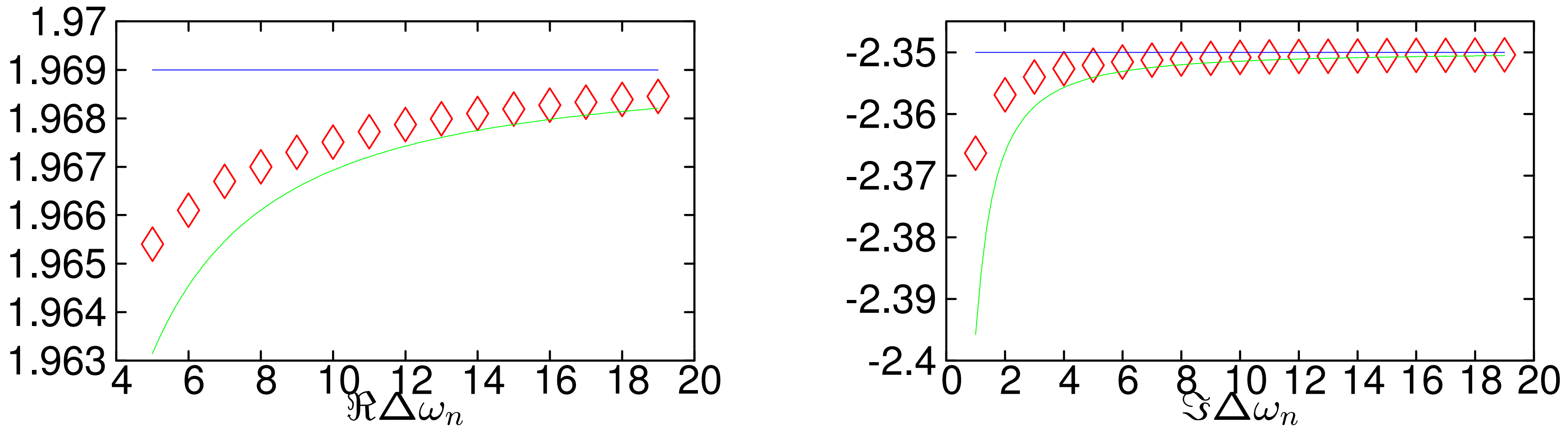


Fig. 2: $r_H = 1$ and $\ell = 0$: zeroth (horizontal line) and first order (curved line) analytical compared with numerical data (diamonds).

$$\omega_n \bar{r}_* = \left(n + \frac{1}{4}\right)\pi + \frac{i}{2}\ln 2 + e^{i\pi/4}\frac{\mathcal{A}_\mathsf{T}\Gamma^4(\frac{1}{4})}{16\pi^2}\sqrt{\frac{\bar{r}_*}{2\mu n}}\,, \quad \mathcal{A}_\mathsf{T} = \frac{3\ell(\ell+1)+1}{6}$$

Numerically for large r_H and $\ell = 0$,

$$\frac{\omega_n}{r_H} = (1.299 - 2.250i)n + 0.573 - 0.419i + \frac{0.102 + 0.0586i}{r_H^2\sqrt{n}}$$

For an intermediate black hole, $r_H = 1$, we obtain

$$\omega_n = (1.969 - 2.350i)n + 0.752 - 0.370i + \frac{0.131 + 0.0916i}{\sqrt{n}}$$

in good agreement with the result of numerical analysis (Fig. 2), including the offset.

For a small black hole, $r_H = 0.2$, we obtain

$$\omega_n = (1.695 - 0.571i)n + 2.182 - 0.615i + \frac{0.489 + 0.251i}{\sqrt{n}}$$

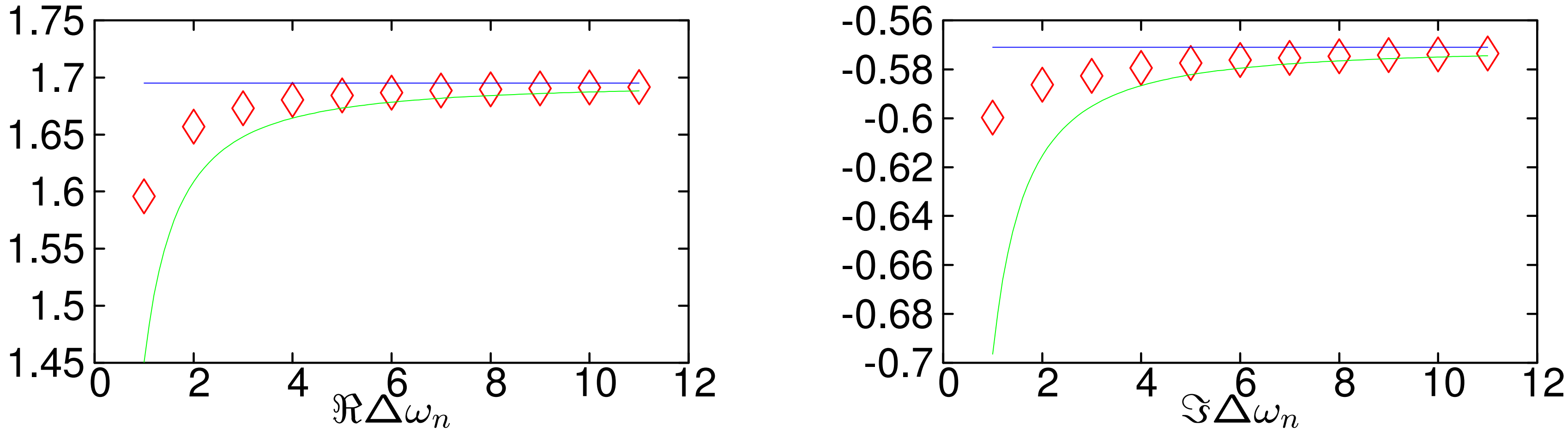


Fig. 3: $r_H = 0.2$ and $\ell = 0$: zeroth (horizontal line) and first order (curved line) analytical compared with numerical data (diamonds).

Vector perturbations

$$\omega_n \bar{r}_* = \left(n + \frac{1}{4}\right)\pi + \frac{i}{2}\ln 2 + e^{i\pi/4}\frac{\mathcal{A}_{\rm V}\Gamma^4(\frac{1}{4})}{48\pi^2}\sqrt{\frac{\bar{r}_*}{2\mu n}}\,, \quad \mathcal{A}_{\rm V} = \frac{\ell(\ell+1)}{2} + \frac{3}{14}$$

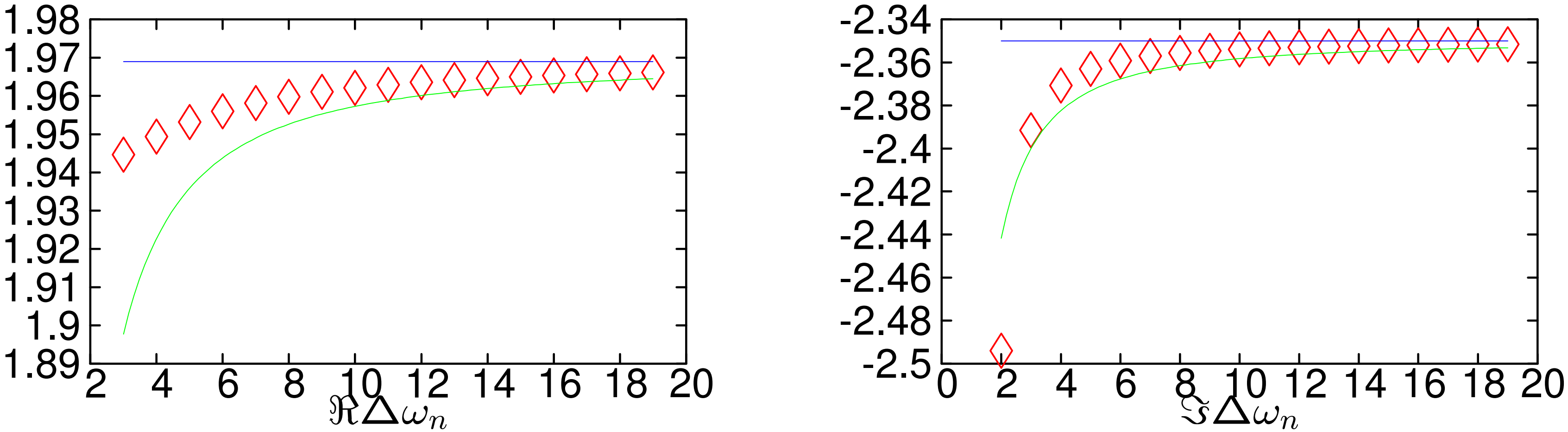


Fig. 4: $r_H = 1$ and $\ell = 2$: zeroth (horizontal line) and first order (curved line) analytical (eq. ()) compared with numerical data (diamonds).

Numerically for large r_H and $\ell = 2$,

$$\frac{\omega_n}{r_H} = (1.299 - 2.250i)n + 0.573 - 0.419i + \frac{8.19 + 6.29i}{r_H^2\sqrt{n}}$$

to be compared with the result of numerical analysis,

$$\left(\frac{\omega_n}{r_H}\right)_{\text{numerical}} \approx (1.299 - 2.25i)n + 0.58 - 0.42i$$

For an intermediate black hole, $r_H = 1$, we obtain

$$\omega_n = (1.969 - 2.350i)n + 0.752 - 0.370i + \frac{0.741 + 0.519i}{\sqrt{n}}$$

and for a small black hole, $r_H = 0.2$, we obtain

$$\omega_n = (1.695 - 0.571i)n + 0.487 - 0.0441i + \frac{1.239 + 0.6357i}{\sqrt{n}}$$

estimates of the offset agree for large r_H but diverge as $r_H \to 0$.

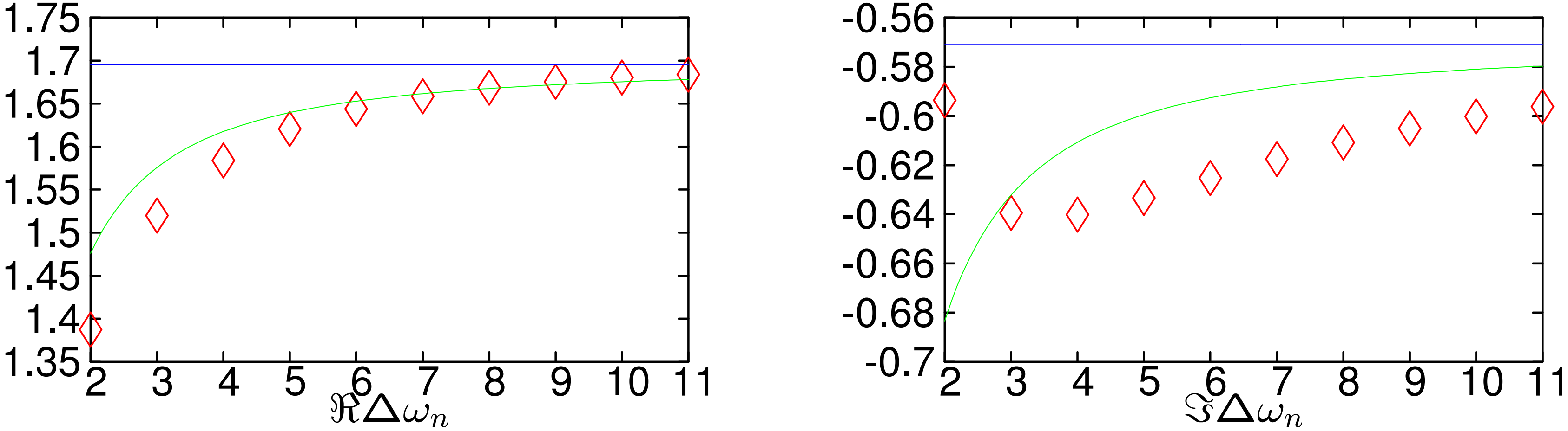


Fig. 5: $r_H = 0.2$ and $\ell = 2$: zeroth (horizontal line) and first order (curved line) analytical (eq. ()) compared with numerical data (diamonds).

Electromagnetic perturbations

electromagnetic potential

$$V_{\text{EM}} = \frac{\ell(\ell+1)}{r^2} f(r).$$

Near the origin,

$$V_{\text{EM}} = \frac{j^2 - 1}{4r_*^2} + \frac{\ell(\ell+1) r_*^{-3/2}}{2\sqrt{-4\mu}} + \dots \ ,$$

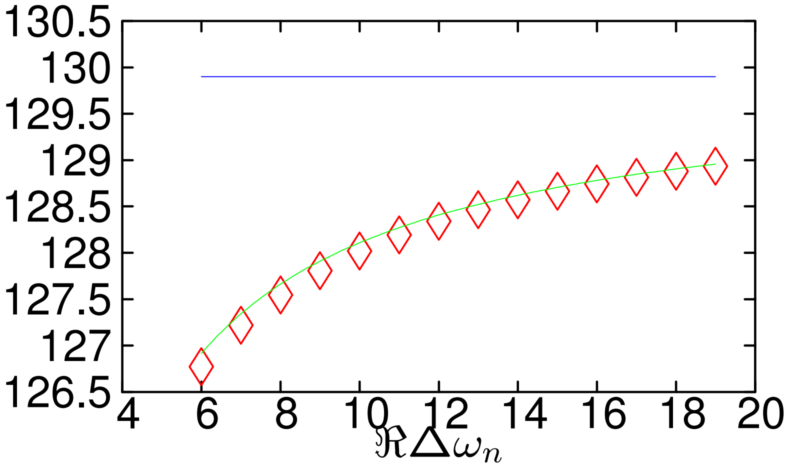
where $j = 1$ - vanishing potential to zeroth order!

► need to include first-order corrections for QNMs.

QNMs

$$\omega \bar{r}_* = n\pi - \frac{i}{4} \ln n + \frac{1}{2i} \ln \left(2(1+i)\mathcal{A}\sqrt{\bar{r}_*} \right) \ , \quad \mathcal{A} = \frac{\ell(\ell+1)}{2\sqrt{-4\mu}}$$

► correction behaves as $\ln n$.



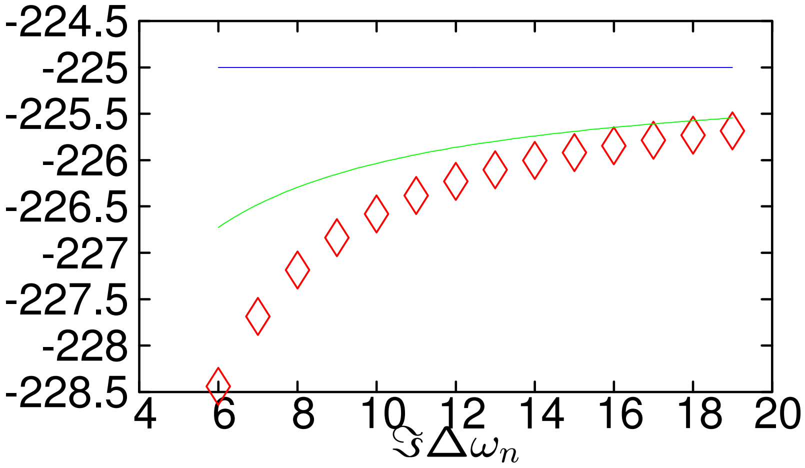


Fig. 6: $r_H = 100$ and $\ell = 1$: zeroth (horizontal line) and first order (curved line) analytical compared with numerical data (diamonds).

For a large black hole, we obtain the spectrum

$$\frac{\Delta\omega_n}{r_H} \approx \frac{3\sqrt{3}(1-i\sqrt{3})}{4}\left(1-\frac{i}{4\pi n}+\dots\right) = 1.299 - 2.25i - \frac{0.179+0.103i}{n} + \dots$$

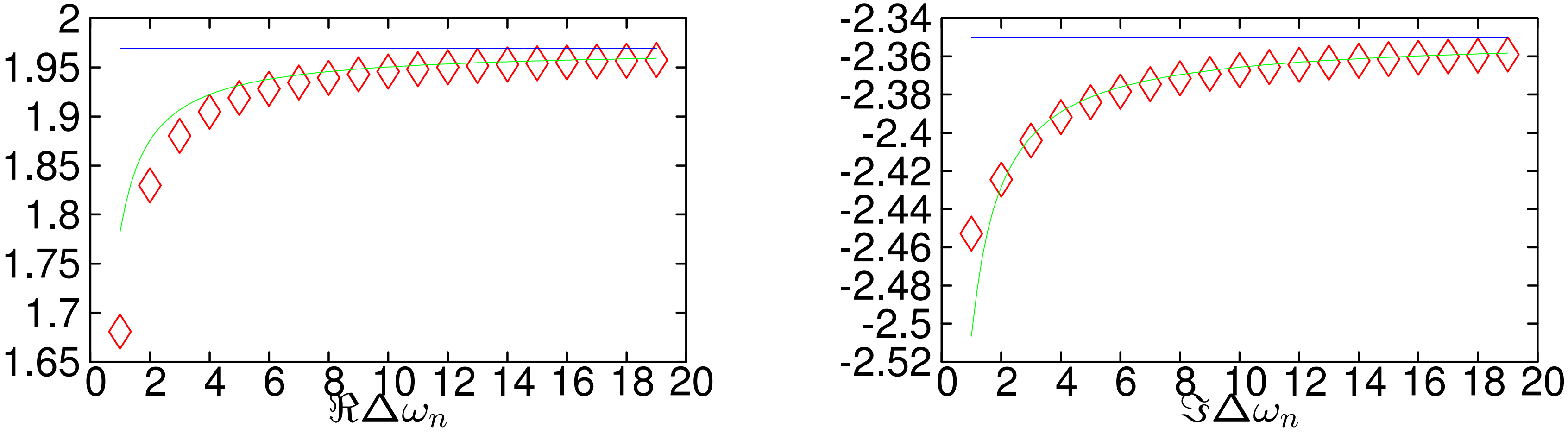


Fig. 7: $r_H = 1$ and $\ell = 1$: zeroth (horizontal line) and first order (curved line) analytical compared with numerical data (diamonds).

For an intermediate black hole, $r_H = 1$,

$$\omega_n = (1.969 - 2.350i)n - (0.187 + 0.1567i)\ln n + \dots$$

and for a small black hole, $r_H = 0.2$,

$$\omega_n = (1.695 - 0.571i)n - (0.045 + 0.135i)\ln n + \dots$$

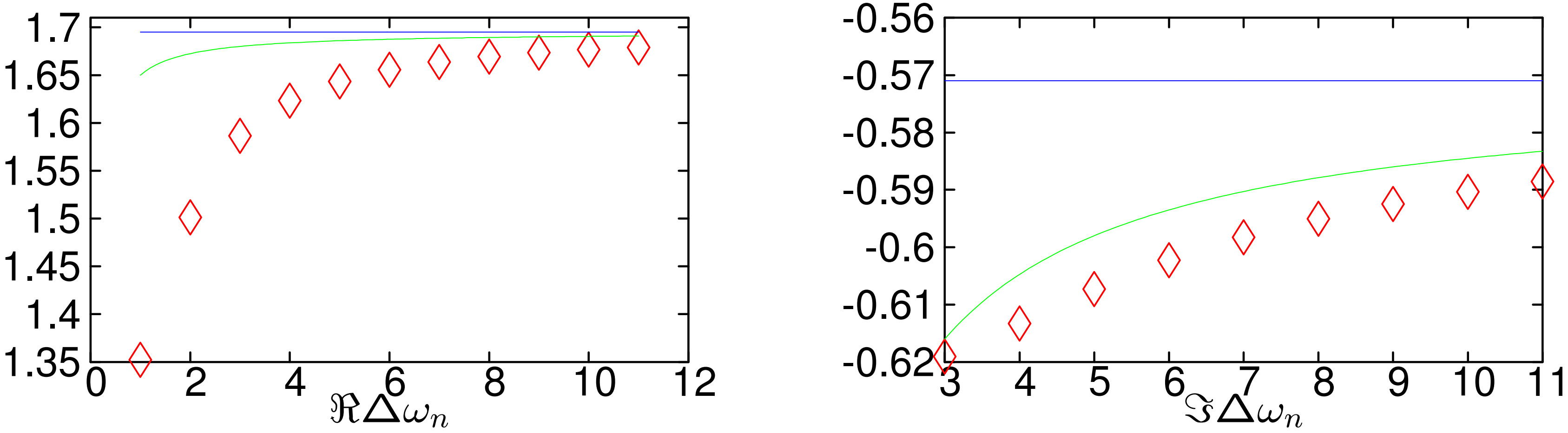


Fig. 8: $r_H = 0.2$ and $\ell = 1$: zeroth (horizontal line) and first order (curved line) analytical compared with numerical data (diamonds).

All first-order analytical results are in good agreement with numerical results.

Unitarity

black-hole perturbations governed by a discrete spectrum of complex eigen-frequencies (QNMs)

$\Rightarrow$ breakdown of unitarity.

In asymptotically AdS spaces, this is puzzling

$\because$ corresponding CFT is unitary.

PLAN

- In 3d, replace the BTZ black hole by a wormhole, following a suggestion by Solodukhin.
- solve the wave equation for a massive scalar field and find an equation for the poles of the propagator.

RESULTS

- rich spectrum of *real* eigen-frequencies.
- throat of wormhole must be $o(e^{-1/G})$, where G is Newton's constant.
- quantum effects which might produce the wormhole are non-perturbative.

imaginary part of QNM is negative

⇒ black hole eventually relaxes back to its original (thermal) equilibrium at (Hawking) temperature T_H.

⇒ leakage of information into the horizon

⇒ breakdown of unitarity

⇒ closely related to Hawking's information loss paradox

resolution will require understanding of quantum gravity beyond semi-classical approximation.

asymptotically AdS space-times

additional tool due to AdS/CFT correspondence:

- complex QNM frequencies are poles of the retarded propagator in CFT
- puzzling: CFT is unitary $\therefore$ propagator should possess *real* poles only.

Poincaré recurrence theorem

- two-point function quasi-periodic with a period

$$t_P \sim e^S$$

  S: entropy.

- For times $t \ll t_P$, system may look like it is decaying back to thermal equilibrium, but for $t \gtrsim t_P$, it should return to its original state (or close) an infinite number of times.
- system will *never* relax back to its original state.

Linear response theory

[Birmingham, Sachs and Solodukhin]

system in thermal equilibrium described by density matrix ρ.
perturbation

$$H' = \int dx J(t,x)\mathcal{O}(t,x)$$

J: external source
change in ensemble average

$$\delta\langle\mathcal{O}(t,x)\rangle = \int_{-\infty}^{\infty} dt' \int dx' J(t',x') G_R(t,x;t',x')$$

in terms of retarded propagator

$$G_R(t,x;t',x') = -i\theta(t-t')\mathrm{Tr}\left(\rho[\mathcal{O}(t,x),\mathcal{O}(t',x')]\right)$$

Fourier transform $\tilde{G}_R(\omega, p)$:

- analytic in upper-half ω-plane.
- discrete energy levels
  $\Rightarrow$ simple poles on real axis
  $\Rightarrow$ meromorphic in lower-half ω-plane
  $\Rightarrow$ oscillatory behavior
- continuous energy levels
  $\Rightarrow$ poles (stable states) or cuts (multi-particle states) on real axis
  $\Rightarrow$ poles (resonances) in lower-half ω-plane

FINITE SYSTEM

Correlator on torus of periods $1/T$ and 1.

EXAMPLE: free fermion ($\Delta = 1$)

Correlator:

$$\langle\psi(w)\psi(0)\rangle = \frac{\partial_w\vartheta_1(0|T)}{\vartheta_\nu(0|iT)}\,\frac{\vartheta_\nu(wT|iT)}{\vartheta_1(wT|iT)} \;\;,\;\; \nu = 3,4$$

$w = i(t+\phi)$, invariant under $w \to w + 1/T$, $w \to w + i$,

$\Rightarrow$ periodic in t, period 1.

poles:

$$w = \frac{m}{T} + in \;\;,\;\; m,n \in \mathbb{Z}$$

As $T \to 0$, oscillating behavior:

$$\langle \psi(w)\psi(0) \rangle \sim \frac{1}{\sin \pi(t+\phi)}$$

As $T \to \infty$, exponential decay

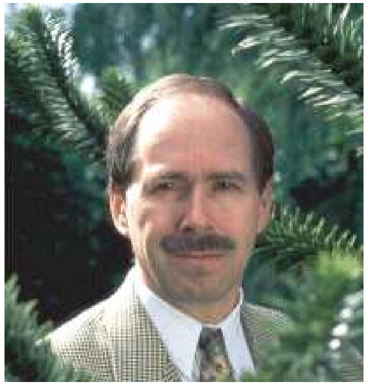
$$\langle \psi(w)\psi(0) \rangle = \frac{\pi T}{4 \sinh \pi T(t+\phi)} \left\{ 1 \pm 2e^{-\pi T} \cosh 2\pi T(t+\phi) + \dots \right\}$$

violation of periodicity ($t \to t+1$) and loss of unitarity? NO!

Two time scales: 1 and $1/T \ll 1$.

- When $t \lesssim 1/T$, system decays
- When $t \sim 1/T$, corrections important
- When $t \gtrsim 1$, periodicity is restored

’t Hooft’s brick wall



divergent physical quantities due to infinite blue shift experienced by an in-falling object near the horizon.

► infinite energy levels

$\therefore$ information loss

$\therefore$ Hawking radiation

$\hookrightarrow$ continuous spectrum due to horizon. Set $r_h = 1$.

place brick wall at distance ϵ from horizon

$$\phi(r) = 0 \ , \ \ r \le 1 + \epsilon$$

discrete energy levels

$$\omega_n \sim \frac{n\pi}{-\ln \epsilon}$$

Free energy $F \sim T_H^3 \frac{A}{\epsilon}$.

Entropy

$$S = -\frac{\partial F}{\partial T} \sim \frac{A}{\epsilon}$$

PROBLEM: Unnatural cutoff (coordinate invariance broken)

subsequently understood that. . .

- infinities may be absorbed by the gravitational parameters
  - ϵ contributes to renormalization of G.
- theory is finite when expressed in terms of physical parameters like any renormalizable field theory.

  [Susskind and Uglum; Demers, Lafrance and Myers]
- form of entropy, including these quantum effects, remains unchanged.

Solodukhin's wormhole

replace black hole by a wormhole

$\Rightarrow$ eliminate horizon and attendant leakage of information.

size of narrow throat $\lambda \sim o(e^{-1/G})$

leading to a Poincaré recurrence time

$$t_P \sim \frac{1}{\lambda} \sim o(e^{1/G})$$

in agreement with expectations.

GOAL

- calculate two-point functions explicitly
- obtain the *real* poles of the propagator, thus demonstrating unitarity.
- Calculate λ

Three dimensions

wave equation for massive scalar of mass m at BTZ black hole

$$\frac{1}{r}\partial_r \left(r^3 \left(1 - \frac{r_h^2}{r^2} \right) \partial_r \Phi \right) - \frac{1}{r^2 - r_h^2}\partial_t^2 \Phi + \frac{1}{r^2}\partial_x^2 \Phi = m^2 \Phi$$

r_h: radius of horizon (set AdS radius $R = 1$)
Let

$$\Phi = e^{i(\omega t - px)}\Psi(y), \qquad \Psi(y) = y^{i\hat{p}}(1-y)^{-i\hat{\omega}}F(y)$$

Solution

$$F(y) = F(a_+, a_-; c; 1-y)$$

where

$$a_\pm = \tfrac{1}{2}\Delta_\pm - i(\hat{\omega} - \hat{p}) \ , \ c = 1 - 2i\hat{\omega} \ \ , \ \ \Delta_\pm = 1 \pm \sqrt{1+m^2}$$

in terms of dimensionless variables

$$\hat{\omega} = \frac{\omega}{2r_h} = \frac{\omega}{4\pi T_H}, \quad \hat{p}^2 = \frac{p}{2r_h} = \frac{p}{4\pi T_H}$$

$T_H = r_h/(2\pi)$: Hawking temperature.

As $y \to \infty$, this function behaves as

$$F(y) \sim \mathcal{A} y^{-a_+} + \mathcal{B} y^{-a_-}$$

where

$$\mathcal{A} = \frac{\Gamma(c)\Gamma(a_- - a_+)}{\Gamma(a_-)\Gamma(c - a_+)} \ , \quad \mathcal{B} = \frac{\Gamma(c)\Gamma(a_+ - a_-)}{\Gamma(a_+)\Gamma(c - a_-)}$$

For desired behavior ($\Psi \sim y^{-\Delta_+/2}$ as $y \to \infty$), set

$$\mathcal{B} = 0$$

This condition implies

$$\hat{\omega} = \pm \hat{p} - i(n + \tfrac{1}{2}\Delta_+ - 1) \quad , \quad n = 1, 2, \dots$$

asymptotically AdS space-times

additional tool due to AdS/CFT correspondence:

- complex QNM frequencies are poles of the retarded propagator in CFT
- puzzling: CFT is unitary $\therefore$ propagator should possess *real* poles only.

AdS/CFT correspondence:

- flux at the boundary ($y \to \infty$) is related to the retarded propagator of the corresponding CFT living on the boundary.

A standard calculation yields

$$\tilde{G}_R(\omega, p) \sim \lim_{y\to\infty} \frac{F'(y)}{F(y)}$$

Explicitly,

$$\begin{aligned} \tilde{G}_R(\omega, p) &\sim \frac{\mathcal{A}}{\mathcal{B}} \\ &\sim |\Gamma(\tfrac{1}{2}\Delta_+ - i(\hat{\omega} - \hat{p}))\Gamma(\tfrac{1}{2}\Delta_+ - i(\hat{\omega} + \hat{p}))|^2 \\ &\quad \times \sin\pi(\tfrac{1}{2}\Delta_+ - i(\hat{\omega} - \hat{p}))\sin\pi(\tfrac{1}{2}\Delta_+ - i(\hat{\omega} + \hat{p})) \end{aligned}$$

Plainly, QNMs (zeroes of $\mathcal{B}$) are poles of the retarded propagator $\because$

$$\tilde{G}_R \sim 1/\mathcal{B}$$

2-point correlator

$$\langle \mathcal{O}(t,x)\mathcal{O}(0,0)\rangle = \frac{(\pi T_H)^{2\Delta_+}}{(\sinh \pi T_H(t-x) \sinh \pi T_H(t+x))^{\Delta_+}}$$

decays exponentially as $t \to \infty$,

$$\langle \mathcal{O}(t,x)\mathcal{O}(0,0)\rangle \sim e^{-2\pi T_H \Delta_+ t}$$

AdS$_3$

associated with zero temperature

Metric

$$ds^2 = -\cosh^2\rho d\tau^2 + d\rho^2 + \sinh^2\rho d\phi^2$$

The boundary on which the corresponding CFT lives is the cylinder $\mathbb{R} \times S^1$.

Upon a change of coordinates,

$$y = \cosh^2\rho \ , \quad x = \frac{\tau}{2\pi T}$$

metric identical to the BTZ black hole metric with $y = r^2/r_h^2$, $r_h = 2\pi T$.

cf. with corresponding CFT:

- write propagator in terms of invariant distance in the embedding

$$\mathcal{P}(X, X') = (X - X')^2$$

where

$$X^0 = \cosh\rho\cos\tau \ , \quad X^3 = \cosh\rho\sin\tau$$
$$X^1 = \sinh\rho\cos\phi \ , \quad X^2 = \sinh\rho\sin\phi$$

and similarly for X'.

propagator on the boundary

$$G(\tau,\phi;\tau',\phi') \sim \lim_{\rho,\rho'\to\infty} \mathcal{P}^{-\Delta_+/2}$$

In this limit,

$$\mathcal{P} \sim \cosh(\tau-\tau') - \cos(\phi-\phi')$$

therefore,

$$G(\tau,\phi;\tau',\phi') \sim \frac{1}{(\cos(\tau-\tau') - \cos(\phi-\phi'))^{\Delta_+/2}}$$

real poles

$$\omega = p + 2(n + \tfrac{1}{2}\Delta_+ - 1) \ , \ \ p\in\mathbb{Z} \ , \ \ n = 1,2,\ldots$$

$\Rightarrow$ oscillatory behavior

CFT calculation

(without reference to the corresponding AdS)

two-point function of a massless scalar on the cylinder $\mathbb{R} \times S^1$ is

$$G_0(\tau, \phi; \tau', \phi') \sim T \sum_{j=-\infty}^{\infty} \int \frac{dk}{2\pi} \, e^{-ik\cdot x} \, \frac{i}{k^2}\bigg|_{k^0=2\pi j T}$$

After integrating over k, summing over j and subtracting an irrelevant (infinite) constant, we obtain

$$G_0(\tau, \phi; \tau', \phi') \sim \ln \mathcal{P}$$

For a scalar operator $\mathcal{O}$ of dimension Δ, the two point function then reads

$$G(\tau, \phi; \tau', \phi') \equiv \langle T(\mathcal{O}(\tau, \phi)\mathcal{O}(\tau', \phi'))\rangle \sim \frac{1}{\mathcal{P}^{\Delta/2}}$$

as before.

SUMMARY

- high temperature limit: BTZ black hole
- zero temperature limit: AdS space (locally equivalent to BTZ black hole)
- intermediate temperature? Hard
  Correlator on torus of periods $1/T$ and 1.

Strong coupling

AdS_3 arises in type IIB superstring theory in the near horizon limit of a large number of D1 and D5 branes.

- Low energy excitations form a gas of strings wound around a circle with winding number $\mathtt{k}$ and target space T^4.
- They are described by a strongly coupled CFT_2 whose central charge is

$$c = 6\mathtt{k} \sim \frac{1}{G} \gg 1$$

At finite temperature, the thermal CFT_2 has entropy

$$S \sim \mathtt{k} \sim \frac{1}{G}$$

BTZ black hole:

If radius of horizon is $o(1)$, then so is area of horizon

$$A \sim 1$$

Bekenstein-Hawking entropy:

$$S = \frac{A}{4G} \sim \frac{1}{G}$$

in agreement with CFT.

Poincaré recurrence time:

$$t_P \sim e^S \sim o(e^{1/G})$$

To understand this, one ought to include contributions to gravity correlators

- beyond the semi-classical approximation

which will modify the black-hole background.

Wormhole metric

$$ds^2 = -(\sinh^2 y + \lambda^2)\, dt^2 + dy^2 + \cosh^2 y\, d\phi^2$$

In the limit $\lambda \to 0$, reduces to BTZ black hole.

no horizon at $y = 0$:

- wormhole has a very narrow throat ($o(\lambda)$) joining two regions of space-time with two distinct boundaries (at $y \to \pm\infty$, respectively).
- Information may flow in both directions through the throat.
- modification significant near the "horizon" point $y = 0$.
- As $y \to 0$, time-like distance is $ds^2 \approx -\lambda^2 dt^2$,
  $\Rightarrow$ time scale of system is $\sim 1/\lambda$.
  $\Rightarrow$ Poincaré recurrence time

$$t_P \sim o(1/\lambda)$$

  as advertised.

λ will be fixed upon comparison with CFT.

wave equation

$$\frac{1}{\cosh y\,(\sinh^2 y+\lambda^2)^{1/2}}\left(\cosh y\,(\sinh^2 y+\lambda^2)^{1/2}\,\Psi'\right)'+\left(\frac{\omega^2}{\sinh^2 y+\lambda^2}+\frac{k^2}{\cosh^2 y}\right)\Psi=m^2\Psi$$

to be solved along the entire real axis ($y\in\mathbb{R}$)

cf. black hole: $y\geq 0$, horizon at $y=0$.

solve wave equation in the small-λ limit ($\lambda\ll 1$).

$\Rightarrow$ quantization condition

$$\left(\frac{2}{\lambda}\right)^{2i\omega}=\frac{\mathcal{B}_-}{\mathcal{B}_+}=\frac{\Gamma(-i\omega)\Gamma(h_++\frac{i}{2}(\omega+k))\Gamma(h_++\frac{i}{2}(\omega-k))}{\Gamma(+i\omega)\Gamma(h_+-\frac{i}{2}(\omega+k))\Gamma(h_+-\frac{i}{2}(\omega-k))}$$

$\Rightarrow$ discrete spectrum of *real* frequencies

For small ω,

$$\omega_n\approx\left(n+\tfrac{1}{2}\right)\frac{\pi}{\ln\frac{2}{\lambda}}\quad,\quad n\in\mathbb{Z}$$

$\Rightarrow$ periodicity with period $L_{eff}\sim\ln(1/\lambda)$.

cf. with CFT (string winding $\mathtt{k}$ times around circle of length $o(1)$)

$L_{eff} \sim \mathtt{k} \sim 1/G$

$$\lambda \sim o(e^{-1/G})$$

as promised.

Notice: $L_{eff} \ll t_P$, $\therefore$ two time scales.

In the limit $\lambda \to 0$ (or, equivalently, $\mathtt{k} \to \infty$),

$\Rightarrow$ spectrum of real frequencies becomes continuous,

$\Rightarrow$ emergence of a horizon.

$\Rightarrow$ QNMs emerge

It should be emphasized that for no other value of λ, no matter how small, do complex poles arise.

Stability

IMPORTANT: study classical stability of solutions to Einstein's eqs of General Relativity.

- solution not stable $\Rightarrow$ cannot be found in nature, unless instability timescale is much larger than the age of our Universe.

EXAMPLE: Schwarzschild spacetime is stable against all kinds of perturbations, massive or massless

- Schwarzschild geometry appropriate to study astrophysical objects.

EXAMPLE: Kerr spacetime (rotating black hole) is stable against massless perturbations but not against massive bosonic fields, but instability timescale is much larger than the age of the Universe.

AdS space

unstable solutions are common especially with extra dimensions

- ADS/CFT correspondence: A black hole corresponds to a thermal state on the CFT.
- $D > 5$ of interest to string theory

- Schwarzschild black holes are stable
- black branes are classically unstable against tensorial gravitational perturbations
  [Gregory and Laflamme]
- ultra-spinning black holes are similarly unstable
  [Emparan and Myers]

4d rotating black hole

[Detweiler]

for massive scalar of mass μ, radial wave equation:

$$\frac{d}{dr}\left(\Delta\frac{dR}{dr}\right)+\left\{\frac{\omega^2(r^2+a^2)^2-4aMm\omega r+m^2a^2}{\Delta}\right.$$
$$\left.-\mu^2r^2-a^2\omega^2-\ell(\ell+1)\right\}R=0,$$

where $\Delta = r^2 - 2Mr + a^2 = (r-r_+)(r-r_-)$, rotation parameter

$$a=\frac{J}{M}$$

for $\mu, \omega \ll 1/M$.

Away from horizon ($r \gg M$), approximate

$$\frac{d^2}{dr^2}(rR)+\left[-k^2+\frac{2M\mu^2}{r}-\frac{\ell(\ell+1)}{r^2}\right]rR=0\ ,\quad k^2=\mu^2-\omega^2$$

Solution in terms of confluent hypergeometric function,

$$R(r)=(2kr)^\ell e^{-kr}U(\ell+1-M\mu^2/k,2(\ell+1),2kr)$$

Near horizon ($r \ll \ell/|k|$), approximate

$$z(z+1)\frac{d}{dz}\left[z(z+1)\frac{dR}{dz}\right] + \left[P^2 - \ell(\ell+1)z(z+1)\right]R = 0$$

where $P = \frac{am-2Mr_+\omega}{r_+-r_-}$, $z = \frac{r-r_+}{r_+-r_-}$.

Solution in terms of hypergeometric function,

$$R(z) = \left(\frac{z}{z+1}\right)^{iP} F(-\ell, \ell+1; 1-2iP; z+1)$$

Matching in overlap region ($M \ll r \ll \ell/|k|$),

$$\omega_n \approx \mu + i\gamma_n \quad , \qquad n \in \mathbb{N}$$

where

$$\gamma_n = \mathcal{C}_{\ell n}\mu(\mu M)^{4(\ell+1)}\frac{am}{M-2\mu r_+}\prod_{j=1}^{\ell}\left[j^2\left(1-\frac{a^2}{M^2}\right)+\left(\frac{am}{M}-2\mu r_+\right)^2\right]$$

$$\mathcal{C}_{\ell n} = \frac{2^{2(2\ell+1)}(2\ell+1+n)!(\ell!)^2}{(\ell+1+n)^{2(\ell+2)}(2\ell+1)^2 n!((2\ell)!)^4}$$

For $m > 0$, $\gamma_n > 0 \Rightarrow$ instability!

Fastest growing mode with $\ell = 1$, $m = 1$, $n = 2$ (2p state) and

$$\tau = \frac{1}{\gamma} = \frac{24}{a\mu^2(\mu M)^7}$$

generally large.

No instability ($\gamma \to 0$, $\tau \to \infty$):

- Schwarzschild, $a \to 0$
- massless perturbation, $\mu \to 0$

higher-dimensional rotating black holes

[Cardoso, Siopsis, Yoshida]

GOAL

investigate *quantitatively* the stability of ultra-spinning black holes.

NB: For $D > 5$, there is no upper bound on rotation parameter

$$a = \frac{J}{M}$$

Use QNMs:

- $D \leq 5$: frequencies have *negative* imaginary part
  $\Rightarrow$ Kerr spacetime is stable.
- $D > 5$: if imaginary part flips sign as $a \to \infty$
  $\Rightarrow$ instability

Concentrate on scalar perturbations in $D = 6$.

Basic Equations

$D = 4 + n$. Wavefunction:

$$\Phi = e^{i\omega t - im\varphi} R(r) S(\vartheta) Y(\Omega)$$

where $Y(\Omega)$ are spherical harmonics.

Angular equation:

$$\frac{1}{\sin\vartheta\cos^n\vartheta}\left(\frac{d}{d\vartheta}\sin\vartheta\cos^n\vartheta\frac{dS}{d\vartheta}\right)+\left[\omega^2a^2\cos^2\vartheta\right.$$
$$\left.-m^2\csc^2\vartheta-j(j+n-1)\sec^2\vartheta+A\right]S=0,$$

radial equation:

$$r^{-n}\frac{d}{dr}\left(r^n\Delta\frac{dR}{dr}\right)+\left\{\frac{\left[\omega(r^2+a^2)-ma\right]^2}{\Delta}\right.$$
$$\left.-\frac{j(j+n-1)a^2}{r^2}-\lambda\right\}R=0,$$

where $\lambda := A-2m\omega a+\omega^2a^2$ and

$$\Delta = r^2+a^2-\mu r^{1-n}$$

horizon at $r=r_H$, $\Delta=0$

► exactly one positive root for arbitrary $a>0$.

$\Rightarrow$ no bound on a

Numerical Computation

Use method by Leaver which makes use of a continued fraction representation

- can determine resonant frequency ω and separation constant A with very high accuracy.

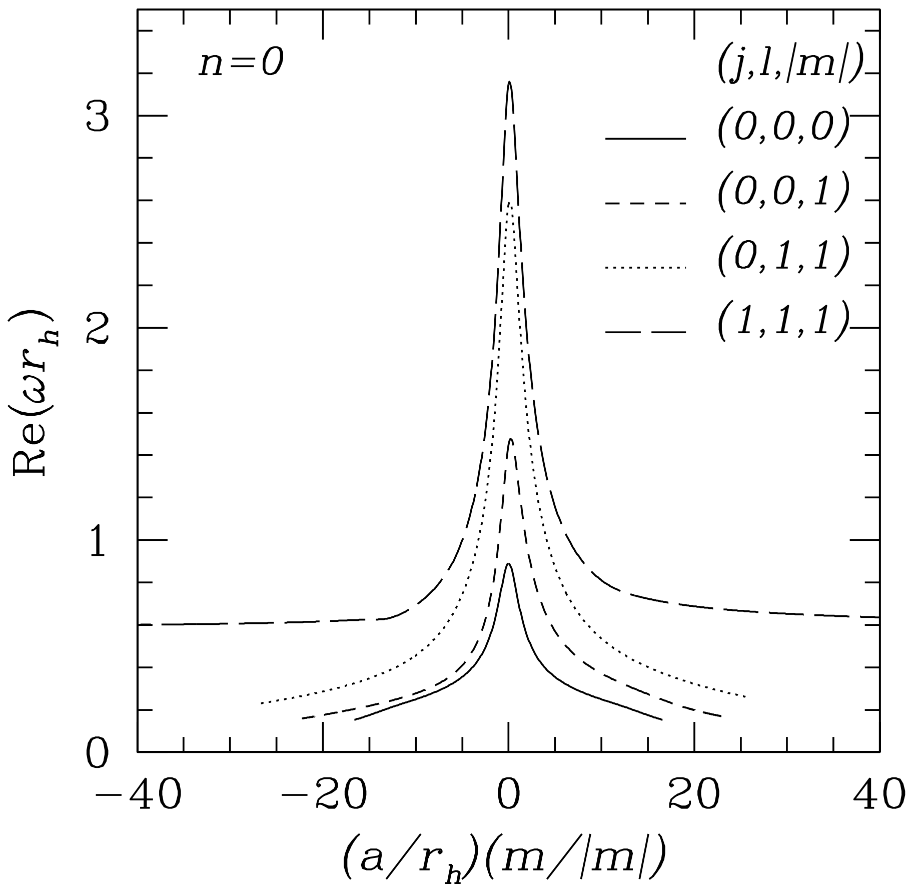
define dimensionless quantities

$$\omega_* := \omega r_H \quad , \quad a_* := a/r_H$$

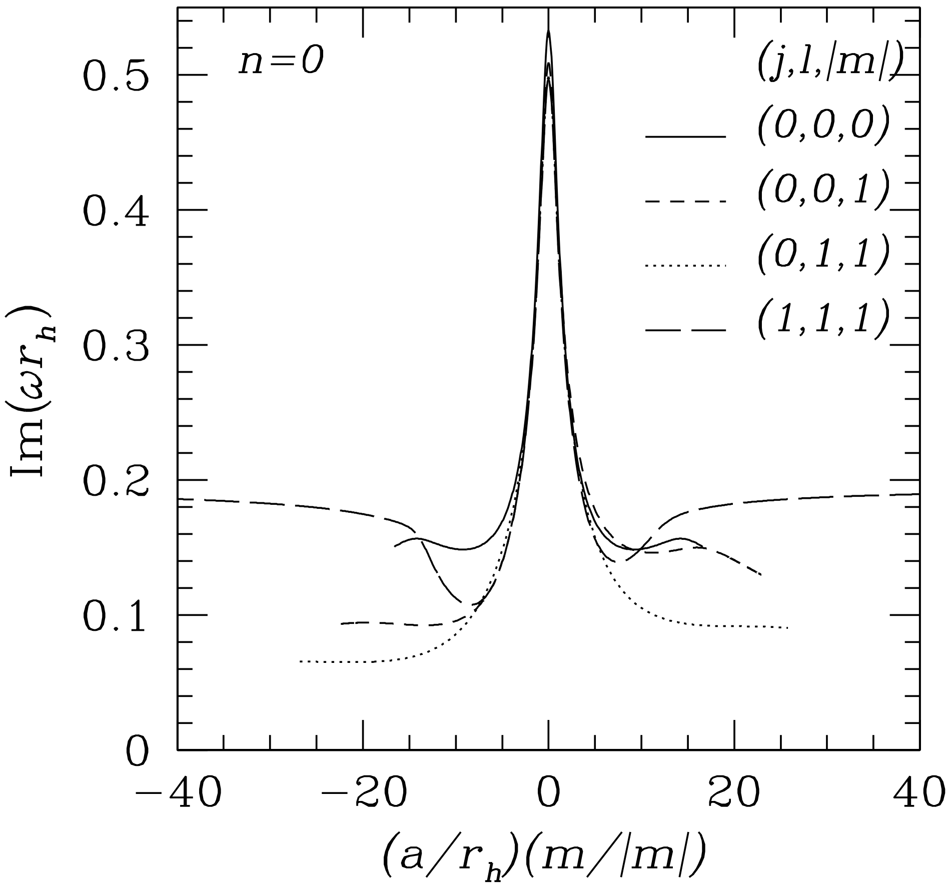
RESULTS

l	j	m	$\omega_{QN}^{\text{Num}} r_H$	$\omega_{QN}^{\text{WKB}} r_H$	% Re	% Im
0	0	0	0.8894+0.5331i	0.7682+0.5265i	13.6	1.2
0	0	1	1.4465+0.5093i	1.3846+0.4933i	4.3	3.1
1	0	1	2.5791+0.4989i	2.5455+0.4942i	1.3	0.9
1	1	1	3.1478+0.4973i	3.1205+0.4944i	0.9	0.6

We compare our numerical results for the QN frequencies of six-dimensional, non-rotating black holes, with results obtained through WKB techniques, and we also indicate the error involved using the WKB approach. The results refer to the fundamental mode of several l, j, m perturbations.



Real part of the fundamental QN frequency as a function of the rotation parameter a for some $l\,,j\,,m$ values. The maximum is reached at zero rotation, and as a increases the real part of ω_{QN} decreases monotonically.



Imaginary part of the fundamental QN frequency as a function of the rotation parameter a for some l , j , m values. Notice that, for all values of a the imaginary part is always positive, which means that even ultra-spinning black holes are stable.

Analytical results

bring radial wave equation into a Schrödinger-like form

$$\Psi(y) = y^{n/2}(g(y))^{1/4}R(y)\ ,\ g(y) = (y^2 + a_*^2)^2 - a_*^2\hat{\Delta}$$

where $\hat{\Delta} = \Delta/r_H^2$.

tortoise coordinate y_* defined by

$$\frac{dy}{dy_*} = \frac{\hat{\Delta}}{\sqrt{g(y)}}$$

wave equation:

$$-\frac{d^2\Psi}{dy_*^2} + V[y(y_*)]\ \Psi = (\omega_* - m\Omega_*)^2\Psi\ .$$

The potential is

$$\frac{g(y)}{\hat{\Delta}}V(y) = A - 2m\omega_*\Omega_* - m^2 a_*^2 \Omega_*^2 + \frac{n(n+2)}{4} + \frac{[j(j+n-1)+n/2(n/2-1)]a^2}{y^2} + \frac{n^2(1+a^2)}{4y^{n+1}} + m\Omega_* \frac{y^2-1}{\hat{\Delta}}[(m\Omega_* - 2\omega_*)(y^2+a^2)+ma] + \frac{1}{4}\left(-\frac{5(g')^2}{4g^2}\hat{\Delta} + \frac{g''}{g}\hat{\Delta} + \frac{g'}{g}\hat{\Delta}'\right)$$

angular velocity of horizon,

$$\Omega_H = \frac{\Omega_*}{r_H} = \frac{a}{r_H^2 + a^2}\,.$$

The potential

- vanishes at the horizon ($y = 1$)
- approaches a constant as $y \to \infty$ ($V \to m\Omega_*(m\Omega_* - 2\omega_*)$)

Limit $a \to 0$

Schwarzschild wave equation

$$-\frac{d^2\Psi}{dy_*^2} + V_0[y(y_*)]\,\Psi = \omega_*^2 \Psi\,,$$

where

$$V_0(y) = \left(1 - \frac{1}{y^{n+1}}\right) \left\{ \frac{L^2 - \frac{1}{4}}{y^2} + \frac{(n+2)^2}{4y^{n+3}} \right\}\,,$$

and $L = 2\ell + j + |m| + \frac{n+1}{2}$.

expand around the maximum of the potential: $V_0'(y_{max}) = 0$,

$$y_{max} = \left(\frac{n+3}{2}\right)^{1/(n+1)} + o(1/L)\,.$$

$$V_0[y(y_*)] \approx \alpha^2 - \beta^2 (y_* - y_*(y_{max}))^2\,,$$

where

$$\alpha^2 = \frac{n+1}{n+3}\left(\frac{2}{n+3}\right)^{2/(n+1)} L^2 + o(1)$$

$$\beta^2 = \frac{(n+1)^3}{(n+3)^2}\left(\frac{2}{n+3}\right)^{4/(n+1)} L^2 + o(1)\,.$$

solutions

$$\Psi_N = H_N(\sqrt{i\beta}x)e^{i\beta x^2/2} \ , \ \ N = 0, 1, 2, \dots$$

where H_N are Hermite polynomials.
corresponding eigenvalues

$$\omega_* = C(n)\left\{L + i\sqrt{n+1}(N + \tfrac{1}{2})\right\} + o(1/L)\,,$$

with

$$C(n) = \sqrt{\frac{n+1}{n+3}}\left(\frac{2}{n+3}\right)^{\frac{1}{n+1}}\,.$$

In agreement with standard WKB approach

Limit $a \to \infty$

the potential to leading order in $1/a$,

$$V_\infty(y) = y^{n-3}\left(1 - \frac{1}{y^{n-1}}\right) \times$$
$$\left\{\left(j + \frac{n-1}{2}\right)^2 - \left(\frac{n-3}{4}\right)^2 + \frac{(n+1)^2}{16y^{n-1}}\right\}$$

well-defined as $a \to \infty$.

For $n > 3$:

- potential is positive and diverges as $y \to \infty$, so subleading terms are needed to estimate the eigenfrequencies.

For $n \leq 3$ (six and seven dimensions),

- ω approaches a constant value independent of a which is easily found by solving the Schrödinger equation. This asymptotic value only depends on j.

In 6d ($n = 2$), the potential exhibits a maximum and may be approximated by an inverted harmonic oscillator potential
► as in the Schwarzschild limit.

The frequencies can be found explicitly as functions of j

j	$\omega_{QN}^{\text{Analy}} r_H$
0	$0 + 0.162i$
1	$0.576 + 0i$
2	$1.078 + 0i$

Results of an analytical WKB type scheme for computing the QN frequencies in the ultra-spinning regime, $a \to \infty$. The results depend only on j. This scheme shows that ω_{QN} asymptotes to a constant value, which is consistent both qualitatively and quantitatively with the numerical results.

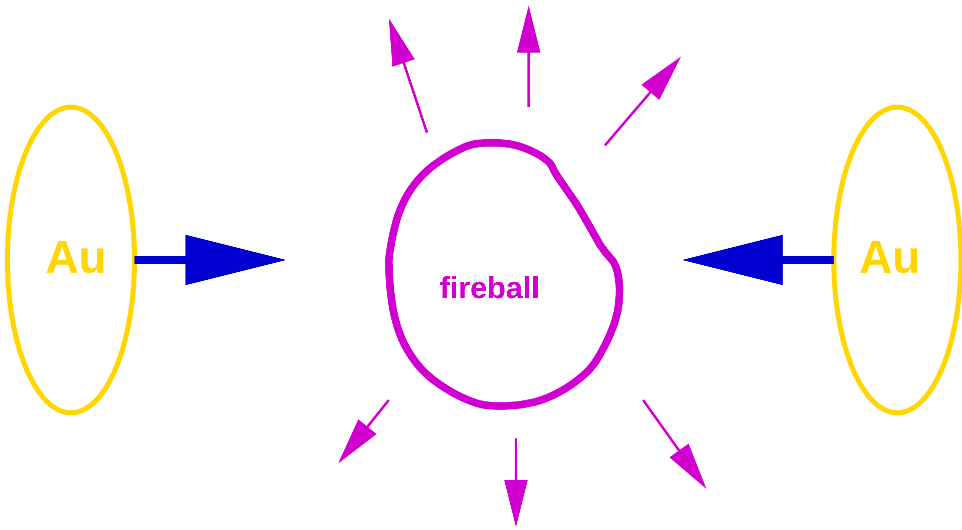
CONCLUSION

- geometry is stable against scalar field perturbations, even if the black hole is ultra-spinning.
- interesting to check stability against gravitational perturbations (Gregory-Laflamme-type instability)

Hydrodynamics

"A second unexpected connection comes from studies carried out using the Relativistic Heavy Ion Collider, a particle accelerator at Brookhaven National Laboratory. This machine smashes together nuclei at high energy to produce a hot, strongly interacting plasma. Physicists have found that some of the properties of this plasma are better modeled (via duality) as a tiny black hole in a space with extra dimensions than as the expected clump of elementary particles in the usual four dimensions of spacetime. The prediction here is again not a sharp one, as the string model works much better than expected. String-theory skeptics could take the point of view that it is just a mathematical spinoff. However, one of the repeated lessons of physics is unity - nature uses a small number of principles in diverse ways. And so the quantum gravity that is manifesting itself in dual form at Brookhaven is likely to be the same one that operates everywhere else in the universe."

– Joe Polchinski



AdS/CFT correspondence and hydrodynamics

[Policastro, Son and Starinets]

correspondence between $\mathcal{N} = 4$ SYM in the large N limit and type-IIB string theory in $\mathrm{AdS}_5 \times \mathrm{S}^5$.

- ► in strong coupling limit of field theory, string theory is reduced to classical supergravity, which allows one to calculate all field-theory correlation functions.
  $\hookrightarrow$ nontrivial prediction of gauge theory/gravity correspondence

entropy of $\mathcal{N} = 4$ SYM theory in the limit of large 't Hooft coupling is precisely 3/4 the value in zero coupling limit.

long-distance, low-frequency behavior of any interacting theory at finite temperature must be described by fluid mechanics (hydrodynamics).

universality: hydrodynamics implies very precise constraints on correlation functions of conserved currents and stress-energy tensor:

► correlators fixed once a few transport coefficients are known.

Hydrodynamics

conserved current: j^μ

chemical potential $\mu = 0$, so in thermal equilibrium

$$\langle j^0 \rangle = 0$$

retarded thermal Green function

$$G^R_{\mu\nu}(\omega, \boldsymbol{q}) = -i \int d^4x \, e^{-iq\cdot x} \, \theta(t) \langle [j_\mu(x), \, j_\nu(0)] \rangle \,,$$

where $q = (\omega, \boldsymbol{q})$, $x = (t, \boldsymbol{x})$

▶ determines response to a small external source coupled to the current.

ω and $\boldsymbol{q}$ small:

- external perturbation varies slowly in space and time
- macroscopic hydrodynamic description for its evolution is possible.

diffusion equation

$$\partial_0 j^0 = D \nabla^2 j^0 \,,$$

where D is a diffusion constant with dimension of length.

$\Rightarrow$ overdamped mode, dispersion relation

$$\omega = -iDq^2 \,,$$

pole at $\omega = -iDq^2$ in the complex ω-plane, in the retarded correlation functions of j^0

stress-energy tensor $T^{\mu\nu}$

$$\partial_0 \tilde{T}^{00} + \partial_i T^{0i} = 0 \,,$$
$$\partial_0 T^{0i} + \partial_j \tilde{T}^{ij} = 0 \,,$$

where

$$\tilde{T}^{00} = T^{00} - \rho, \qquad \rho = \langle T^{00} \rangle \,,$$
$$\tilde{T}^{ij} = T^{ij} - p\delta^{ij} = -\frac{1}{\rho + p}\left[\eta\left(\partial_i T^{0j} + \partial_j T^{0i} - \frac{2}{3}\delta^{ij}\partial_k T^{0k}\right) + \zeta\delta^{ij}\partial_k T^{0k}\right],$$

ρ (p): energy density (pressure)

η (ζ): shear (bulk) viscosity.

two types of eigenmodes:

- the shear modes - transverse fluctuations of momentum density T^{0i}, with a purely imaginary eigenvalue

$$\omega = -iDq^2 \ , \quad D = \frac{\eta}{\rho + p} ,$$

- sound wave - simultaneous fluctuation of energy density T^{00} and longitudinal component of momentum density T^{0i}, with dispersion relation

$$\omega = u_s q - \frac{i}{2} \frac{1}{\rho + p} \left(\zeta + \frac{4}{3} \eta \right) q^2 , \qquad u_s^2 = \frac{\partial p}{\partial \rho} .$$

conformal theory $\Rightarrow$ stress-energy tensor is traceless, so

$$\rho = 3p \ , \quad \zeta = 0 \ , \quad u_s = \frac{1}{\sqrt{3}}$$

Gravity

The non-extremal 3-brane background is a solution of type-IIB low energy equations of motion.

In the near-horizon limit $r \ll R$, the metric becomes

$$ds_{10}^2 = \frac{(\pi T R)^2}{u}\left(-f(u)dt^2 + dx^2 + dy^2 + dz^2\right) + \frac{R^2}{4u^2 f(u)}du^2 + R^2 d\Omega_5^2\,,$$

where $T = \frac{r_0}{\pi R^2}$ is Hawking temperature, $u = \frac{r_0^2}{r^2}$, $f(u) = 1 - u^2$.

The horizon corresponds to $u = 1$, spatial infinity to $u = 0$.

gauge theory/gravity correspondence:

- background metric with non-extremality parameter r_0 is dual to $\mathcal{N} = 4$ $SU(N)$ SYM at finite temperature T in the limit of $N \to \infty$, $g_{YM}^2 N \to \infty$.

retarded Green function

$$G_{\mu\nu,\lambda\rho}(\omega,\boldsymbol{q}) = -i\int d^4x\, e^{-iq\cdot x}\,\theta(t)\langle[T_{\mu\nu}(x),\, T_{\lambda\rho}(0)]\rangle\,.$$

▶ Deduce

$$G_{xy,xy}(\omega,\boldsymbol{q}) = -\frac{N^2T^2}{16}\left(i\,2\pi T\omega + q^2\right)\,.$$

shear viscosity of strongly coupled $\mathcal{N}=4$ SYM plasma (Kubo formula)

$$\eta = \lim_{\omega\to 0}\frac{1}{2\omega}\int dt\, d\boldsymbol{x}\, e^{i\omega t}\,\langle[T_{xy}(x),\, T_{xy}(0)]\rangle = \frac{\pi}{8}N^2T^3\,.$$

▶ Deduce correlators

$$\begin{aligned}
G_{tx,tx}(\omega,\boldsymbol{q}) &= \frac{N^2\pi T^3 q^2}{8(i\omega-\mathcal{D}q^2)} + \mathcal{O}(\mathfrak{w}^2,\mathfrak{w}\mathfrak{q}^2,\mathfrak{q}^4)\,,\\
G_{tx,xz}(\omega,\boldsymbol{q}) &= -\frac{N^2\pi T^3 \omega q}{8(i\omega-\mathcal{D}q^2)} + \mathcal{O}(\mathfrak{w}^2,\mathfrak{w}\mathfrak{q}^2,\mathfrak{q}^4)\,,\\
G_{xz,xz}(\omega,\boldsymbol{q}) &= \frac{N^2\pi T^3 \omega^2}{8(i\omega-\mathcal{D}q^2)} + \mathcal{O}(\mathfrak{w}^2,\mathfrak{w}\mathfrak{q}^2,\mathfrak{q}^4)\,,
\end{aligned}$$

where $\mathcal{D} = \frac{1}{4\pi T}$

Deduce η:

- recall from hydrodynamics $\mathcal{D} = \frac{\eta}{\rho+p}$.

Entropy:

$$s = \frac{3}{4} s_0 = \frac{\pi^2}{2} N^2 T^3 \,,$$

where s_0 is entropy at zero coupling.

From $s = \frac{\partial P}{\partial T}$, $\rho = 3p$, deduce $\rho + p = \frac{\pi^2}{2} N^2 T^4$, $\quad \therefore$

$$\eta = \frac{\pi}{8} N^2 T^3 \,, \quad \frac{\eta}{s} = \frac{1}{4\pi}$$

- agrees with Kubo formula.
- no agreement unless $s = \frac{3}{4} s_0$.

behavior of η as a function of the 't Hooft coupling

$$\eta = f_\eta(g^2_{\mathsf{YM}}N)N^2T^3$$

where $f_\eta(x) \sim \frac{1}{-x^2 \ln x}$ for $x \ll 1$ and $f_\eta(x) = \frac{\pi}{8}$ for $x \gg 1$.

- At weak coupling,

$$\frac{\eta}{s} \gg \frac{1}{4\pi}$$

Conformal soliton flow

the holographic image on Minkowski space of the global AdS$_5$-Schwarzschild black hole is a spherical shell of plasma first contracting and then expanding.

▶ conformal map from $S^{d-2} \times \mathbb{R}$ to $(d-1)$-dim Minkowski space

[Friess, Gubser, Michalogiorgakis, Pufu]

QNMs $\Rightarrow$ properties of plasma

- $$\frac{v_2}{\delta} = \frac{1}{6\pi}\mathrm{Re}\,\frac{\omega^4 - 40\omega^2 + 72}{\omega^3 - 4\omega}\sin\frac{\pi\omega}{2}$$
  - $v_2 = \langle \cos 2\phi \rangle$ at $\theta = \frac{\pi}{2}$ (mid-rapidity), average with respect to energy density at late times
  - $\delta = \frac{\langle y^2 - x^2 \rangle}{\langle y^2 + x^2 \rangle}$ (eccentricity at time $t = 0$).

  Numerically, $\frac{v_2}{\delta} = 0.37$, *cf.* with result from RHIC data, $\frac{v_2}{\delta} \approx 0.323$

  [PHENIX Collaboration, arXiv:nucl-ex/0608033]

- thermalization time

$$\tau = \frac{1}{2|\mathrm{Im}\,\omega|} \approx \frac{1}{8.6 T_{\mathrm{peak}}} \approx 0.08 \text{ fm/c} \quad , \qquad T_{\mathrm{peak}} = 300 \text{ MeV}$$

cf. with RHIC result $\tau \sim$ 0.6 fm/c

[Arnold, Lenaghan, Moore, Yaffe, Phys. Rev. Lett. **94** *(2005) 072302]*

Not in agreement, but encouragingly small

► perturbative QCD yields $\tau \gtrsim$ 2.5 fm/c.

[Baier, Mueller, Schiff, Son; Molnar, Gyulassy]

Analytical calculation of low-lying QNMs

[G. S., hep-th/0702079]

Vector perturbations

introduce the coordinate

$$u = \left(\frac{r_H}{r}\right)^{d-3}$$

wave equation

$$-(d-3)^2 u^{\frac{d-4}{d-3}} \hat{f}(u) \left(u^{\frac{d-4}{d-3}} \hat{f}(u) \Psi' \right)' + \hat{V}_{\mathsf{V}}(u)\Psi = \hat{\omega}^2 \Psi \quad , \quad \hat{\omega} = \frac{\omega}{r_H}$$

where prime denotes differentiation with respect to u and

$$\hat{f}(u) \equiv \frac{f(r)}{r^2} = 1 - u^{\frac{2}{d-3}} \left(u - \frac{1-u}{r_H^2} \right)$$

$$\hat{V}_{\mathsf{V}}(u) \equiv \frac{V_{\mathsf{V}}}{r_H^2} = \hat{f}(u) \left\{ \hat{L}^2 + \frac{(d-2)(d-4)}{4} u^{-\frac{2}{d-3}} \hat{f}(u) - \frac{(d-1)(d-2)\left(1 + \frac{1}{r_H^2}\right)}{2} u \right\}$$

where $\hat{L}^2 = \frac{\ell(\ell+d-3)}{r_H^2}$

First consider large black hole limit $r_H \to \infty$ keeping $\hat{\omega}$ and $\hat{L}$ fixed (small).
Factoring out the behavior at the horizon ($u = 1$)

$$\Psi = (1-u)^{-i\frac{\hat{\omega}}{d-1}} F(u)$$

the wave equation simplifies to

$$\mathcal{A}F'' + \mathcal{B}_{\hat{\omega}}F' + \mathcal{C}_{\hat{\omega},\hat{L}}F = 0$$

where

$$\begin{aligned}
\mathcal{A} &= -(d-3)^2 u^{\frac{2d-8}{d-3}}(1-u^{\frac{d-1}{d-3}}) \\
\mathcal{B}_{\hat{\omega}} &= -(d-3)[d-4-(2d-5)u^{\frac{d-1}{d-3}}]u^{\frac{d-5}{d-3}} - 2(d-3)^2 \frac{i\hat{\omega}}{d-1} \frac{u^{\frac{2d-8}{d-3}}(1-u^{\frac{d-1}{d-3}})}{1-u} \\
\mathcal{C}_{\hat{\omega},\hat{L}} &= \hat{L}^2 + \frac{(d-2)[d-4-3(d-2)u^{\frac{d-1}{d-3}}]}{4} u^{-\frac{2}{d-3}} \\
&\quad - \frac{\hat{\omega}^2}{1-u^{\frac{d-1}{d-3}}} + (d-3)^2 \frac{\hat{\omega}^2}{(d-1)^2} \frac{u^{\frac{2d-8}{d-3}}(1-u^{\frac{d-1}{d-3}})}{(1-u)^2} \\
&\quad -(d-3)\frac{i\hat{\omega}}{d-1} \frac{[d-4-(2d-5)u^{\frac{d-1}{d-3}}]u^{\frac{d-5}{d-3}}}{1-u} - (d-3)^2 \frac{i\hat{\omega}}{d-1} \frac{u^{\frac{2d-8}{d-3}}(1-u^{\frac{d-1}{d-3}})}{(1-u)^2}
\end{aligned}$$

solve perturbatively:

$$(\mathcal{H}_0 + \mathcal{H}_1)F = 0$$

where

$$\begin{aligned} \mathcal{H}_0 F &\equiv \mathcal{A}F'' + \mathcal{B}_0 F' + \mathcal{C}_{0,0} F \\ \mathcal{H}_1 F &\equiv (\mathcal{B}_{\hat{\omega}} - \mathcal{B}_0)F' + (\mathcal{C}_{\hat{\omega},\hat{L}} - \mathcal{C}_{0,0})F \end{aligned}$$

Expanding the wavefunction perturbatively,

$$F = F_0 + F_1 + \dots$$

at zeroth order we have

$$\mathcal{H}_0 F_0 = 0$$

whose acceptable solution is

$$F_0 = u^{\frac{d-2}{2(d-3)}}$$

regular at horizon ($u = 1$) and boundary ($u = 0$, or $\Psi \sim r^{-\frac{d-2}{2}} \to 0$ as $r \to \infty$).

Wronskian

$$\mathcal{W} = \frac{1}{u^{\frac{d-4}{d-3}}(1 - u^{\frac{d-1}{d-3}})}$$

Another linearly independent solution

$$\check{F}_0 = F_0 \int \frac{\mathcal{W}}{F_0^2}$$

unacceptable $\because$ diverges at both horizon ($\check{F}_0 \sim \ln(1-u)$ for $u \approx 1$) and boundary ($\check{F}_0 \sim u^{-\frac{d-4}{2(d-3)}}$ for $u \approx 0$, or $\Psi \sim r^{\frac{d-4}{2}} \to \infty$ as $r \to \infty$).

At first order we have

$$\mathcal{H}_0 F_1 = -\mathcal{H}_1 F_0$$

whose solution may be written as

$$F_1 = F_0 \int \frac{\mathcal{W}}{F_0^2} \int \frac{F_0 \mathcal{H}_1 F_0}{\mathcal{A}\mathcal{W}}$$

The limits of the inner integral may be adjusted at will

$\because$ this amounts to adding an arbitrary amount of the unacceptable solution.

To ensure regularity at the horizon, choose one of the limits at $u = 1$

- integrand is regular at the horizon, by design.

at the boundary ($u = 0$),

$$F_1 = \breve{F}_0 \int_0^1 \frac{F_0 \mathcal{H}_1 F_0}{\mathcal{A}\mathcal{W}} + \text{regular terms}$$

The coefficient of the singularity ought to vanish,

$$\int_0^1 \frac{F_0 \mathcal{H}_1 F_0}{\mathcal{A}\mathcal{W}} = 0$$

$\Rightarrow$ constraint on the parameters (dispersion relation)

$$\mathbf{a}_0 \hat{L}^2 - i\mathbf{a}_1 \hat{\omega} - \mathbf{a}_2 \hat{\omega}^2 = 0$$

After some algebra, we arrive at

$$\mathbf{a}_0 = \frac{d-3}{d-1} \;, \quad \mathbf{a}_1 = d-3$$

The coefficient $\mathbf{a}_2$

- may also be found explicitly for each dimension d,

- it cannot be written as a function of d in closed form.
- it does not contribute to the dispersion relation at lowest order.
- E.g., for $d = 4, 5$, we obtain, respectively

$$\mathbf{a}_2 = \frac{65}{108} - \frac{1}{3}\ln 3 \ , \quad \frac{5}{6} - \frac{1}{2}\ln 2$$

quadratic in $\hat{\omega}$ eq. has two solutions,

$$\hat{\omega}_0 \approx -i\frac{\hat{L}^2}{d-1} \ , \quad \hat{\omega}_1 \approx -i\frac{d-3}{\mathbf{a}_2} + i\frac{\hat{L}^2}{d-1}$$

In terms of frequency ω and quantum number ℓ,

$$\omega_0 \approx -i\frac{\ell(\ell+d-3)}{(d-1)r_H} \ , \quad \frac{\omega_1}{r_H} \approx -i\frac{d-3}{\mathbf{a}_2} + i\frac{\ell(\ell+d-3)}{(d-1)r_H^2}$$

The smaller of the two, ω_0,

- is inversely proportional to the radius of the horizon,
- is not included in the asymptotic spectrum.

The other solution, ω_1,

- is a crude estimate of the first overtone in the asymptotic spectrum.
- shares important features with asymptotic spectrum:
  - it is proportional to r_H
  - dependence on ℓ is $O(1/r_H^2)$.

The approximation may be improved by including higher-order terms

- ▶ Inclusion of higher orders also increases the degree of the polynomial in the dispersion relation whose roots then yield approximate values of more QNMs.
- ▶ this method reproduces the asymptotic spectrum albeit not in an efficient way.

Include finite size effects:

$\hookrightarrow$ use perturbation (assuming $1/r_H$ is small) and replace $\mathcal{H}_1$ by

$$\mathcal{H}_1' = \mathcal{H}_1 + \frac{1}{r_H^2}\mathcal{H}_H$$

where

$$\mathcal{H}_H F \equiv \mathcal{A}_H F'' + \mathcal{B}_H F' + \mathcal{C}_H F$$

$$\begin{aligned}
\mathcal{A}_H &= -2(d-3)^2 u^2 (1-u) \\
\mathcal{B}_H &= -(d-3)u\left[(d-3)(2-3u) - (d-1)\frac{1-u}{1-u^{\frac{d-1}{d-3}}} u^{\frac{d-1}{d-3}}\right] \\
\mathcal{C}_H &= \frac{d-2}{2}\left[d-4-(2d-5)u-(d-1)\frac{1-u}{1-u^{\frac{d-1}{d-3}}} u^{\frac{d-1}{d-3}}\right]
\end{aligned}$$

Interestingly, zeroth order wavefunction F_0 is eigenfunction of $\mathcal{H}_H$,

$$\mathcal{H}_H F_0 = -(d-2) F_0$$

$\therefore$ first-order finite-size effect is simple shift of angular momentum

$$\hat{L}^2 \to \hat{L}^2 - \frac{d-2}{r_H^2}$$

$\therefore$ QNMs of lowest frequency are modified to

$$\omega_0 = -i\frac{\ell(\ell+d-3)-(d-2)}{(d-1)r_H} + O(1/r_H^2)$$

For $d = 4, 5$, we have respectively,

$$\omega_0 = -i\frac{(\ell-1)(\ell+2)}{3r_H} \quad , \qquad -i\frac{(\ell+1)^2-4}{4r_H}$$

in agreement with numerical results

[Cardoso, Konoplya and Lemos; Friess, Gubser, Michalogiorgakis and Pufu]

AdS/CFT correspondence

dual to AdS Schwarzschild bh: gauge theory fluid on boundary of AdS ($S^{d-2} \times \mathbb{R}$).

consider the fluid dynamics ansatz

$$u_i = \mathcal{K} e^{-i\Omega\tau} \mathbb{V}_i$$

u_i: (small) velocity of a point in the fluid, $\mathbb{V}_i$: vector harmonic on S^{d-2}.

Demanding that this ansatz satisfy standard eqs of linearized hydrodynamics,

$\Rightarrow$ constraint on the frequency of the perturbation Ω which yields

$$\Omega = -i\frac{\ell(\ell+d-3)-(d-2)}{(d-1)r_H} + O(1/r_H^2)$$

[Michalogiorgakis and Pufu]

in perfect agreement with its dual counterpart.

Scalar perturbations

$\widehat{V}_{\mathsf{V}}$ replaced by

$$\begin{aligned}
\widehat{V}_{\mathsf{S}}(u) &= \frac{\widehat{f}(u)}{4}\left[\widehat{m}+\left(1+\frac{1}{r_H^2}\right)u\right]^{-2} \\
&\times \left\{d(d-2)\left(1+\frac{1}{r_H^2}\right)^2 u^{\frac{2d-8}{d-3}} - 6(d-2)(d-4)\widehat{m}\left(1+\frac{1}{r_H^2}\right)u^{\frac{d-5}{d-3}}\right. \\
&+(d-4)(d-6)\widehat{m}^2 u^{-\frac{2}{d-3}} + (d-2)^2\left(1+\frac{1}{r_H^2}\right)^3 u^3 \\
&+2(2d^2-11d+18)\widehat{m}\left(1+\frac{1}{r_H^2}\right)^2 u^2 \\
&+\frac{(d-4)(d-6)\left(1+\frac{1}{r_H^2}\right)^2}{r_H^2}u^2 - 3(d-2)(d-6)\widehat{m}^2\left(1+\frac{1}{r_H^2}\right)u \\
&\left.-\frac{6(d-2)(d-4)\widehat{m}\left(1+\frac{1}{r_H^2}\right)}{r_H^2}u + 2(d-1)(d-2)\widehat{m}^3 + d(d-2)\frac{\widehat{m}^2}{r_H^2}\right\}
\end{aligned}$$

where $\widehat{m} = 2\frac{\ell(\ell+d-3)-(d-2)}{(d-1)(d-2)r_H^2} = \frac{2(\ell+d-2)(\ell-1)}{(d-1)(d-2)r_H^2}$

In the large black hole limit $r_H \to \infty$ with $\hat{m}$ fixed, potential simplifies

$$\hat{V}_S^{(0)}(u) = \frac{1-u^{\frac{d-1}{d-3}}}{4(\hat{m}+u)^2}\Big\{d(d-2)u^{\frac{2d-8}{d-3}} - 6(d-2)(d-4)\hat{m}u^{\frac{d-5}{d-3}}$$
$$+(d-4)(d-6)\hat{m}^2u^{-\frac{2}{d-3}} + (d-2)^2u^3$$
$$+2(2d^2-11d+18)\hat{m}u^2 - 3(d-2)(d-6)\hat{m}^2u + 2(d-1)(d-2)\hat{m}^3\Big\}$$

- additional singularity due to double pole of scalar potential at $u = -\hat{m}$.
- desirable to factor out the behavior not only at the horizon, but also at the boundary and the pole of the scalar potential,

$$\Psi = (1-u)^{-i\frac{\hat{\omega}}{d-1}} \frac{u^{\frac{d-4}{2(d-3)}}}{\hat{m}+u} F(u)$$

$\therefore$ wave equation

$$\mathcal{A}F'' + \mathcal{B}_{\hat{\omega}}F' + \mathcal{C}_{\hat{\omega}}F = 0$$

where

$$\begin{aligned}
\mathcal{A} &= -(d-3)^2 u^{\frac{2d-8}{d-3}}(1-u^{\frac{d-1}{d-3}}) \\
\mathcal{B}_{\hat{\omega}} &= -(d-3)u^{\frac{2d-8}{d-3}}(1-u^{\frac{d-1}{d-3}})\left[\frac{d-4}{u} - \frac{2(d-3)}{\hat{m}+u}\right] \\
&\quad -(d-3)[d-4-(2d-5)u^{\frac{d-1}{d-3}}]u^{\frac{d-5}{d-3}} - 2(d-3)^2\frac{i\hat{\omega}}{d-1}\frac{u^{\frac{2d-8}{d-3}}(1-u^{\frac{d-1}{d-3}})}{1-u} \\
\mathcal{C}_{\hat{\omega}} &= -u^{\frac{2d-8}{d-3}}(1-u^{\frac{d-1}{d-3}})\left[-\frac{(d-2)(d-4)}{4u^2} - \frac{(d-3)(d-4)}{u(\hat{m}+u)} + \frac{2(d-3)^2}{(\hat{m}+u)^2}\right] \\
&\quad -\left[\left\{d-4-(2d-5)u^{\frac{d-1}{d-3}}\right\}u^{\frac{d-5}{d-3}} + 2(d-3)\frac{i\hat{\omega}}{d-1}\frac{u^{\frac{2d-8}{d-3}}(1-u^{\frac{d-1}{d-3}})}{1-u}\right]\left[\frac{d-4}{2u} - \frac{d-3}{\hat{m}+u}\right] \\
&\quad -(d-3)\frac{i\hat{\omega}}{d-1}\frac{[d-4-(2d-5)u^{\frac{d-1}{d-3}}]u^{\frac{d-5}{d-3}}}{1-u} - (d-3)^2\frac{i\hat{\omega}}{d-1}\frac{u^{\frac{2d-8}{d-3}}(1-u^{\frac{d-1}{d-3}})}{(1-u)^2} \\
&\quad +\frac{\hat{V}_S^{(0)}(u)-\hat{\omega}^2}{1-u^{\frac{d-1}{d-3}}} + (d-3)^2\frac{\hat{\omega}^2}{(d-1)^2}\frac{u^{\frac{2d-8}{d-3}}(1-u^{\frac{d-1}{d-3}})}{(1-u)^2}
\end{aligned}$$

Define zeroth-order wave equation $\mathcal{H}_0 F_0 = 0$, where

$$\mathcal{H}_0 F \equiv \mathcal{A}F'' + \mathcal{B}_0 F'$$

Acceptable zeroth-order solution

$$F_0(u) = 1$$

- plainly regular at all singular points ($u = 0, 1, -\hat{m}$).
- corresponds to a wavefunction vanishing at the boundary ($\Psi \sim r^{-\frac{d-4}{2}}$ as $r \to \infty$).

Wronskian

$$\mathcal{W} = \frac{(\hat{m} + u)^2}{u^{\frac{2d-8}{d-3}}(1 - u^{\frac{d-1}{d-3}})}$$

Unacceptable solution: $\check{F}_0 = \int \mathcal{W}$

- can be written in terms of hypergeometric functions.
- for $d \geq 6$, has a singularity at the boundary, $\check{F}_0 \sim u^{-\frac{d-5}{d-3}}$ for $u \approx 0$, or $\Psi \sim r^{\frac{d-6}{2}} \to \infty$ as $r \to \infty$.
- for $d = 5$, acceptable wavefunction $\sim r^{-1/2}$; unacceptable $\sim r^{-1/2} \ln r$
- for $d = 4$, roles of F_0 and $\check{F}_0$ reversed; results still valid.
- $\check{F}_0$ is also singular (logarithmically) at the horizon ($u = 1$).

Working as in the case of vector modes, we arrive at the first-order constraint

$$\int_0^1 \frac{\mathcal{C}_{\hat{\omega}}}{\mathcal{A}\mathcal{W}} = 0$$

$\because \quad \mathcal{H}_1 F_0 \equiv (\mathcal{B}_{\hat{\omega}} - \mathcal{B}_0) F_0' + \mathcal{C}_{\hat{\omega}} F_0 = \mathcal{C}_{\hat{\omega}}$

$\therefore$ dispersion relation

$$\mathbf{a}_0 - \mathbf{a}_1 i\hat{\omega} - \mathbf{a}_2 \hat{\omega}^2 = 0$$

After some algebra, we obtain

$$\mathbf{a}_0 = \frac{d-1}{2} \frac{1+(d-2)\hat{m}}{(1+\hat{m})^2} \ , \quad \mathbf{a}_1 = \frac{d-3}{(1+\hat{m})^2} \ , \quad \mathbf{a}_2 = \frac{1}{\hat{m}} \{1 + O(\hat{m})\}$$

For small $\hat{m}$, the quadratic equation has solutions

$$\hat{\omega}_0^{\pm} \approx -i\frac{d-3}{2}\,\hat{m} \pm \sqrt{\frac{d-1}{2}\,\hat{m}}$$

related to each other by $\hat{\omega}_0^{+} = -\hat{\omega}_0^{-*}$

▶ general symmetry of the spectrum.

Finite size effects at first order amount to a shift of the coefficient $\mathbf{a}_0$ in the dispersion relation

$$\mathbf{a}_0 \to \mathbf{a}_0 + \frac{1}{r_H^2}\mathbf{a}_H$$

after some tedious but straightforward algebra, we obtain

$$\mathbf{a}_H = \frac{1}{\hat{m}}\{1 + O(\hat{m})\}$$

The modified dispersion relation yields the modes

$$\hat{\omega}_0^{\pm} \approx -i\frac{d-3}{2}\,\hat{m} \pm \sqrt{\frac{d-1}{2}\,\hat{m} + 1}$$

in terms of the quantum number ℓ,

$$\omega_0^{\pm} \approx -i(d-3)\,\frac{\ell(\ell+d-3)-(d-2)}{(d-1)(d-2)r_H} \pm \sqrt{\frac{\ell(\ell+d-3)}{d-2}}$$

in agreement with numerical results

[Friess, Gubser, Michalogiorgakis and Pufu]

- imaginary part inversely proportional to r_H, as in vector case
- finite real part independent of r_H
  $\hookrightarrow$ speed of sound $v_s = \frac{1}{\sqrt{d-2}}$ (due to conformal invariance)

AdS/CFT correspondence

perturb gauge theory fluid on the boundary of AdS ($S^{d-2} \times \mathbb{R}$) using the ansatz

$$u_i = \mathcal{K} e^{-i\Omega\tau} \nabla_i \mathbb{S} \quad , \quad \delta p = \mathcal{K}' e^{-i\Omega\tau} \mathbb{S}$$

u_i: (small) velocity of a point in the fluid,
δp: pressure perturbation,
$\mathbb{S}$: scalar harmonic on S^{d-2}.
Demanding that this ansatz satisfy eqs of linearized hydrodynamics,
$\Rightarrow$ frequency of perturbation Ω in perfect agreement with our analytic result.

Tensor perturbations

Unlike the other two cases, asymptotic spectrum is entire spectrum.
In large bh limit, wave equation

$$-(d-3)^2(u^{\frac{2d-8}{d-3}} - u^3)\Psi'' - (d-3)[(d-4)u^{\frac{d-5}{d-3}} - (2d-5)u^2]\Psi'$$
$$+ \left\{ \hat{L}^2 + \frac{d(d-2)}{4}u^{-\frac{2}{d-3}} + \frac{(d-2)^2}{4}u - \frac{\hat{\omega}^2}{1-u^{\frac{d-1}{d-3}}} \right\}\Psi = 0$$

For zeroth-order eq., set $\hat{L} = 0 = \hat{\omega}$
$\hookrightarrow$ two solutions are ($\Psi = F_0$ at zeroth order)

$$F_0(u) = u^{\frac{d-2}{2(d-3)}} \quad , \quad \check{F}_0(u) = u^{-\frac{d-2}{2(d-3)}} \ln\left(1 - u^{\frac{d-1}{d-3}}\right)$$

Neither behaves nicely at both ends ($u = 0, 1$)

$\therefore$ both are unacceptable.

$\therefore$ impossible to build a perturbation theory to calculate small frequencies.

in agreement with numerical results and in accordance with the

AdS/CFT correspondence

- there is no ansatz that can be built from tensor spherical harmonics $\mathbb{T}_{ij}$ satisfying the linearized hydrodynamic eqs because of the conservation and tracelessness properties of $\mathbb{T}_{ij}$.

CONCLUSIONS

- Quasi-normal modes are a powerful tool in understanding hydrodynamic behavior of gauge theory fluid at strong coupling
- RHIC's fireball can be described by a dual black hole
- RHIC and LHC may probe black holes and provide information on string theory as well as non-perturbative QCD effects.