Generalized connections and tensors for Courant algebroids

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based on

work in progress with Th. Chatzistavrakidis, C. Hull, S. Lavau, P. Schupp

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Background

Dualities & fluxes in string/M theory in the framework of double field theory.

Duff ‘90, Tseytlin ‘90, Siegel ‘93, Hull, Zwiebach ‘09, Hohm, Hull, Zwiebach ‘10

Symmetries of DFT - global $G = O(d, d)$
- local diffeomorphisms plus gauge trafos of $2-$form

$\rightsquigarrow$ use generalized geometry Hitchin ’02; Gualtieri ’04 and Courant algebroids Courant ’90; Liu, Weinstein, Xu ’97; Roytenberg ’99; Ševera ’17

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- Establish the geometric origin for the structures appearing in DFT $\rightsquigarrow$ DFT algebroid Chatzistavrakidis, Khoo, LJ, Szabo '18; Grewcoe, LJ '20
- Utilize the relation between Courant algebroids and QP2 manifolds Roytenberg '02 to study world volume theory corresponding to DFT Chatzistavrakidis, Khoo, LJ, Szabo '18.

In general - AKSZ construction for topological sigma models

Alexandrov, Schwarz, Zaboronsky, Kontsevich '95

$\rightsquigarrow$ Solution of classical master equation from QP-structures

- QP1 $\leftrightarrow$ Poisson sigma model Ikeda '93; Schaller, Strobl '94
- QP2 $\leftrightarrow$ Courant sigma model Ikeda '02, Hofman '02, Roytenberg '06.

Geometrical content:

- dg-manifold data cf. Maxim's talk
- auxiliary connection Roytenberg '06; Chatzistavrakidis, LJ '23

Plan

Main question $\rightsquigarrow$ Generalized connection and curvature for Courant algebroid?

In this talk

- Is there fundamental theorem of generalized Riemannian geometry?
- Can we find geometric conditions that result in interesting gravity models?

Probably not in this talk

- What is a graded manifold formulation of generalized geometry structures?

Courant algebroids & gen'd connections

- Courant algebroid: $(E, \circ, \rho : E \to TM, \langle \cdot, \cdot \rangle \equiv \eta)$, $(\Gamma(E), \circ)$ is a (left) Leibniz algebra,

$$\eta(e, e' \circ e') = \frac{1}{2}\rho(e)\eta(e', e') = \eta(e \circ e', e') \, .$$

- Generalized connection ("E-on-E" connection): $\nabla^E : \Gamma(E) \times \Gamma(E) \to \Gamma(E)$,

$$\nabla^E_{fe} e' = f \nabla^E_e e' \quad \text{and} \quad \nabla^E_e f e' = f \nabla^E_e e' + \rho(e) f \, e' \, , \; e, e' \in \Gamma(E), \; f \in C^\infty(M) \, .$$

η-compatibility: $\big(\nabla^E \eta\big)(e, e', e'') = 0 = \rho(e)\eta(e', e'') - \eta(\nabla^E_e e', e'') - \eta(e', \nabla^E_e e'')$.

Courant algebroids & gen'd connections

- Proposals for torsion and curvature tensors, Gualtieri; Jurco, Vysoky; cf. Hohm, Zwiebach

$$T^{\nabla^E}(e,e',e'') = \eta(\nabla^E_e e' - \nabla^E_{e'} e - e \circ e', e'') + \eta(\nabla^E_{e''} e, e')\,,$$

$$R^{\nabla^E}(\hat{e},\hat{e}',e,e') = \frac{1}{2}\Big(R_0^{\nabla^E}(\hat{e},\hat{e}',e,e') + R_0^{\nabla^E}(e',e,\hat{e}',\hat{e})\Big) + \eta\big(K(e,e'),K(\hat{e},\hat{e}')\big)\,.$$

$$\Big(R_0^{\nabla^E}(e,e') = [\nabla^E_e, \nabla^E_{e'}] - \nabla^E_{e\circ e'} \quad \text{and} \quad \eta(K(e,e'),\hat{e}) = \eta(\nabla^E_{\hat{e}} e, e')\Big)\,.$$

- Work well in practice. No Koszul formula. Bianchi identities? Symmetrization?

Courant algebroids & gen'd connections

- Graded geometric description of E as symplectic submanifold of $M_2 = T^*[2]E[1]$.
Roytenberg
- A canonical degree 2 symplectic structure Ω and a homological vector field Q.

$$|Q| = 1 \quad \text{and} \quad \{Q, Q\} = 0\,.$$

Compatibility of the graded symplectic and Q structures.

$$L_Q\Omega = 0\,.$$

A degree 3 Hamiltonian function.

$$Q = \{\Theta, -\}\,, \quad \{\Theta, \Theta\} = 0\,.$$

- The Dorfman bracket is a derived bracket, together with ρ and η are given as

$$e \circ e' = \{\{\Theta, e\}, e'\}$$

$$\rho(e)f = \{\{\Theta, e\}, f\}\,,$$

$$\eta(e, e') = \{e, e'\}\,.$$

Courant algebroids & lie 2-algebroids

- In general, split Qn manifolds correspond to Lie n-algebroids.
  Voronov; Sheng, Zhu; Bonavolonta, Poncin; ...
- A Courant algebroid is not a split QP2 manifold, in general.

* Price to pay: an η-compatible TM-on-E connection $\nabla : \mathfrak{X}(M) \times \Gamma(E) \to \Gamma(E)$.
* Then the split graded vector bundle $T^*[1]M \oplus E$ admits a Lie 2-algebroid structure.
* The brackets are given as higher derived brackets ($C^\infty(M)$-linear for $k \geq 3$)

$$\ell_k(e_1, \ldots, e_k) = \{\ldots\{\{Q^{(k-1)}, e_1\}, e_2\}\ldots e_k\}\,,$$

  where the arity k $Q^{(k)}$ w.r.t. the (unweighted) Euler vector field is

$$\{Q^{(k)}, \varepsilon\} = kQ^{(k)}\,.$$

- For Q1 manifolds, only $k = 1$, giving the Lie bracket of the Lie (1-)algebroid.

Courant algebroids & lie 2-algebroids

- For Courant algebroids, there are $Q^{(k)}$ for $k = 0, 1, 2$.
- Denoting the canonically induced E-on-E connection from the TM-on-E as

$$\dot{\nabla}^E_e e' = \nabla_{\rho(e)} e' \,,$$

the Lie 2-algebroid on $T^*[1]M \oplus E$ is given by the anchor ρ and the brackets

$$\ell_1 = -\rho^\sharp$$
$$\ell_2(e, e') = e \circ e' - \eta^\sharp(\dot{\nabla}^E_- e, e')$$
$$\ell_2(e, \sigma) = L_{\rho(e)}(\sigma) - (\nabla_- e, \rho^*(\sigma))$$
$$\ell_3(e, e', e'') = -\eta(S^\nabla(e, e')(-), e'')$$

- Here $S^\nabla \in \Gamma(E \otimes E \otimes E \otimes TM)$ is the basic curvature tensor for the TM-on-E

$$\mathcal{S}^\nabla(e_1, e_2, e_3)X = \eta(\nabla_X(e_1 \circ e_2) - \nabla_X e_1 \circ e_2 - e_1 \circ \nabla_X e_2 - \nabla_{\overline{\nabla}^E_{e_2} X} e_1 + \nabla_{\overline{\nabla}^E_{e_1} X} e_2, e_3)$$
$$+ \eta(\nabla_{\overline{\nabla}^E_{e_3} X} e_1, e_2) \,, \quad \text{where} \quad \overline{\nabla}^E_e X = \rho(\nabla_X e) + [\rho(e), X] \,.$$

Another choice of ∇ gives L_∞ quasi-isomorphic Lie 2-algebroid.

Torsion & curvature revisited

- Equipped with the ℓ_2 bracket and its properties, define tensors (by construction)

$$T_{\nabla}^{\nabla^E}(e,e') = \nabla^E_e e' - \nabla^E_{e'} e - \ell_2(e,e')\,,$$

$$R_{\nabla}^{\nabla^E}(e,e') = [\nabla^E_e, \nabla^E_{e'}] - \nabla^E_{\ell_2(e,e')}\,.$$

- n.b.: for $\dot{\nabla}^E$ (only), the torsion is identical to the Gualtieri torsion.

  Also, for $\dot{\nabla}^E$ the "naive" R_0 is a tensor

- First and second Bianchi identities simply follow from the construction.

$$R_{\nabla}^{\nabla^E}(e,e')e'' + \circlearrowright = \left(\mathrm{d}^{\nabla^E} T_{\nabla}^{\nabla^E} + \ell_1\ell_3\right)(e,e',e'')\,,$$

$$d^{\nabla^E} R_{\nabla}^{\nabla^E} + \nabla^E_{\ell_1\ell_3} = 0\,.$$

Gen'd metric & Koszul formula

- Generalized metric: $G(e, e') = \eta(e, \tau(e'))$, for $\tau \in \mathrm{End}(E)$ with $\tau^2 = \mathrm{id}$.
- For a Lie 2-algebroid, unique gen'd metric compatible ($\nabla^E G = 0$) gen'd connection:

$$G(\nabla^E_e e', e'') = \frac{1}{2}\Big(\rho(e)G(e', e'') + \rho(e')G(e, e'') - \rho(e'')G(e, e')$$

$$- G((T^{\nabla^E} + \ell_2)(e, e''), e') - G((T^{\nabla^E} + \ell_2)(e', e''), e) + G((T^{\nabla^E} + \ell_2)(e, e'), e'')\Big) .$$

Koszul formula for fixed torsion (e.g. zero).

Toward gravity models

- There is a Ricci tensor and two ways to construct a Ricci scalar.

$$\mathrm{Ric}^{\nabla^E}_{\nabla}(e, e') = \mathrm{Tr}\big(R^{\nabla^E}_{\nabla}(-, e, -, e')\big)\,,$$

$$\mathcal{R}^G = \mathrm{Tr}_G(\mathrm{Ric}^{\nabla^E}_{\nabla})\,,$$

$$\mathcal{R}^\eta = \mathrm{Tr}_\eta(\mathrm{Ric}^{\nabla^E}_{\nabla})\,.$$

- For the data of a Lie2oid (with ∇), the gen'd metric G, a ∇^E and a volume form ω:

$$S = \int \omega\left(\mathcal{R}^G + \lambda\mathcal{R}^\eta\right)\,, \quad \lambda \in \mathbb{R}\,.$$

cf. Jurco, Moučka, Vysoky for a Palatini analog using different curvature and torsion tensors

Geometric conditions for (super)gravity

- Specialize to the standard (exact) CAoid with H-twisted Dorfman bracket.

$$E = TM \oplus T^*M\,, \quad \Gamma(E) \ni e = (X,\xi)\,, \quad \rho(e) = X\,, \quad H \in \Omega^3_{\mathrm{cl}}(M)\,.$$

$$e \circ e' = \big([X,X'], L_X\xi' - \iota_{X'}\mathrm{d}\xi - H(X,X',-)\big)\,.$$

- It comes from $T^*[2]T[1]M$ and we consider a TM-on-E connection that splits it.

- We impose geometric conditions that fix ∇, giving ∇^E via Koszul for fixed torsion.

- Take into account a (pseudo)Riemannian volume form (for a metric g)

  motivated by physics, the dilaton field

$$\omega = e^{-2\phi}\sqrt{-g}\,\mathrm{d}^D x\,, \quad \phi \in C^\infty(M)\,.$$

Geometric conditions for (super)gravity

A. Metricity conditions: use $G = \mathrm{diag}(g, g^{-1})$.

$$\nabla\eta = 0\,, \qquad \nabla G = 0\,, \qquad \nabla^E G = 0\,.$$

B. Conditions that fix ∇. Define $\rho_g : E \to TM$, with $\rho_g[(X,\xi)] = X + g^{-1}(\xi)$.

$$T^{\dot{\nabla}^E}(X, Y) = 0\,, \quad X, Y \in \Gamma(TM)\,,$$

$$L_{\rho_g(e)}\omega = \mathrm{Tr}\big(\rho_g(\nabla_- e), -\big)\omega$$

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$$\Gamma_{\mu a}{}^{b} = \begin{pmatrix} \mathring{\Gamma}_{\mu\nu}{}^{\rho} & -\frac{1}{3}H_{\mu}{}^{\nu\rho} + \frac{4}{(D-1)}\delta_{\mu}^{[\nu}\partial^{\rho]}\phi \\ -\frac{1}{3}H_{\mu\nu\rho} + \frac{4}{(D-1)}g_{\mu[\nu}\partial_{\rho]}\phi & -\mathring{\Gamma}_{\mu\nu}{}^{\rho} \end{pmatrix}$$

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Γ. The trace of the torsion is fixed to be suitably proportional to $\mathrm{d}\phi$.

- Then the action functional takes the form (for $\lambda = 0$, i.e. only the G-trace,)

$$S = \int \mathrm{d}^D x \sqrt{-g}\, e^{-2\phi}\left(R^{\mathrm{LC}} - \frac{1}{12}H^2 + 4\Box_g\phi - 4(\partial\phi)^2\right)\,.$$

- This is precisely the action that originates from the β-functions of the 2D σ-model.
  without the criticality

Summary & outlook

- An alternative route to define torsion and curvature in generalized geometry.
  via Lie 2-algebroids, paying the price of a connection, buying advantages like canonical definitions, Bianchi identities & ...
- Analog of the fundamental theorem of Riemannian geometry for gen'd connections.
- A set of geometric conditions resulting in physically motivated gravity models.

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- Ultimately, a full formulation directly within graded geometry would be desirable.
  "Generalized gravity as ordinary gravity on a graded manifold"?
- As a first step, characterize a torsion-free degree zero connection on graded manifold. work in progress with Th. Chatzistavrakidis and D. Roytenberg
- Analysis so far indicates that gen'd connections and their tensors are obtained from ordinary connections and their ordinary tensors (plus the Atiyah cocycle) on dg manifolds.