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Quantum causality in kappa Minkowski

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based on *Quantum causality in kappa -Minkowski and related constraints,*
N. Franco, K. Hersent, V Maris, and J-C Wallet.

Causality in Lorentzian Manifold

Lorentzian Manifold and metric :

$$(M, g)$$

Points which characterize events :

$$x, y \in \mathcal{M}$$

Tangents vecteurs for causal evolution

$$\gamma : \mathbb{R} \to \mathcal{M}, \quad g(\gamma'(t), \gamma'(t)) \leq 0$$

y is in the causal futur of x if :

$$\exists \gamma, \quad t_1 \leq t_2 \in \mathbb{R}, \quad \gamma(t_1) = x, \quad \gamma(t_2) = y$$

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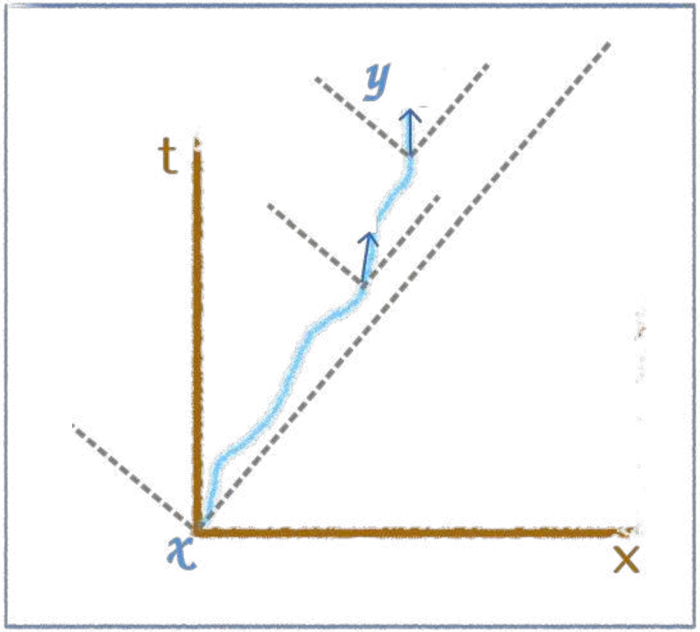


Proxlem : We do not have a Manifold, Points and Tangent vectors in non commutative geometry

Algebraic version of Causality

Defintion [Causal function] :

A non decreasing function $f : \mathcal{M} \to \mathbb{R}$ along every future directed causal curve.

Franco, Eckstein 2014 [1212.5171]

Proposition [Algebraic causality] :

Let (M, g) be a globally hyperbolic Lorentzien Manifold.

Let $\mathcal{C}(\mathcal{M})$ be the set of smooth bounded causal functions.

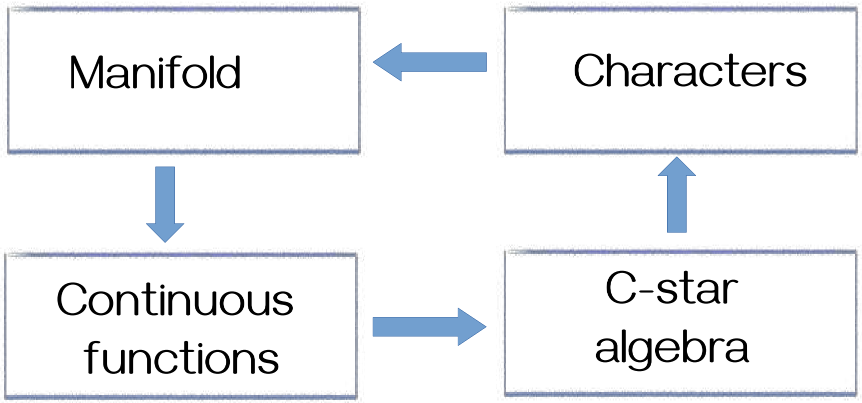
Then, the causal structure of the manifold is completely determined by :

$$\forall x, y \in \mathcal{M}, \quad x \preceq y \iff \forall f \in \mathcal{C}(\mathcal{M}), \quad f(x) \leq f(y)$$

Where $\preceq$ is the causal order on the manifold

« Points » in Noncommutative Geometry

Commutative case :



[Gelfand transform] :

$$\Phi : \mathcal{A} \to C_0(\Delta(\mathcal{A}))$$
$$a \longrightarrow \hat{a}; \quad \hat{a}(\chi) = \chi(a)$$

With $\mathcal{A}$ a C-star algebra, $\Delta(\mathcal{A})$ the set of characters.

Théorème [Gelfand Naimark] :
For a commutative C-star algebra the previous transform is a star isomorphisme and a isometry.

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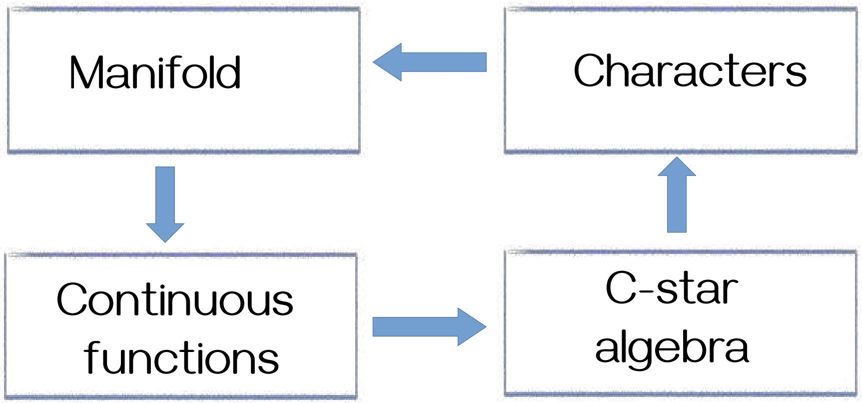

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<u>Noncommutative case :</u>

Starting with a noncommutative C-star algebra the equivalent of points are the **pure states**

Lorentzian spectral triple $\{\mathcal{A}, \widetilde{\mathcal{A}}, \pi, \mathcal{H}, \mathcal{D}, \mathcal{J}\}$

Strohmaier 2002 [0110001]
Franço 2014 [11210.6575]

A non-unital noncommutative C* algebra $\mathcal{A}$ and a preferred unitalization $\tilde{\mathcal{A}}$

An Hilbert space $\mathcal{H}$

A faithfull * representation π of $\mathcal{A}$ and $\tilde{\mathcal{A}}$ in the space of bounded operators $\mathcal{B}(\mathcal{H})$

An unbound Dirac operator $\mathcal{D}$ densely defined in $\mathcal{H}$ such that :

$$\forall a \in \widetilde{\mathcal{A}}, [\mathcal{D}, \pi(a)] \in B(\mathcal{H})$$

A bounded operator « the fundamental symmetry » $\mathcal{J}$ such that :

$$\mathcal{J}^2 = 1, \quad \mathcal{J}^\dagger = \mathcal{J}, \quad \forall a \in \widetilde{\mathcal{A}}, [\mathcal{J}, \pi(a)] = 0, \quad \mathcal{D}^\dagger \mathcal{J} = -\mathcal{J}\mathcal{D}$$

Lorentzian signature

Action of the fundamental symmetry on the Hilbert product :

$$(\;\;,\;\;)_{\mathcal{J}} = < \;\;, \mathcal{J} \; >$$

Equivalent of
a Wick rotation

Krein
product

Lorentzian signature

Action of the fundamental symmetry on the Hilbert product : $(\ \ ,\ \ )_{\mathcal{J}} = \langle\ \ , \mathcal{J}\ \rangle$

Equivalent of a Wick rotation

Krein product

Let $\mathcal{T}$ be a self-adjoint operator and $\mathcal{N}$ a positive operator of $\mathcal{A}$ with dense domaine in $\mathcal{H}$:

$$\mathcal{J} = -\mathcal{N}[\mathcal{D}, \mathcal{T}] \qquad (1 + \mathcal{T})^{-1/2} \in \tilde{\mathcal{A}}$$

$\mathcal{T}$ is viewed as a noncommutative generalization of time function. The left condition ensures a Lorentzian-type signature.

Causality in noncommutative geometry

Let $\{\mathcal{A}, \widetilde{\mathcal{A}}, \pi, \mathcal{H}, \mathcal{D}, \mathcal{J}\}$ be a Lorentzian Spectral Triple.

Besnard, 2009[0804.3551]
Franco, Eckstein 2014 [1212.5171]

Definition [Causal cone] :

$$\forall a, b \in \widetilde{\mathcal{A}}, \quad \lambda \in \mathbb{R}_+, \quad x \in \mathbb{R}, \quad \forall \phi \in \mathcal{H},$$

$$a^\dagger = a, \quad a + b \in \mathcal{C}, \quad \lambda a \in \mathcal{C}, \quad 1x \in \mathcal{C}, \quad \overline{\text{span}_{\mathbb{C}}} = \overline{\widetilde{\mathcal{A}}}$$

$g(\gamma'(t), \gamma'(t)) \leq 0$

$$< \phi, \mathcal{J}[\mathcal{D}, \pi(a)]\phi > \leq 0$$

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$g(\gamma'(t), \gamma'(t)) \leq 0$

$$< \phi, \mathcal{J}[\mathcal{D}, \pi(a)]\phi > \leq 0$$

The causal evolution between two pure states is given by :

$$\forall \omega, \eta \in \mathcal{S}(\widetilde{\mathcal{A}}), \ \omega \leq \eta \Longleftrightarrow \forall a \in C, \ \omega(a) \leq \eta(a)$$

$\mathcal{M}_\kappa$: Kappa-Minkowski (in 1+1 dimensions)

Universal enveloping algebra $[x_0, x_1] = ix_1/\kappa$ with Lie group G, the affine group of the real line.

Durhuus,Sitarz 2011 [1104.0206]
Poulain, Wallet, 2018 [1801.02715]

$$(f \star g)(x_0, x) = \int \frac{dp_0}{2\pi} dy_0 \; e^{-iy_0p_0} f(x_0 + y_0, x) g(x_0, e^{-p_0/\kappa} x)$$ star product

$$f^\dagger(x_0, x) = \int \frac{dp_0}{2\pi} dy_0 \; e^{-iy_0p_0} f(x_0 + y_0, e^{-p_0/\kappa} x),$$ involution

With f and g in $\mathcal{A}_x = (\mathfrak{E}(\mathbb{R}, \mathfrak{E}(\mathbb{R})), \star, \dagger)$. Thanks to a Fourier transform we can use another set of variables (p_0,p_1) and functions in $\mathcal{A}_p = C_c^\infty(\mathbb{R}, C_c^\infty(\mathbb{R}))$.

Motivations:
- At the commutative limit we recover the usual Minkowski space.
- Associated to a deformed symmetry space (kappa-Poincaré).

$\mathcal{P}_\kappa$: Kappa-Poincaré algebra

Symmetry space of kappa Minkowski, it is an Hopf algebra with a bicrossproduct structure.

Majid and Ruegg 1994 [9405107]

Generators : $P_0, P_1, N, \mathcal{E} := e^{-P_0/\kappa}$

twist

- Satisfy commutation relations
- Hopf algebra structure

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- Satisfy commutation relations
- Hopf algebra structure

We define $X_0 = (1-\mathcal{E}), \quad X = P,$ the twisted derivative which satisfy a twisted Liebniz rule :

$$X_\mu(f \star g) = X_\mu(f) \star g + (\mathcal{E} \triangleright f) \star X_\mu(g)$$

← twist

At commutative limit, we find back the usual derivative and Leibniz rule.

Representations and Hilbert space

Unitary irreductible representations of the affine group G :

Gelfand Naimark 1947
Khalil 1974 [117244213]

$$\pi_U : \mathcal{G} \to \mathcal{B}(L^2(\mathbb{R}, ds))$$

$$(\pi_{U\pm}(p_0, p_1)\phi)(s) = e^{\pm i p_1 e^{-s}} \phi(s + p_0)$$

$$\mathcal{H}_{-,0,+} := (L^2(\mathbb{R}), ds) \otimes \mathbb{C}^2$$

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Bounded representations of the C* algebra $\mathcal{A}$:

$$\pi_\nu : \mathcal{A}_p \to \mathcal{B}(L^2(\mathbb{R}, ds)), \quad \nu = -1, 0, 1$$

$$(\pi_\nu(f)\phi)(s) = \int d^2p\ e^{p_0} \mathcal{F}f(p_0, p_1) e^{i\nu p_1 e^{-s}} \phi(p_0 + s)$$

(p_0,p_1) variables

$$\pi_\nu : \mathcal{A}_x \to \mathcal{B}(L^2(\mathbb{R}, ds)), \quad \nu = -1, 0, 1$$

$$(\pi_\nu(f)\phi)(s) = \int du dv\ f(v, ve^{-s}) e^{-iv(u-s)} \phi(u)$$

(x_0,x_1) variables

Representation and Hilbert space

A suitable Hilbert space is then :

$$\mathcal{H} := \mathcal{H}_+ \oplus \mathcal{H}_0 \oplus \mathcal{H}_-$$

With associated faithfull representation :

$$\pi = (\pi_+ \oplus \pi_0 \oplus \pi_-) \otimes \mathbb{I}_2$$

Scalar product associated for $\phi^{(\nu)}, \psi^{(\nu)} \in L^2(\mathbb{R}, ds) \otimes \mathbb{C}^2$

$$\langle \Phi, \Psi \rangle_{\mathcal{H}} = \sum_{\nu=+,0,-} \langle \phi^{(\nu)}, \psi^{(\nu)} \rangle_{\mathcal{H}_\nu} \quad \Rightarrow \quad \langle \phi^{(\nu)}, \psi^{(\nu)} \rangle_{\mathcal{H}_\nu} = \int ds \phi^{(\nu)\dagger}(s) \psi^{(\nu)}(s)$$

Dirac operator and twisted commutator

The Dirac operator is define as :

$$\mathcal{D} = -i\gamma^\mu X_\mu \otimes \mathbb{I}_3 = \begin{pmatrix} 0 & X_- \\ X_+ & 0 \end{pmatrix} \otimes \mathbb{I}_3 := D \otimes \mathbb{I}_3$$

$$[a, b] = a \star b - b \star a \quad \Rightarrow \quad \forall a \in \tilde{\mathcal{A}}, \quad [\mathcal{D}, \pi(a)] \notin \mathcal{B}(\mathcal{H})$$

$$[a, b]_\mathcal{E} = a \star b - \mathcal{E} \triangleright (b) \star a \quad \Rightarrow \quad \forall a \in \tilde{\mathcal{A}}, \quad [\mathcal{D}, \pi(a)]_\mathcal{E} \in \mathcal{B}(\mathcal{H})$$

with $\mathcal{E} := e^{-P_0/\kappa}$

Matassa 2012 [1212.3462]

The Lorentzian Spectral Triple satisfies twisted conditions :

$$\mathcal{D}^\dagger = -\mathcal{J}\rho^{-1}\mathcal{D}\rho\mathcal{J}, \quad \rho := (\mathcal{E} \otimes \mathbb{I}_2) \otimes \mathbb{I}_3 \quad [\mathcal{D}, \mathcal{T}]_\mathcal{E} = -\mathcal{N}\mathcal{J}$$

Operators of the twisted Lorentzian Spectral Triple

We define the fundamental symmetry and the time operator as :

$$\mathcal{J} = i\gamma^0 \otimes \mathbb{I}_3$$

$$\mathcal{T} = -i \oplus_\nu \left(\pi_\nu(x_0) \otimes \mathbb{I}_2\right)$$

Those operators verify all conditions of a twisted Lorentzian Spectral Triple.

We also introduce position and momentum operators as : $x_0 := P, \quad x_1 := e^{-Q}$

With the representations $\pi_\pm(x_0) = -i\frac{d}{ds}, \quad \pi_\pm(x_1) = m(\pm e^s)$

Functions of the causal cone

Reminder :

Causal cone : convex cone of hermitian elements a of the algebra such that :

$$\forall \psi \in \mathcal{H}, \quad \langle \psi, \mathcal{J}[\mathcal{D}, \pi(a)]_{\mathcal{E}} \psi \rangle \leq 0$$

twist

Causal evolution :

$$\forall \omega, \eta \in \mathcal{S}(\widetilde{\mathcal{A}}), \; \omega \leq \eta \Longleftrightarrow \forall a \in C, \; \omega(a) \leq \eta(a)$$

We will derive an explicit form of those conditions using unitary representations of kappa Minkowski. Then, we will give a family of functions in the causal cone and study the evolution of pure states for those causal functions.

Functions of the causal cone

Functions of the causal cone should verify : $\langle \phi, \pi_\nu(X_\pm(a))\phi \rangle \geq 0, \quad \nu = -1, 0, 1$

Explicitly : $$\langle \phi, \pi_\pm(X_\pm(a))\phi \rangle = -i \int ds\,du\, \left(\left(1 - e^{s-u}\right)\tilde{a}(u - s, \nu e^{-s}) \mp \partial_\beta \tilde{a}(u - s, \nu e^{-s}) \right) \phi(u)\overline{\phi(s)} \geq 0,$$

Causal functions :

- Time function $T := x_0$
- Lambda light cone coordinates $a_\lambda(x_0, x_1) = x_0 \pm \lambda x_1 \quad \lambda \in [-1, 1]$
- Split functions $a(x_0, x_1) := h(x_0) + g(x_1)$ with $iX_0(h) = 1 \quad |g'(\nu e^{-s})| \geq 1$

Evolution of pure states

- Pure states are defined by : $\omega_{\pm}^{\Phi} = \langle \Phi, \pi_{\pm}(a)\Phi \rangle, \quad \Phi \in \mathcal{H}_{\pm}$

Explicitly : $$\omega_{\pm}^{\Phi}(a) = \int duds \; a(-s, \pm e^{-s})\overline{\Phi}(s)\Phi(u)$$

- Causality relation between two pure states :

$$\omega_{\pm}^{\Phi_1} \preceq \omega_{\pm}^{\Phi_2} \iff \forall a \in \mathcal{C}_x, \quad \omega_{\pm}^{\Phi_1}(a) \leq \omega_{\pm}^{\Phi_2}(a)$$

Explicitly :

$$\int dsdu \; \tilde{a}(u-s, \pm e^{-s}) \left[\overline{\Phi}_2(s)\Phi_2(u) - \overline{\Phi}_1(s)\Phi_1(u)\right] \geq 0$$

Quantum constraints

Evolution of pure states $\int ds du\, \tilde{a}(u-s, \pm e^{-s})\, \left[\overline{\Phi}_2(s)\Phi_2(u) - \overline{\Phi}_1(s)\Phi_1(u)\right] \geq 0$

+ expression for causal function $a(x_0, x_1) = x_0 \pm x_1$

= quantum constraint $\langle \Phi_2|P|\Phi_2\rangle - \langle \Phi_1|P|\Phi_2\rangle \geq |\langle \Phi_2|X|\Phi_2\rangle - \langle \Phi_1|X|\Phi_1\rangle|$

Denoting the variation of the expectation value by $\delta\langle \mathcal{A} \rangle$, we get :

$$\delta\langle P\rangle \geq |\delta\langle X\rangle|$$

Interpreted as a quantum analogous of the classical speed of light limit

$$\delta t \geq |\delta x|$$

Results and discussion

- Causality has been poorly investigated in noncommutative geometry; some work on microcausality has been done.

  Mercati, Sergola 2018 [1801.01765]

- Here, we obtain quantum analogs of the speed of light.
  1) We don't yet know the complete set of causal functions, which can give other quantum constraints.
  2) At the commutative limit, we lose a dimension with the unitary representation of the real line.

- Futur work :
  1)Take a representation with good commutative limit, for example, the GNS representation,
  2) Investigate phenomenological consequences.

Thanks for your attention