Entropy Bounds in the Causal Sets Approach to Quantum Gravity

Petros Wallden*

21 September 2007
4th Agean Summer School, Mytilene

*Raman Research Institute, Bangalore.

Contents

- Introduction to Causal Sets
- Entropy Bounds
- Entanglement Entropy
- Where We Stand
- Causal Sets and Entropy Bounds
- Causal Sets and Black Hole Entropy
- Summary + Conclusions

Causal Sets: Definition

- Fundamentally Discrete Approach to Quantum Gravity.
  An individual causal set, corresponds to a discrete version of a *spacetime*.

- Mathematically it is:

  (a) Partially Ordered Set

  (b) The order relation is the *causal* relation.

  (c) Locally finite

- Partially Ordered Set: a set $\mathcal{P}$ with relation $\prec$, such that $\forall \quad x, y, z \in \mathcal{P}$:

  (a) $x \prec y$ and $y \prec z \Rightarrow x \prec z$: Transitivity

(b) $x \prec y$ and $y \prec x \Rightarrow x = y$: Acyclicity

- Locally Finite: $\{z|x \prec z \prec y\}$ is a finite set for all $x, y \in \mathcal{P}$.

- The metric of a globally hyperbolic space-time can be reconstructed uniquely from its causal relations up to a conformal factor (Malament).

- Locally finite $\rightarrow$ the remaining degree of freedom corresponding to the scale factor can be recovered by counting the number of elements.

Causal Sets: Kinematics

- Links: $x \prec y$ are linked if $\nexists \quad z| \quad x \prec z \prec y$.

- Chain C: $\quad \forall \quad x, y \in C \quad x \prec y$ or $y \prec x$.

- Proper time: Given $x \prec y$, we define the 'proper'-time $d_t(x, y) := \max |C_i \cap J^+(x) \cap J^-(y)|$, and $|A|$ is set cardinality. Is the maximum steps needed to go from x to y.

- Maximal (minimal) Elements: $x \in \mathcal{C}$, such that $\nexists \quad y \in \mathcal{C}$ where $x \prec y$.

- Alexandrov Neighborhood: Given $x \prec y$, we define $A(x, y) := \{J^+(x) \cap J^-(y)\}$

- 'Spacelike Distance': Given unrelated x, y we define $d_s(x, y) := \min d_t(u, v)$, where $u \in$

$J^-(x) \cap J^-(y)$ and $v \in J^+(x) \cap J^+(y)$. It is the minimum distance of their common past to their common future (some subtlety).

- Faithful Embedding: Preserves causal relations. The number of elements n mapped into *any* Alexandrov neighborhood, is equal to its spacetime volume V up to poisson fluctuations. Leads to continuum *approximation*.

- Dimension in Causets.

- Lorentz Invariance $\longrightarrow$ Poisson Distribution

$$P(n) = \frac{(\rho V)^n e^{-\rho V}}{n!} \tag{1}$$

  Note: lattice is NOT Lorentz invariant.

- Sprinkling in a Lorentzian manifold with some density $\rho \propto \alpha\hbar$ where $\alpha \sim O(1)$

Causal Sets: Dynamics

- Classical (stochastic) Dynamics: Sequential Growth

  (a) General Covariance

  (b) 'Bell's Causality'

- Quantum Dynamics:

  (a) Sum Over Causal Sets

  (b) Quantum Measure Theory

- 'Phenomenology': (heuristic) but natural *prediction* of the Cosmological Constant (Sorkin 1991).

Entropy Bounds

- Hawking Radiation: $T = \frac{\kappa \hbar}{2\pi}$

- Entropy of Black Hole: $S = \frac{A}{4\hbar G}$

- Generalized Second Law of Thermodynamics (GSL): $\delta S_{BH} + \delta S_{mat} \geq 0$

- General Validity of GSL $\rightarrow$ Entropy Bounds Geroch process, Susskind process and BH evaporation.

- Bekenstein Entropy Bound

$$S_{matter} \leq 2\pi E R \tag{2}$$

E, is the mass-energy of the matter, R is the radius of the smallest sphere that fits around the matter.
Physical assumptions are:
(a)Weakly gravitating matter and
(b) Asymptotically flat space.
Controversy (Unruh, Wald), for saturation of bound and the effect of Unruh radiation.

- Spherical (Susskind) Bound

$$S_{matter} \leq A/4 \tag{3}$$

where A is the area of the smallest sphere enclosing the matter.
Physical assumptions:
(a) Spherical Symmetry *or*
(b) Weak Gravity

Bekenstein is stronger for gravitational stable systems (2M$\leq R$) in 4 dim but for $D > 4$ Bekenstein gives only $S \leq \frac{D-2}{8} A$.

- Covariant (Bousso) Bound

$$S[L(B)] \leq A(B)/4 \tag{4}$$

The entropy of *any* light-sheet L of a surface (codimension 2) B does not exceed the area of this surface.
Light-sheet is defined as the area spanned by *in-going* light rays starting from B.
Implies spherical bound in its range of validity (and GSL holds for BH formation), but needs further analysis for Geroch process.

Entanglement Entropy

- Von Neumann Entropy: $S_\rho = \mathrm{Tr}(\rho \ln \rho)$ (zero for pure states).

- Composite System: $\mathcal{H}_A \otimes \mathcal{H}_B \rightarrow$ effective state $\rho_A = \mathrm{Tr}_B(\rho_{AB})$
  It is mixed if the state is entangled.

- Important property: $S_{\rho_A} = S_{\rho_B}$

- If subsystem is a subspace of the configuration space (e.g. interior of a horizon), the only thing the interior and the exterior share, is their common boundary. Thus heuristically we expect the *entanglement* entropy to be proportional to (some power of) the boundary.

- Consider a lattice of coupled harmonic oscillators, as a discretization of the scalar field.

$$H = \frac{1}{2}\left(\sum_i p_i^2 + \sum_{ij} x_i V_{ij} x_j\right) \qquad (5)$$

  Tracing-out a subregion of the lattice, results to Von Neumann entropy proportional to the 'area' divided by the discreteness cut-off (Sorkin).

- Entanglement entropy diverges in the continuum limit.

Where We Stand

- BH entropy as a (semi-)classical result.
- BH entropy as entanglement entropy.
- BH entropy as fundamental degrees of freedom of a full Quantum Gravity Theory.
- Indications for discreteness:
  (a) Entanglement entropy diverges for continuum spacetimes.
  (b) Counting fundamental degrees of freedom of BH gives finite answer $\longrightarrow$ (naively) finite number of ‘atoms’ of spacetime.

Entropy Bounds in Causets

- Bekenstein Bound: Assume we have matter with support on a (spacelike) subset of the causal set $V = \{x_i\}$. We define $R = \min d_t(u, v)$ where $u \in \cap_i J^-(x_i)$ and $v \in \cap_i J^+(x_i)$. (smallest distance between common future and common past).

  In weak gravity limit, this corresponds to the radius of the smaller sphere containing the matter and we can use this quantity for the bound $S_{matter} \leq 2\pi ER$.

  May attempt generalization to strong gravity regime.

- Spherical Bound:

  (a) Entropy contained in Σ must 'flow outside' by crossing the boundary of its future domain of dependence $D^+(\Sigma)$.

(b) Proposal: Maximal entropy contained in Σ is the number of maximal elements in the future of dependence,

$$S_{max} =< n >= \int_{D^+(\Sigma)} e^{-V(x)} dx \qquad (6)$$

where $V(x) := J^+(x) \cap D^+(\Sigma)$.

(c) The resulting number $S_{max} = A/4$ up to a constant of $O(1)$. This constant depends *only* on the dimension, and not on the shape of Σ or geometry of space-time.

(d) The constant is fixed by the exact discreteness scale (which in any case turns out to be of plank order). As we consider higher dimensions, it converges to one giving the exact result (Rideout and Zohren).

Black Hole Entropy in Causets

Count 'atoms' of the horizon.

- Initial suggestion: Count links between common past $J^{-}(H) \cap J^{-}(\Sigma)$ and common future $J^{+}(H) \cap J^{+}(\Sigma)$ of the horizon H and the spacelike surface Σ. It corresponds to information flowing in the horizon from outside. A further condition of maximality was imposed in order to avoid counting irrelevant non-local links (Dou and Sorkin).

  Got same proportionality constant for different spacetimes (static and far of equilibrium as well).

  Unfortunately the kind of 'non-local' links where not counted out in non-spherical symmetric cases.

- Alternative definition of 'atoms': Consider 'diamonds'. Points x, y, u, v where $x, y \in J^{-}(H) \cap J^{-}(\Sigma)$ and $u, v \in J^{+}(H) \cap J^{+}(\Sigma)$, i.e. x, y are outside the BH and to the past of Σ while u, v are inside and to the future of Σ. Furthermore we require all relations $u \prec x, u \prec y, x \prec v$ and $y \prec v$ to be links.

  It is clear that it does *not* suffer from the previous 'non-local' problems. However, details have not yet been worked out.

- Entanglement entropy $\rightarrow$ need QFT on a causal set (work in progress).

Summary and Conclusions

- Introduction to Causal Sets.
- Review of existing Entropy Bounds.
- Entanglement Entropy.
- Bekenstein Bound as causal sets property ('proper time' between common past and common future).
- Spherical Bound as number of maximal elements in the future domain of dependence.
- Black Hole entropy as number of 'atoms' of the horizon.