Gravitational Plane-wave Scattering by a Rotating Black Hole

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Aegean Summer School 07

Outline

1. Introduction
   - Plane wave scattering
   - Geodesics and approximations
   - Schwarzschild scattering
2. Analysis
   - Partial wave expansion
   - On-axis scattering
   - Helicity non-conservation
3. New Results
   - Absorption
   - Scattering cross sections
   - Discussion

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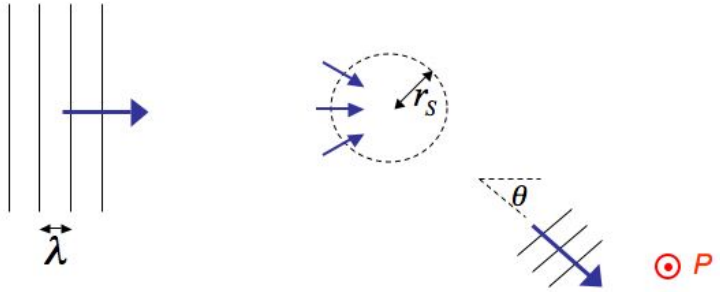
Time-Independent Plane Wave Scattering

An infinite monochromatic plane wave impinges on an isolated black hole:



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Dimensionless Parameters:

- BH coupling : $M\omega \sim r_s/\lambda$
- BH rotation : $0 \leq a \leq 1$

Physical Observables:

- σ_a : absorption cross section.
- $\frac{d\sigma}{d\Omega}$: differential scattering cross section.
- $0 < P < 1$: partial polarisation.

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field	spin	equations
scalar	$s = 0$	[Klein-Gordon]
• neutrino	$s = 1/2$	[Dirac]
photon	$s = 1$	[Maxwell]
gravitational	$s = 2$	[Teukolsky]

- *Usually* massless (neutrino?)
- Classical fields ('first-quantised')
- 'Weak' : neglect back-reaction
- Linear ($s = 0, 1/2, 1$) or,
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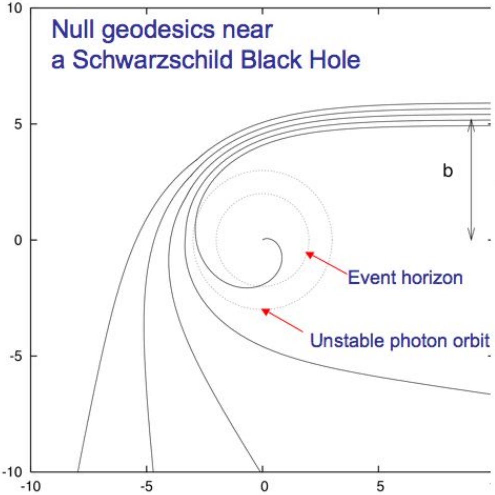
Non-rotating BH: Null Geodesics



$$b_c = 3\sqrt{3}M$$

Horizon at
r = 2M

Deflection-angle Approximations:

- Weak-field deflection:

$$\Delta\theta \approx 4M/b$$

$$\Rightarrow \lim_{\theta\to 0} \frac{d\sigma}{d\Omega} \sim 1/\theta^4$$

- Strong-field deflection: Unstable orbit at $r = 3M$

$$\Delta\theta \sim -\ln\left[(b - b_c)/3.48M\right]$$

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In the long-wavelength limit (low coupling $M\omega \ll 1$) :

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scalar	$\frac{1}{\sin^4(\theta/2)}$	
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gravitational	$\frac{\cos^8(\theta/2)+\sin^8(\theta/2)}{\sin^4(\theta/2)}$	: anomalous extra term!

General rule:

$$\lim_{\lambda\to\infty}\left(\frac{1}{M^2}\frac{d\sigma}{d\Omega}\right) \sim \frac{\cos^{4s}(\theta/2)}{\sin^4(\theta/2)} + \delta_{s,2}\sin^4(\theta/2)$$

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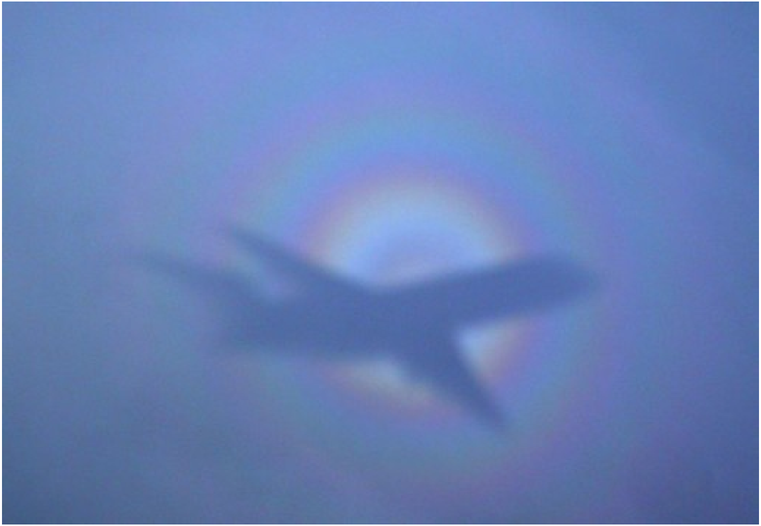
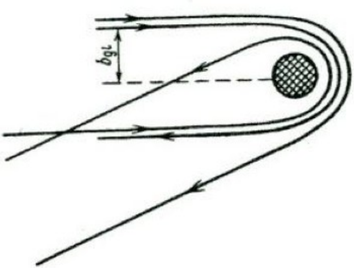
Strong Field: Glory Scattering



Glory Scattering in Optics (II)

Glory Scattering Approximation

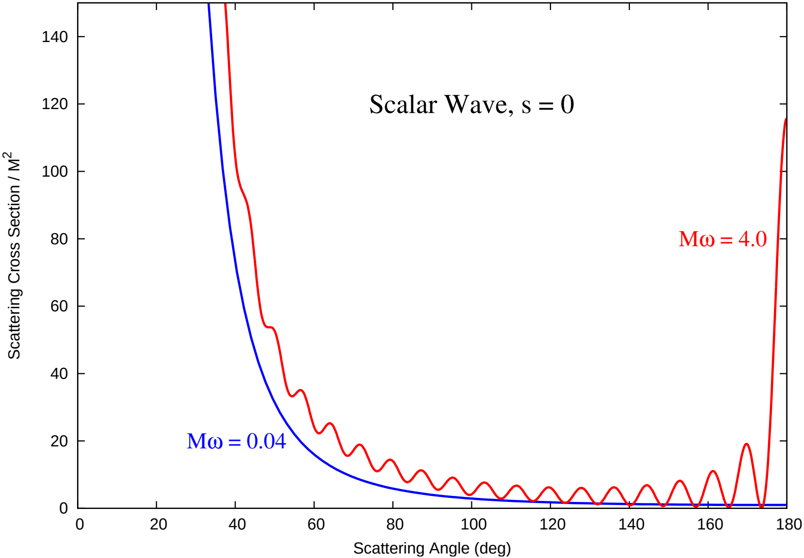


- Semi-classical (WKB) approximation:

$$\lim_{\theta\to\pi} \frac{d\sigma}{d\Omega} \sim 2\pi M^2 \times 4.3M\omega \times J_{2|s|}\,(5.3465\, M\omega\, \sin\theta)$$

- Zero flux in backward direction for polarised fields ($s \neq 0$).

Schwarzschild Scattering Cross Section $\frac{d\sigma}{d\Omega}$



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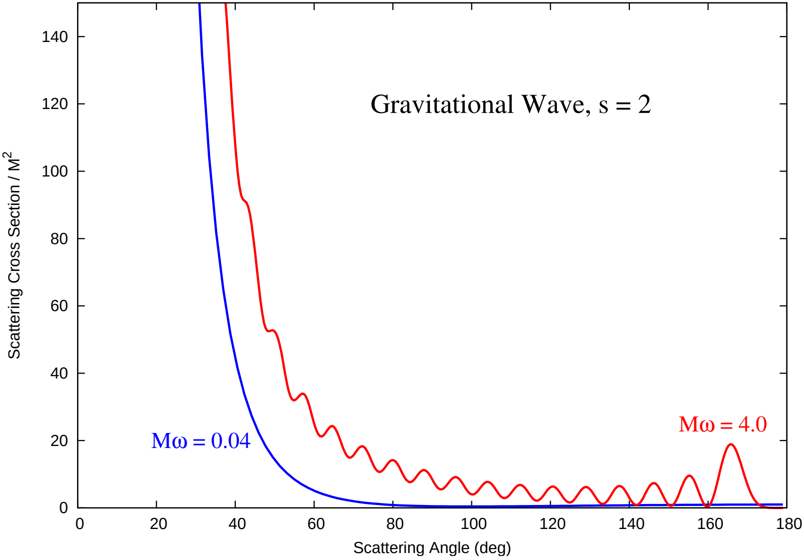
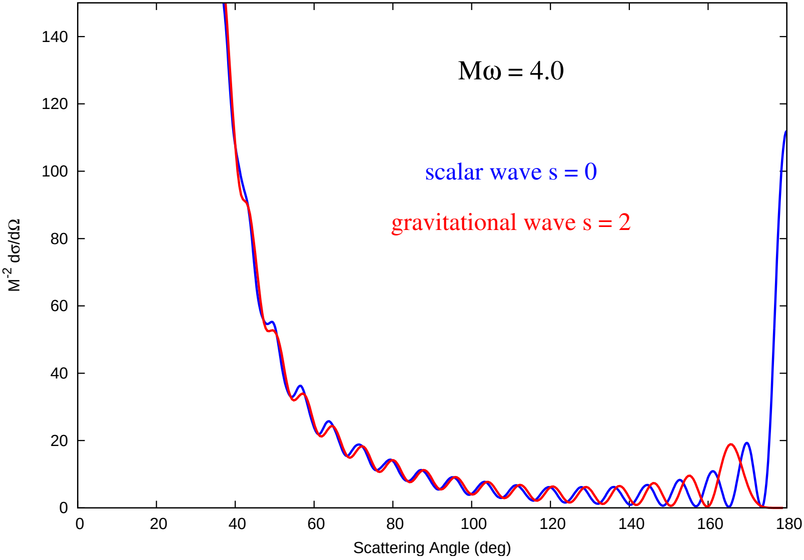
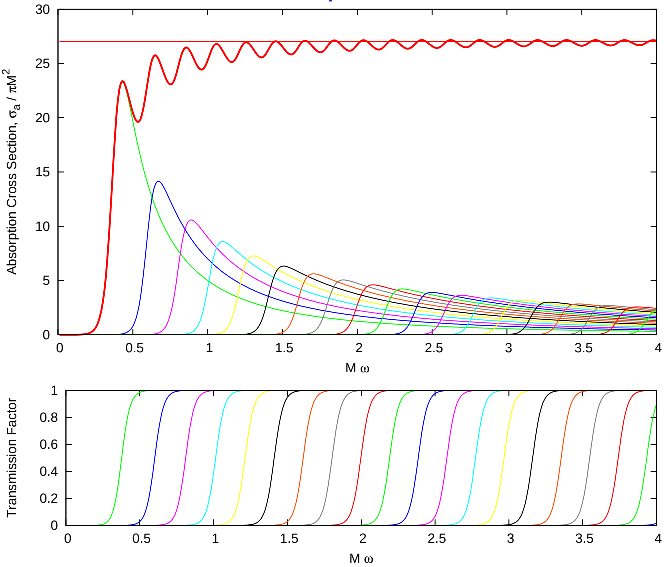


Schwarzschild Scattering: $s = 0$ versus $s = 2$



Schwarzschild Absorption Cross Section σ_a



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Rotating BH: On-axis Incidence

Scattering cross section [Futterman, Handler & Matzner 1988] :

$$\frac{d\sigma}{d\Omega} = |f(\theta)|^2 + |g(\theta)|^2$$

$$f(\theta) = \frac{\pi}{i\omega} \sum_{l,P=\pm 1} \left[\exp(2i\delta_{l2}^{P}) - 1\right] {}_{-2}S_l^2(0)\, {}_{-2}S_l^2(\theta)$$

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- The phase shifts δ^{P}_{lm} are parity-dependent:

$$\exp(2i\delta^{+}_{lm}) = \frac{\mathrm{Re}(C) + 12iM\omega}{\mathrm{Re}(C) - 12iM\omega} \exp(2i\delta^{-}_{lm})$$

where

$$|\mathrm{Re}(C)|^2 = \left((\lambda+2)^2 + 4a\omega m - 4a^2\omega^2\right)(\lambda^2 + 36a\omega m - 36a^2\omega^2) + (2\lambda+3)(96a^2\omega^2 - 48a\omega m) - 144a^2\omega^2$$

- Hence $g \neq 0$ and helicity is not conserved!
- $\Rightarrow$ Non-zero flux in backward-scattering direction. For $a = 0$,

$$\frac{d\sigma}{d\Omega}(\theta = \pi) = |g(\theta)|^2 = M^2$$

- $\Rightarrow$ (Linearised) grav. field on BH background does not behave as pure spin-2 field

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Computing Phase Shifts

Numerical Method:

- Solve radial Teukolsky equation with ingoing boundary condition at (outer) horizon, $R(r) \sim |r - r_+|^{-i\tilde{\omega}r}$.
- 'Peeling' behaviour ($R_{out} \sim r^3 e^{i\omega r^*}$ and $R_{in} \sim r^{-1} e^{-i\omega r^*}$) makes things numerically awkward.
- Use Sasaki and Nakamura transformation, $\chi(R, R')$, so that $\chi_{out} \sim e^{i\omega r^*}$ and $\chi_{in} \sim e^{-i\omega r^*}$
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Calculating Spin-Weighted Spheroidal Harmonics

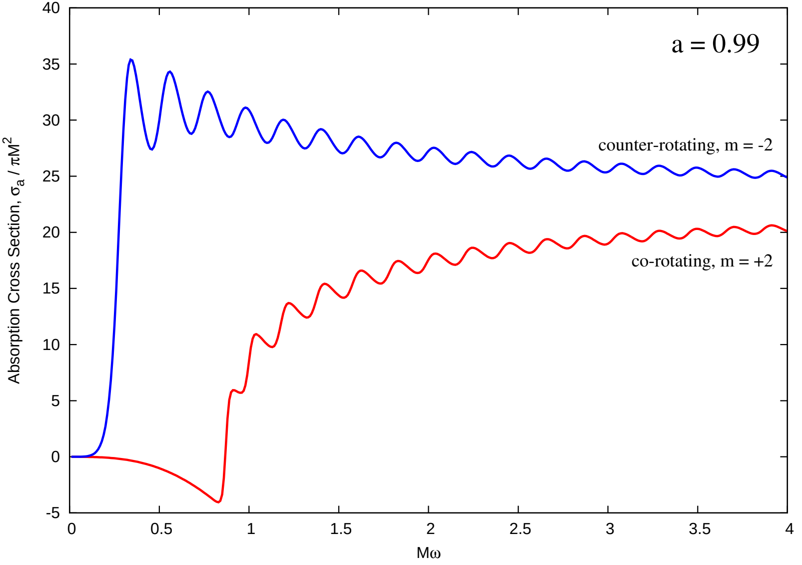
Method:

- Expand spheroidal harmonics ${}_{-2}S_l^2$ in spherical harmonics ${}_{-2}Y_l^m$:

$$ {}_{-2}S_l^m(\theta; a\omega) = \sum_j b_{lj} \, {}_{-2}Y_j^m(\theta) $$

- Solve eigenvalue equation: $\mathbf{A}\,\mathbf{b} = -E_{lm}\,\mathbf{b}$
- $\mathbf{A}$ is quin-diagonal $\Rightarrow$ fast method
- Compute eigenfunctions using recurrence relations [S.A. Hughes (2000)] .

Kerr Absorption Cross Section: $a = 0.99$



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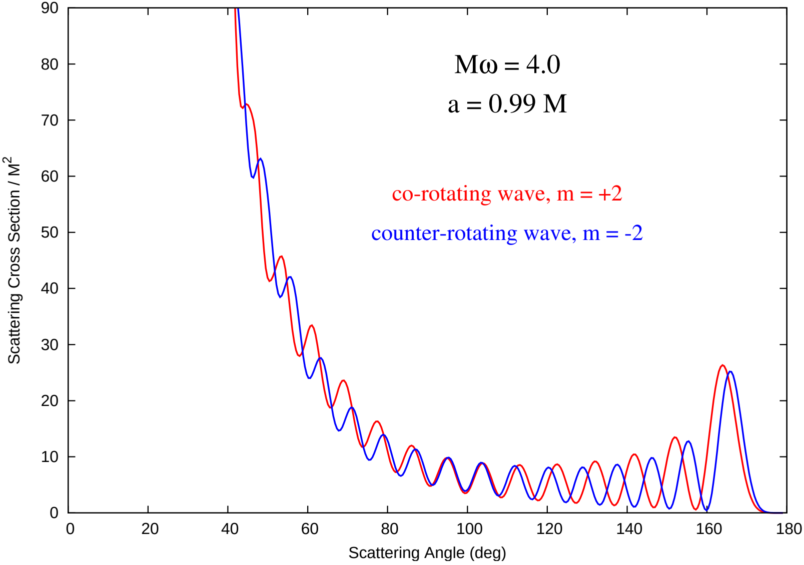
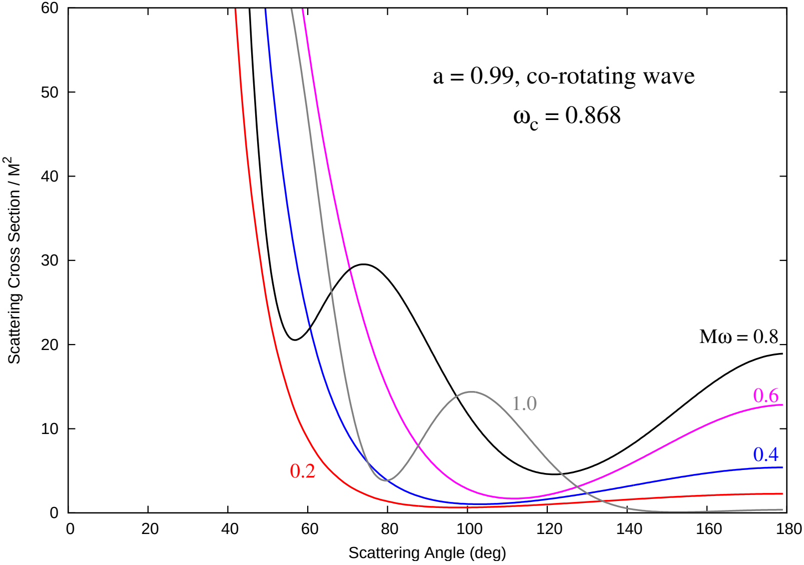


Superradiant Kerr Scattering: $a = 0.99$



Flux in Backward Direction

- Back-scattered flux carries ang. momentum with same sense as BH spin.

- Backward flux mainly from superradiant mode ($l = 2$, $m = 2$)

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Conclusions

- BH grav. wave scattering violates helicity conservation.
- Long-range force $\Rightarrow$ divergence on-axis $d\sigma/d\Omega \sim 1/\theta^4$
- Unstable orbits $\Rightarrow$ glory scattering oscillations.
- BH rotation $\Rightarrow$ enhanced back-scattering ($\sim \times 20$).
- Further work: diffraction patterns from higher-dimensional BHs?

References:

- Monograph: *Scattering from black holes*. Futterman, Handler & Matzner (1988) Cambridge.
- Long-wavelength: P.J. Westervelt *PRD* **3** (1971) p2319; P.C. Peters *PRD* **13** (1976) p775; W.K. de Logi & S.J. Kovacs *PRD* **16** (1977) p237.
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- Radial & angular eqns: S.A. Hughes *PRD* **61** (2000) 084004; M. Sasaki & T. Nakamura *Prog. Theor. Phys.* **67** (1982) p1788.
- Scalar scattering from rotating BH: K. Glampedakis & N. Andersson. *Class. Q. Grav.* **18** (2001) p1939.