Three-particle entanglement in particle decay and scattering

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2024/8/30 @ Corfu Workshop on Standard Model and Beyond

Entanglement

"not one but rather *the* characteristic trait of quantum mechanics"

[Erwin Schrödinger, 1935]

- Entanglement plays an important role in quantum information theory, quantum many body systems, quantum gravity, AdS/CFT (Holography)
- hasn't been discussed much in QFT and Particle Physics (until recently)

➡ **What is the role of entanglement in QFT and Particle Physics?**

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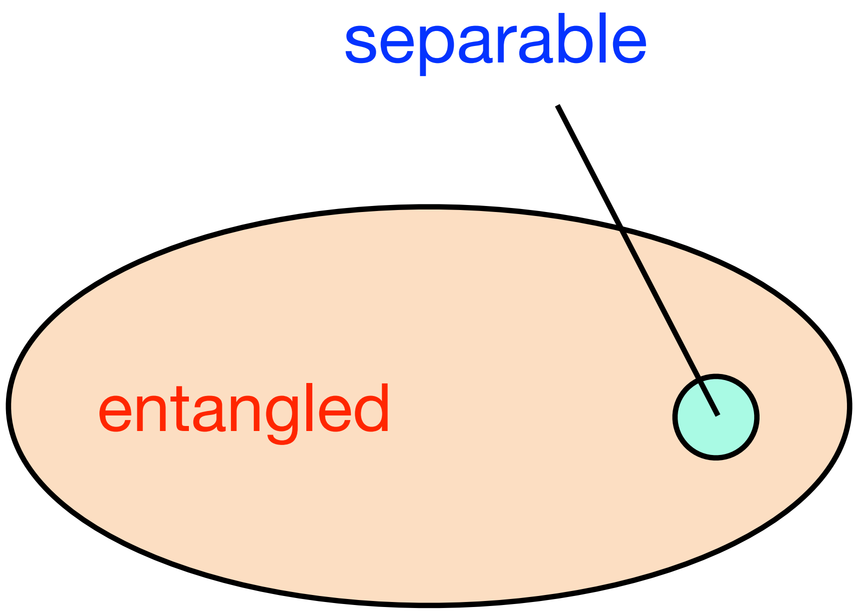
Recent progress:

- Relation between entanglement and on-shell amplitudes — Cheung, He, Sivaramakrishnan [2304.13052]
- entanglement (entropy) growth <==> positivity of $\mathrm{Im}[\mathcal{A}]$ — Aoude, Elor, Remmen, Sumensari [2402.16956]
- entanglement <==> cross-section (area law) — Law, Yin [2405.08056]
- Observation of spin entanglement in the $t\bar{t}$ pair — ATLAS [2311.07288], CMS [2406.03976]

This talk → 3-particle entanglement in particle physics

Definition



- **Definition in pure states:**

$|\Psi_A\rangle \otimes |\Psi_B\rangle$ separable state

$|00\rangle + |11\rangle$ **not** separable ↔ entangle state

General state $c_{00}|00\rangle + c_{01}|01\rangle + c_{10}|10\rangle + c_{11}|11\rangle$ is separable iff $c_{00}c_{11} - c_{01}c_{10} = 0$

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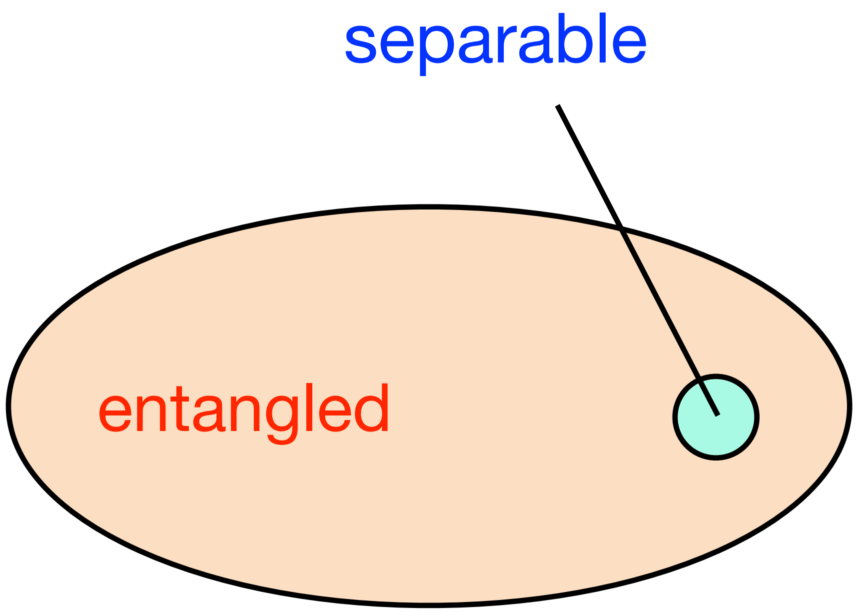

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- Definition in mixed states: $\rho = \sum_i p_i |\psi_i\rangle\langle\psi_i|$ $\left(p_i \geq 0,\ \sum_i p_i = 1\right)$

$$\rho_{\text{sep}} = \sum_i q_i \cdot \rho_i^A \otimes \rho_i^B$$ separable

entangled if the state ρ does not admit such a factorisation

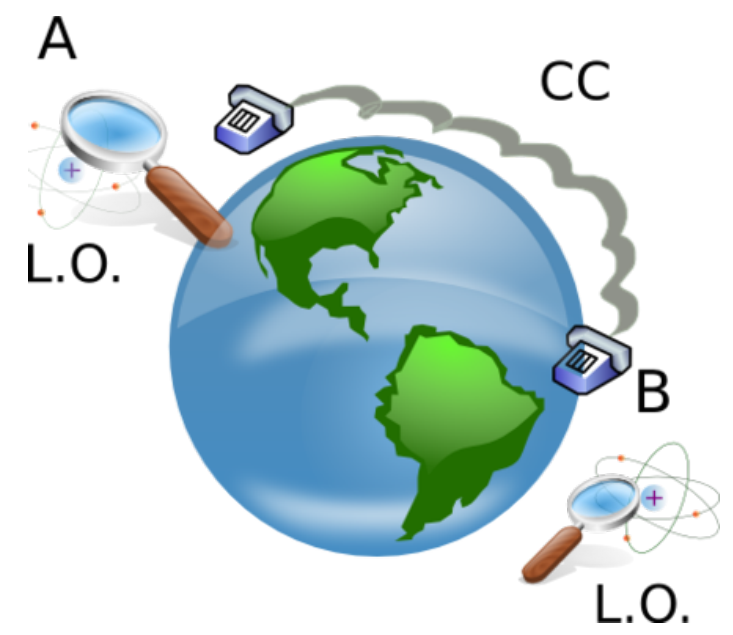
Entanglement as a Resource

- Separable states will never be entangled by Local Operation and Classical Communication (LOCC)

- LOCC introduces an *order* among states

$\Rightarrow$ Entanglement is *defined* s.t. it decreases under LOCC

maximally entangled $\rho_{\max} \xmapsto{\text{LOCC}} \rho \xmapsto{\text{LOCC}} \rho_{\text{sep}}$ separable state



ex) $|00\rangle + |11\rangle \xrightarrow[\text{measurement}]{} |00\rangle$

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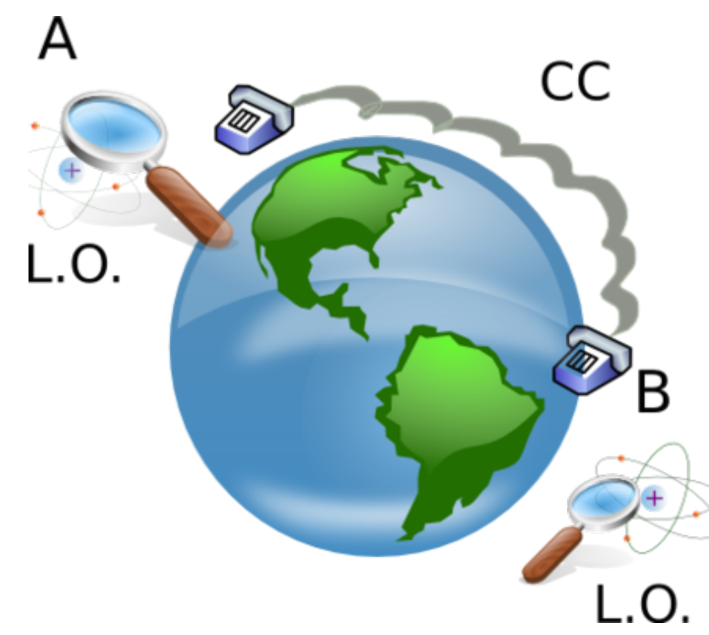

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- **Entanglement measures** (monotones), $E(\rho)$

ex) $|00\rangle + |11\rangle \underset{\text{measurement}}{\longrightarrow} |00\rangle$

  - monotonically decreases under LOCC
  - $E(\rho) = 0$ if ρ is separable and $E(\rho) > 0$ otherwise
  - $E(\rho) = 1$ if ρ is maximally entangled states

(- $E(\rho_1 \otimes \rho_2) = E(\rho_1) + E(\rho_2)$, $\quad E\left(\sum_k p_k \rho_k\right) \leq \sum_k p_k E\left(\rho_k\right)$)

Entanglement Measures

- Entanglement measures are often defined nicely for pure states $|\Psi\rangle_{AB}$

Von Neumann Entropy: $S_V(|\Psi\rangle_{\mathrm{AB}}) = -\mathrm{Tr}[\rho_A \log_2 \rho_A]$

$$\rho_A = \mathrm{Tr}_B\left(|\Psi\rangle\langle\Psi|_{AB}\right)$$

Linear Entropy: $S_L(|\Psi\rangle_{\mathrm{AB}}) = 2[1 - \mathrm{Tr}(\rho_A^2)]$

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- Entanglement measures for mixed states $\rho_{AB} = \sum_i p_k |\Psi_k\rangle\langle\Psi_k|$

Entanglement of Formation: $E_F(\rho_{\mathrm{AB}}) = \inf_{p_k, |\Psi_k\rangle} [p_k S_V(\Psi_k)]$

Concurrence: $\mathcal{C}(\rho_{\mathrm{AB}}) = \inf_{p_k, |\Psi_k\rangle} \left[p_k \sqrt{S_L(\Psi_k)}\right]$

minimising over all possible decompositions

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minimising over all possible decompositions

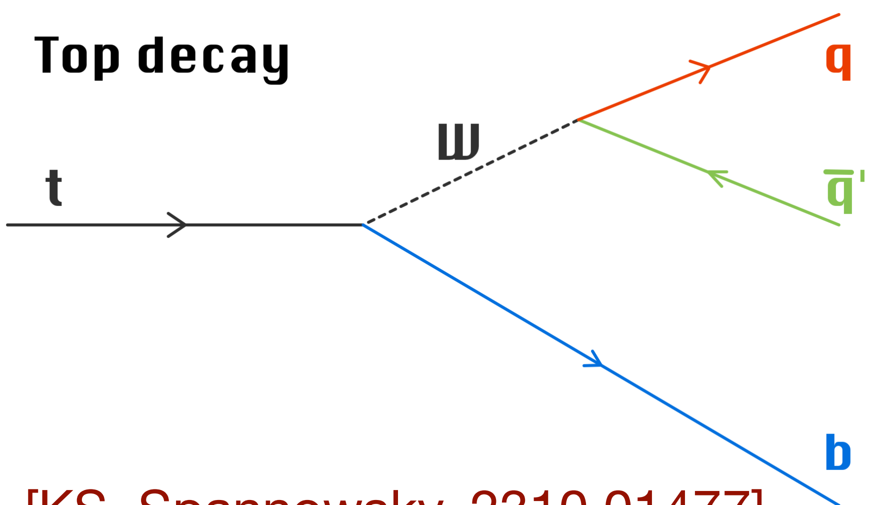
- For 2-qubit systems, the concurrence admits the analytical expression: [Wootters '98]

$$\mathcal{C}[\rho] = \max(0, \eta_1 - \eta_2 - \eta_3 - \eta_4) \in [0, 1]$$

$\eta_1 \geq \eta_2 \geq \eta_3 \geq \eta_4$ are eigenvalues of $\sqrt{\rho\tilde{\rho}}$ with $\tilde{\rho} \equiv (\sigma_y \otimes \sigma_y)\rho^*(\sigma_y \otimes \sigma_y)$.

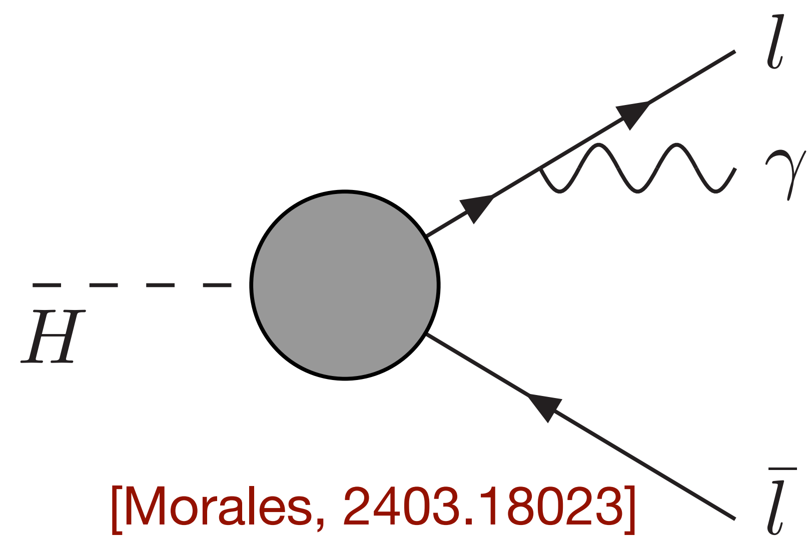
Three-particle entanglement?

$$2 \to 222$$



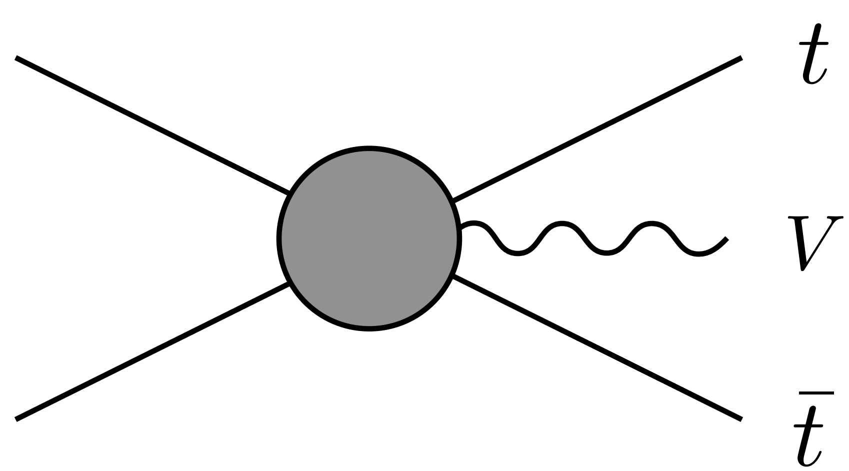
[KS, Spannowsky, 2310.01477]

$$0 \to 222$$



[Morales, 2403.18023]

$$22 \to 223$$



[Subba, Rahaman, 2404.03292]

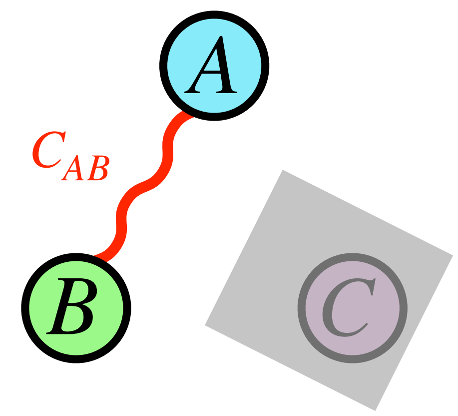
Three qubit system: 222

- $2^3 = 8$ basis kets:

$$|\Psi_{ABC}\rangle = c_{000}|000\rangle + c_{001}|001\rangle + c_{010}|010\rangle + \cdots$$

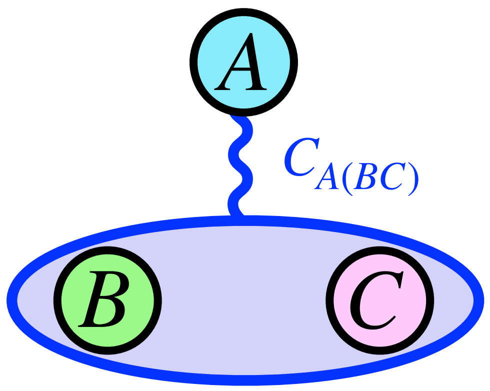
- Entanglement among 2-individual particles:

$$\mathcal{C}_{AB}[|\Psi_{ABC}\rangle] = \mathcal{C}[\rho_{AB}] \qquad \rho_{AB} = \mathrm{Tr}_C|\Psi\rangle\langle\Psi|_{ABC}$$



- Entanglement among one-to-other:

$$|\Psi_{ABC}\rangle = |0\rangle_A \otimes (c_{000}|00\rangle_{BC} + c_{001}|01\rangle_{BC} + \cdots)$$
$$+|1\rangle_A \otimes (c_{100}|00\rangle_{BC} + c_{101}|01\rangle_{BC} + \cdots)$$



$$\mathcal{C}_{A(BC)}[|\Psi_{ABC}\rangle] = \sqrt{2[1 - \mathrm{Tr}\rho_{BC}^2]} \qquad \rho_{BC} = \mathrm{Tr}_A|\Psi\rangle\langle\Psi|_{ABC}$$

Classification

Fully-separable: $|\phi^{\mathrm{fs}}\rangle_{A|B|C} = |\alpha\rangle_A \otimes |\beta\rangle_B \otimes |\gamma\rangle_C,$

$\mathcal{C}_{ij} = \mathcal{C}_{i(jk)} = 0$

Bi-separable: $|\phi^{\mathrm{bs}}\rangle_{A|BC} = |\alpha\rangle_A \otimes |\delta\rangle_{BC}$

$|\phi^{\mathrm{bs}}\rangle_{B|AC} = |\beta\rangle_B \otimes |\delta\rangle_{AC}$

$|\phi^{\mathrm{bs}}\rangle_{C|AB} = |\gamma\rangle_C \otimes |\delta\rangle_{AB}$

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$\mathcal{C}_{BC}, \mathcal{C}_{B(AC)}, \mathcal{C}_{C(AB)} \neq 0$

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complement

Genuinely **M**ultipartite **E**ntangled (**GME**):

$$|GHZ_3\rangle = \frac{1}{\sqrt{2}}(|000\rangle + |111\rangle)$$

$$|W_3\rangle = \frac{1}{\sqrt{3}}(|100\rangle + |010\rangle + |001\rangle$$

maximally GME

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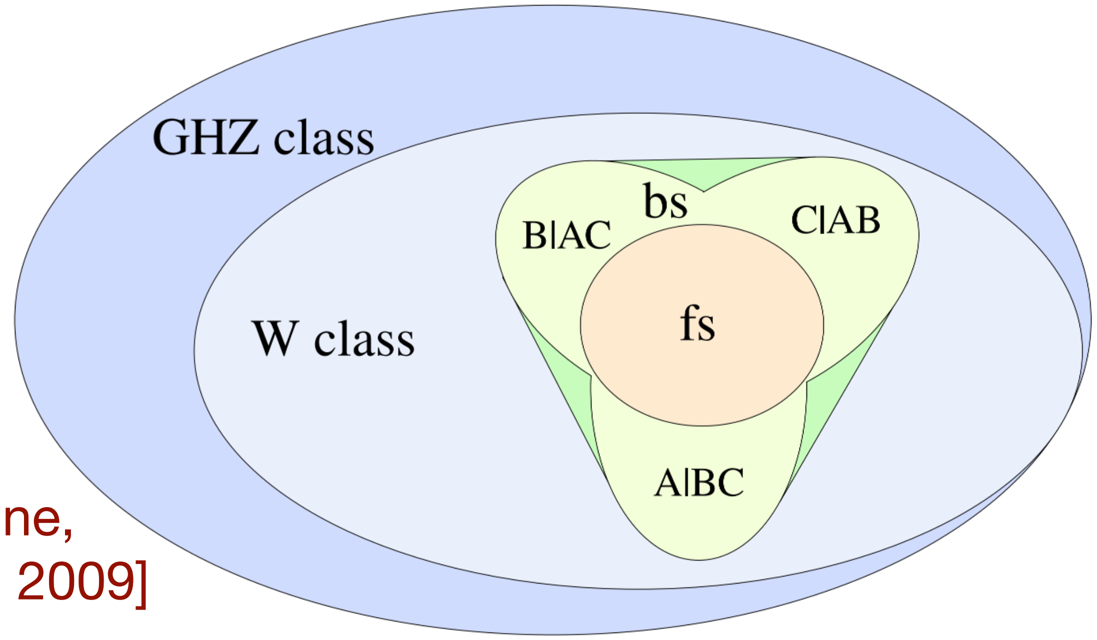
- All GME states can be classified either *GHZ* or *W* classes! [Dur, Vidal, Cirac 2000]

$$|\phi^{\rm GME}\rangle \longrightarrow \hat{A} \otimes \hat{B} \otimes \hat{C}\,|\phi^{\rm GME}\rangle = \begin{cases} |GHZ_3\rangle \\ |W_3\rangle \end{cases}$$

$\hat{A} \in I(\mathcal{H}_A), \hat{B} \in I(\mathcal{H}_B), \hat{C} \in I(\mathcal{H}_C)$

$I(\mathcal{H})$: set of intertible operators in $\mathcal{H}$



[Guhne, Toch 2009]

Monogamy

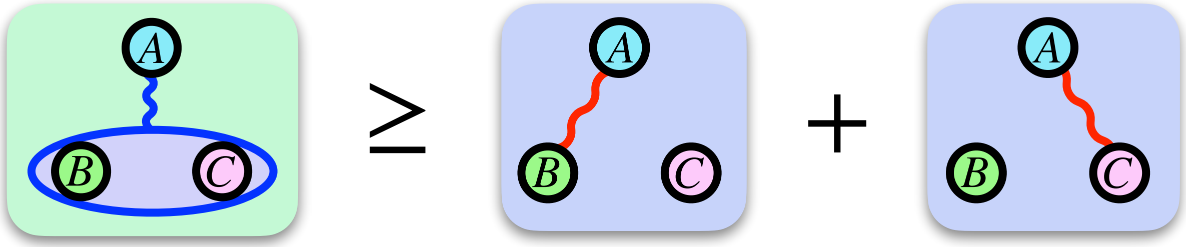
- All 3-qubit pure states can be transformed by a local unitary to

$$|\psi\rangle = \lambda_0|000\rangle + \lambda_1 e^{i\theta}|100\rangle + \lambda_2|101\rangle + \lambda_3|110\rangle + \lambda_4|111\rangle, \qquad \lambda_i \geq 0, \sum_i \lambda_i^2 = 1 \text{ and } \theta \in [0;\pi]$$

- **1-2** and **1-1** entanglements are related to by the **monogamy** relations:

[Coffman, Kundu, Wootters '99]

$$\mathcal{C}^2_{A(BC)} = \mathcal{C}^2_{AB} + \mathcal{C}^2_{AC} + \tau \qquad \tau = 4\lambda_0^2\lambda_4^2 \geq 0 \longleftarrow$$ **3-tangle**



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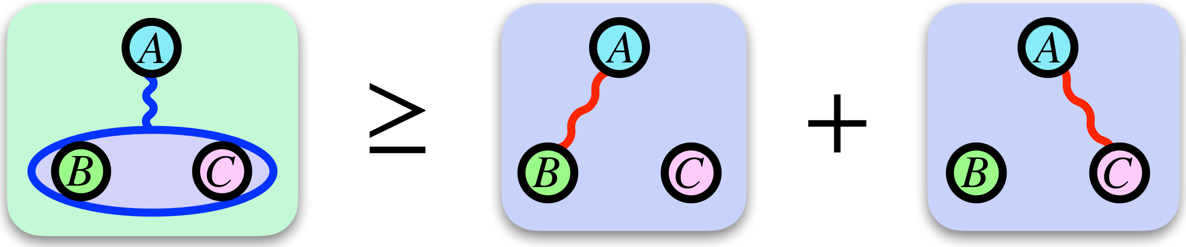

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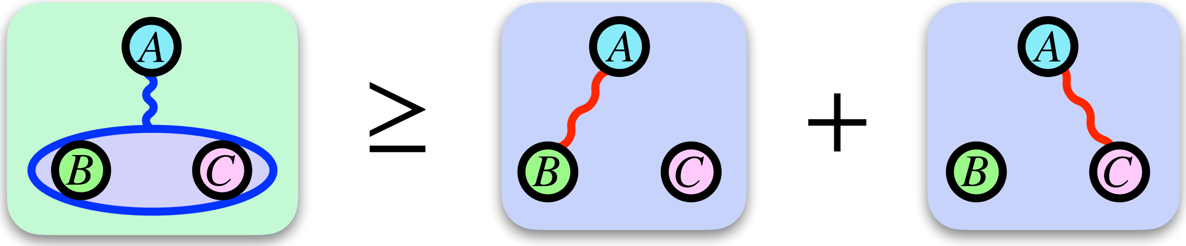

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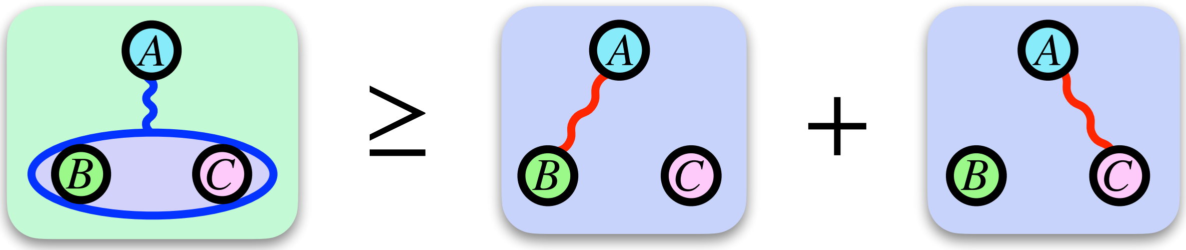
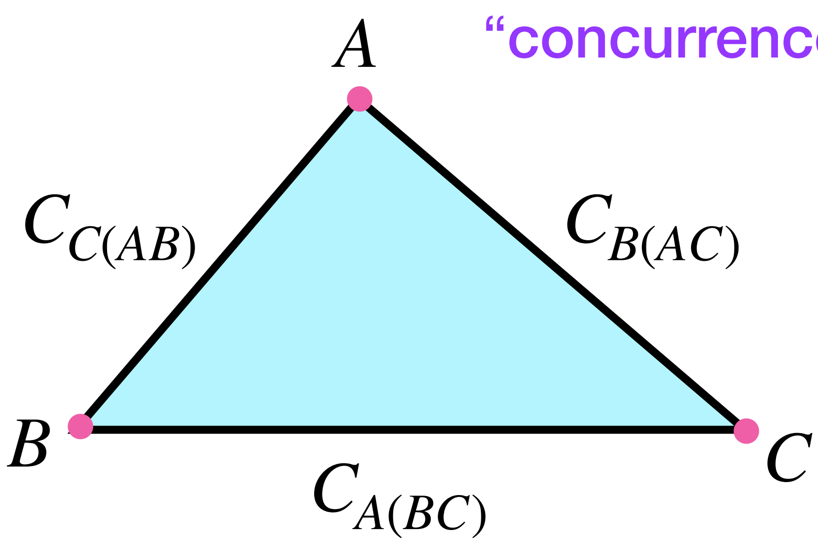


- 1-2 concurrence inequalities

$$\mathcal{C}^2_{A(BC)} + \mathcal{C}^2_{B(AC)} \geq \mathcal{C}^2_{C(AB)}$$

$$\downarrow$$

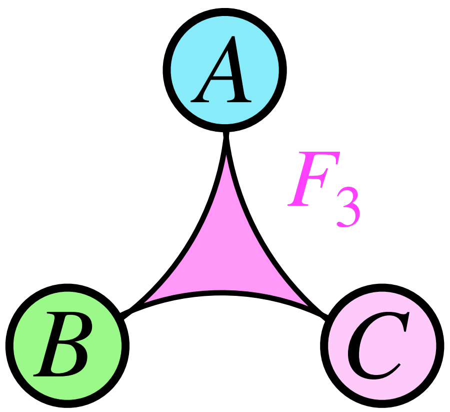
$$\mathcal{C}_{A(BC)} + \mathcal{C}_{B(AC)} \geq \mathcal{C}_{C(AB)} \quad \Rightarrow$$



“concurrence triangle”

GME measure

Genuine Multi-particle Entanglement (GME) measure: [Dur, Vidal, Cirac '00, Ma, Chen, Chen, Spengler, Gabriel, Huber '11, Xie, Eberly '21]

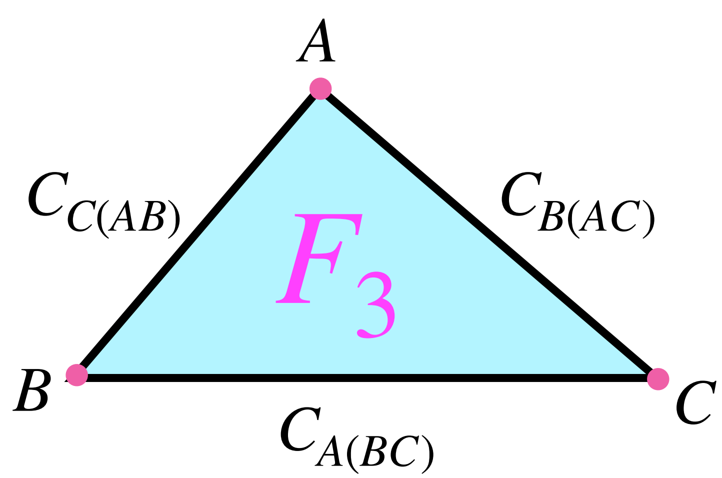


The measure should satisfy:

(1) vanishes for all fully- and bi-separable states

(2) positive for all GME states

(3) non-increasing under LOCC



- The **area** of the "**concurrence triangle**" satisfies (1), (2), (3) !

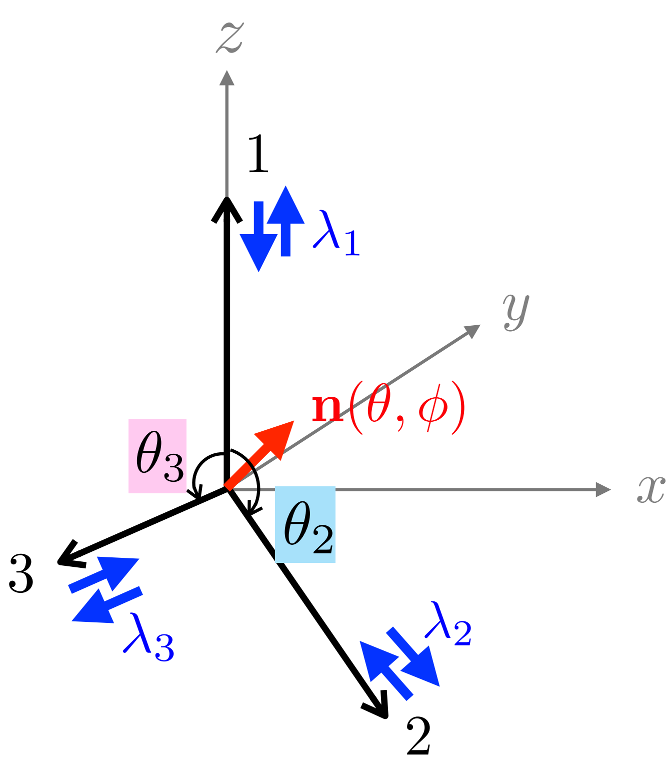
[Jin, Tao, Gui, Fei, Li-Jost, Qiao (2023)]

$$F_3 \equiv \left[\tfrac{16}{3} Q\big(Q - \mathcal{C}_{A(BC)}\big)\big(Q - \mathcal{C}_{B(AC)}\big)\big(Q - \mathcal{C}_{C(AB)}\big)\right]^{\frac{1}{2}} \in [0,1]$$

$$Q \equiv \frac{1}{2}\left[\mathcal{C}_{A(BC)} + \mathcal{C}_{B(AC)} + \mathcal{C}_{C(AB)}\right]$$

3-body decay: $\psi_0 \to \psi_1 \bar{\psi}_2 \psi_3$

[KS, M.Spannowsky 2310.01477]



Assumptions:

- all particles have spin 1/2
- all final particles 1,2,3 are massless

Kinematics:

- rest frame of the initial particle 0
- p_1 is in the z-axis
- decay is in the x-z plane

$$p_1^\mu = p_1(1, 0, 0, 1)$$
$$p_2^\mu = p_2(1, \sin\theta_2, 0, \cos\theta_2)$$
$$p_3^\mu = p_3(1, -\sin\theta_3, 0, \cos\theta_3)$$

$\mathbf{n}(\theta, \phi)$: polarisation of initial spin

$\lambda_1, \lambda_2, \lambda_3 \in (+, -)$: helicities of 1,2,3

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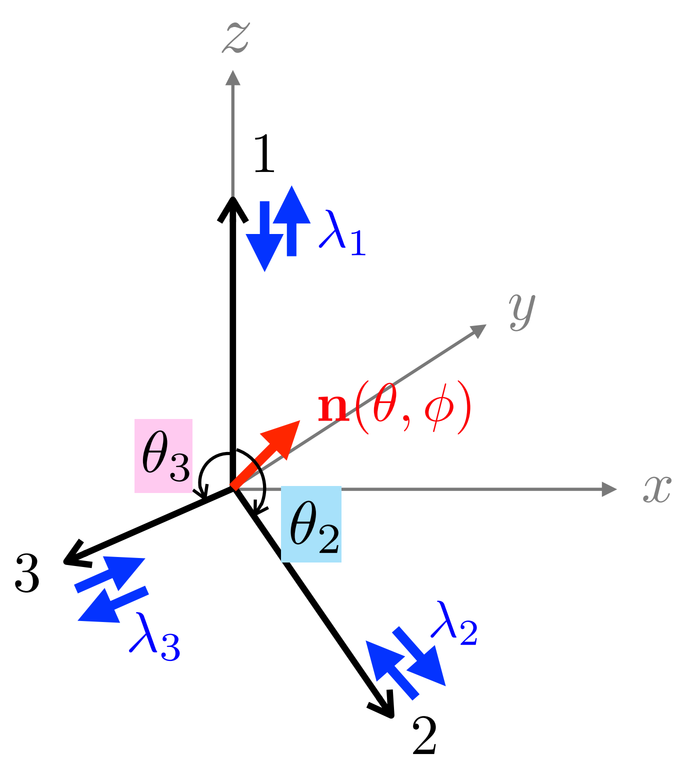

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initial state

$$|\mathbf{n}(\theta, \phi)\rangle$$

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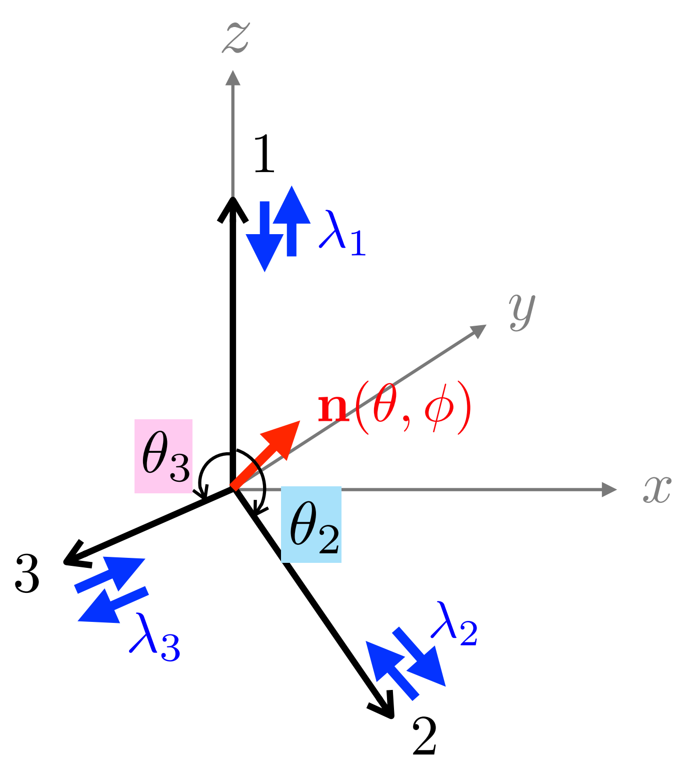

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$$\hat{\mathbf{1}} = \sum_{\lambda_1,\lambda_2,\lambda_3} |\lambda_1, \lambda_2, \lambda_3\rangle\langle\lambda_1, \lambda_2, \lambda_3|$$

amplitude

$$\mathcal{M}^{\mathbf{n}}_{\lambda_1,\lambda_2,\lambda_3} = \langle\lambda_1, \lambda_2, \lambda_3|\mathbf{n}(\theta, \phi)\rangle$$

initial state

$$|\mathbf{n}(\theta, \phi)\rangle = \sum_{\lambda_1,\lambda_2,\lambda_3} \mathcal{M}^{\mathbf{n}}_{\lambda_1,\lambda_2,\lambda_3} |\lambda_1, \lambda_2, \lambda_3\rangle + \cdots$$

final state

3-body decay: $\psi_0 \to \psi_1 \bar{\psi}_2 \psi_3$

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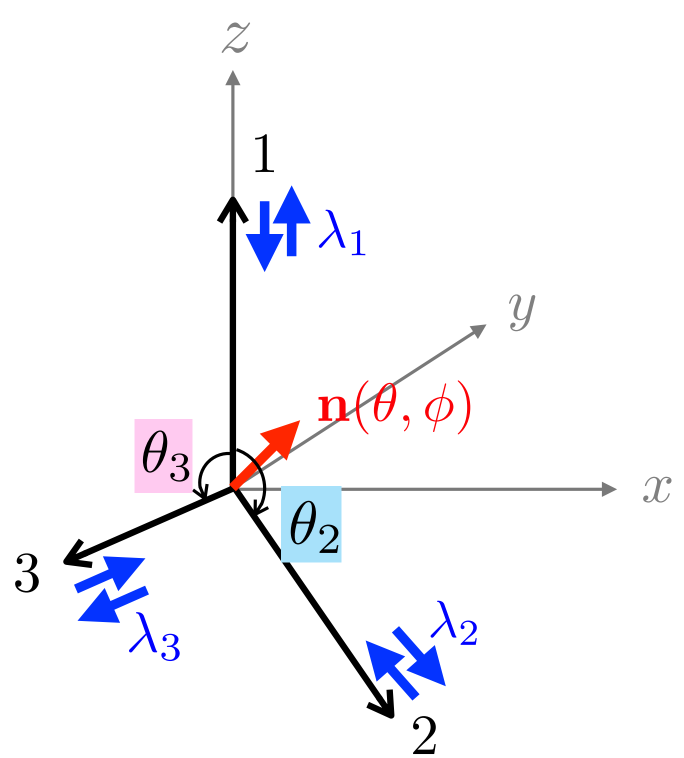

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$$p_3^\mu = p_3(1,-\sin\theta_3,0,\cos\theta_3)$$

$\mathbf{n}(\theta,\phi)$: polarisation of initial spin

$\lambda_1,\lambda_2,\lambda_3 \in (+,-)$: helicities of 1,2,3

$$\hat{1} = \sum_{\lambda_1,\lambda_2,\lambda_3} |\lambda_1,\lambda_2,\lambda_3\rangle\langle\lambda_1,\lambda_2,\lambda_3|$$

amplitude

$$\mathcal{M}^{\mathbf{n}}_{\lambda_1,\lambda_2,\lambda_3} = \langle\lambda_1,\lambda_2,\lambda_3|\mathbf{n}(\theta,\phi)\rangle$$

initial state

$$|\mathbf{n}(\theta,\phi)\rangle = \sum_{\lambda_1,\lambda_2,\lambda_3} \mathcal{M}^{\mathbf{n}}_{\lambda_1,\lambda_2,\lambda_3}|\lambda_1,\lambda_2,\lambda_3\rangle = |\Psi\rangle \leftarrow$$

pure (entangled) **3-spin state**

Interaction

- Consider **most general** Lorentz invariant 4-fermion interactions

$$\mathcal{L}_{\text{int}} = (\bar{\psi}_1 \Gamma_A \psi_0)(\bar{\psi}_3 \Gamma_B \psi_2)$$

$$\Gamma_{A/B} \in \{1, \gamma^5, \gamma^\mu, \gamma^\mu\gamma^5, \sigma^{\mu\nu}\}$$

$$\psi_0 \to \psi_1 \bar{\psi}_2 \psi_3$$

❖ **Scalar-type**

$$[\bar{\psi}_1(c_S + ic_A\gamma_5)\psi_0][\bar{\psi}_3(d_S + id_A\gamma_5)\psi_2]$$

$$c \equiv c_S + ic_A = e^{i\delta_1}$$
$$d \equiv d_S + id_A = e^{i\delta_2}$$

❖ **Vector-type**

$$[\bar{\psi}_1\gamma_\mu(c_L P_L + c_R P_R)\psi_0][\bar{\psi}_3\gamma^\mu(d_L P_L + d_R P_R)\psi_2]$$

$$P_{R/L} = \frac{1 \pm \gamma^5}{2}$$
$$c_L, c_R, d_L, d_R \in \mathbb{R}$$

❖ **Tensor-type**

$$[\bar{\psi}_1(c_M + ic_E\gamma_5)\sigma^{\mu\nu}\psi_0][\bar{\psi}_3(d_M + id_E\gamma_5)\sigma_{\mu\nu}\psi_2]$$

$$c \equiv c_M + ic_E = e^{i\omega_1}$$
$$d \equiv d_M + id_E = e^{i\omega_2}$$

[KS, M.Spannowsky 2310.01477]

Scalar

$$\mathcal{L}_{\rm int} = \ [\bar{\psi}_1(c_S + ic_A\gamma_5)\psi_0][\bar{\psi}_3(d_S + id_A\gamma_5)\psi_2]$$

$$c \equiv c_S + ic_A = e^{i\delta_1}$$
$$d \equiv d_S + id_A = e^{i\delta_2}$$

$$\Rightarrow \ |\Psi\rangle \ = \ \frac{cd}{\sqrt{2}} \cdot e^{i\phi} s_{\frac{\theta}{2}} |- - -\rangle - \frac{cd^*}{\sqrt{2}} \cdot e^{i\phi} s_{\frac{\theta}{2}} |- + +\rangle + \frac{c^* d}{\sqrt{2}} \cdot c_{\frac{\theta}{2}} |+ - -\rangle - \frac{c^* d^*}{\sqrt{2}} \cdot c_{\frac{\theta}{2}} |+ + +\rangle$$

[KS, M.Spannowsky 2310.01477]

Scalar

$$\mathcal{L}_{\rm int} = [\bar{\psi}_1(c_S + ic_A\gamma_5)\psi_0][\bar{\psi}_3(d_S + id_A\gamma_5)\psi_2]$$

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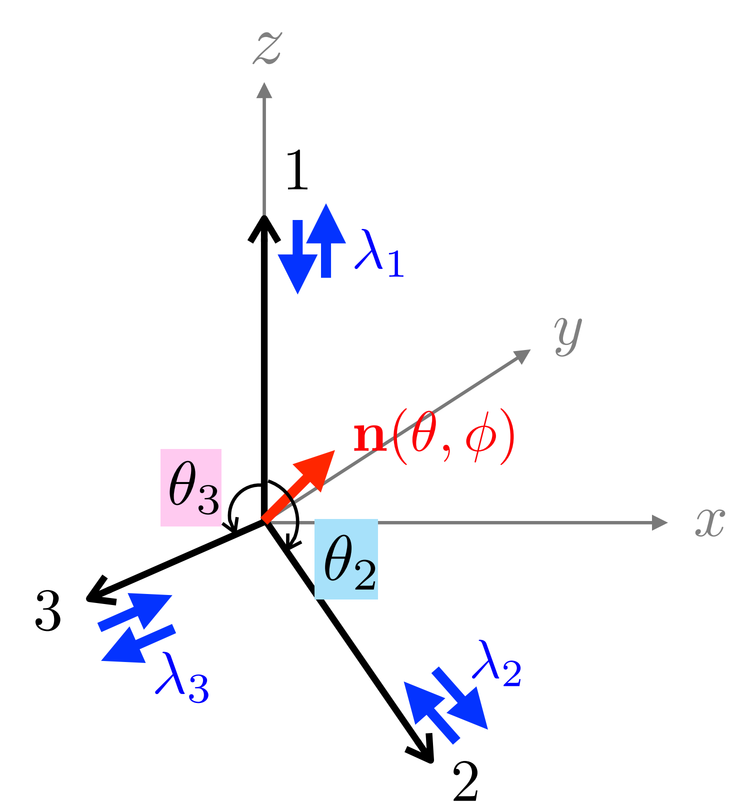
independent of final state momenta θ_2, θ_3



[KS, M.Spannowsky 2310.01477]

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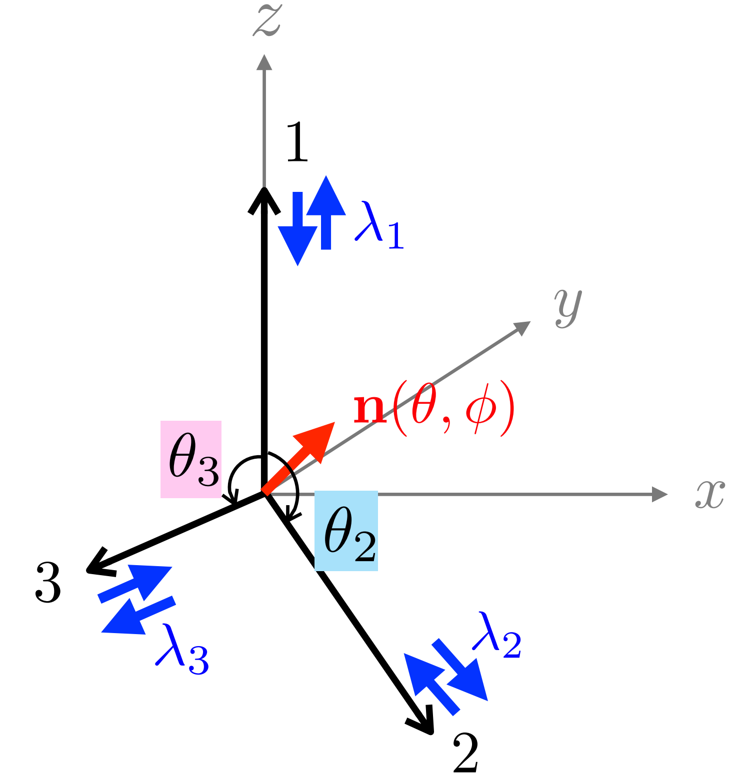
independent of final state momenta θ_2, θ_3

$$= \left[c e^{i\phi} s_{\frac{\theta}{2}} |-\rangle_1 + c^* c_{\frac{\theta}{2}} |+\rangle_1 \right] \otimes \frac{1}{\sqrt{2}} \left[d|--\rangle_{23} - d^*|++\rangle_{23} \right]$$

bi-separable



[KS, M.Spannowsky 2310.01477]

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independent of final state momenta θ_2, θ_3

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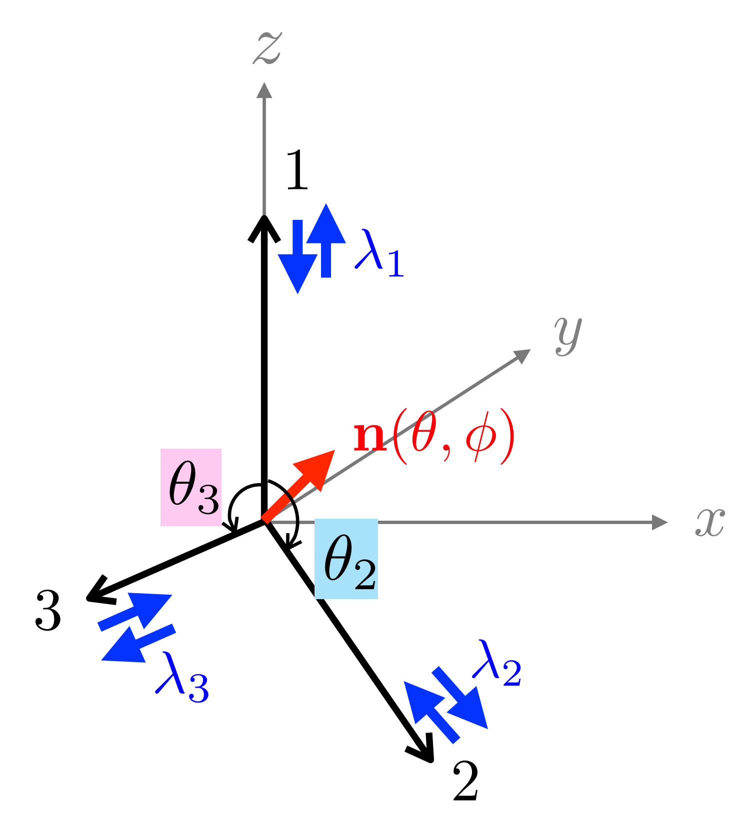
bi-separable

$$\Rightarrow F_3 = 0$$



[KS, M.Spannowsky 2310.01477]

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$$\mathcal{L}_{\rm int} = \; [\bar{\psi}_1(c_S + ic_A\gamma_5)\psi_0][\bar{\psi}_3(d_S + id_A\gamma_5)\psi_2]$$

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$d \equiv d_S + id_A = e^{i\delta_2}$

$$|\Psi\rangle \;=\; \frac{cd}{\sqrt{2}} \cdot e^{i\phi} s_{\frac{\theta}{2}} |-\,--\rangle - \frac{cd^*}{\sqrt{2}} \cdot e^{i\phi} s_{\frac{\theta}{2}} |-\,++\rangle + \frac{c^*d}{\sqrt{2}} \cdot c_{\frac{\theta}{2}} |+\,--\rangle - \frac{c^*d^*}{\sqrt{2}} \cdot c_{\frac{\theta}{2}} |+\,++\rangle$$

independent of final state momenta θ_2, θ_3

$$= \left[ce^{i\phi}s_{\frac{\theta}{2}}|-\rangle_1 + c^* c_{\frac{\theta}{2}}|+\rangle_1\right] \otimes \tfrac{1}{\sqrt{2}}\left[d|--\rangle_{23} - d^*|++\rangle_{23}\right]$$

bi-separable

$$\Rightarrow F_3 = 0$$

✤ **1** is **not entangled** with **2** and **3** in any way: $\mathcal{C}_{12} = \mathcal{C}_{13} = \mathcal{C}_{1(23)} = 0$

[KS, M.Spannowsky 2310.01477]

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independent of final state momenta θ_2, θ_3

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- **1** is **not entangled** with **2** and **3** in any way: $\mathcal{C}_{12} = \mathcal{C}_{13} = \mathcal{C}_{1(23)} = 0$

- **2** and **3** are **maximally entangled**: $\mathcal{C}_{23} = 1$

[KS, M.Spannowsky 2310.01477]

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$$c \equiv c_S + ic_A = e^{i\delta_1}$$
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independent of final state momenta θ_2, θ_3

$$= \left[ce^{i\phi} s_{\frac{\theta}{2}} |-\rangle_1 + c^* c_{\frac{\theta}{2}} |+\rangle_1\right] \otimes \frac{1}{\sqrt{2}} \left[d|--\rangle_{23} - d^*|++\rangle_{23}\right]$$

bi-separable

$$\Rightarrow F_3 = 0$$

- **1** is **not entangled** with **2** and **3** in any way: $\mathcal{C}_{12} = \mathcal{C}_{13} = \mathcal{C}_{1(23)} = 0$

- **2** and **3** are **maximally entangled**: $\mathcal{C}_{23} = 1$

- Due to **monogamy**, **2** and **3** must be **maximally entangled** with the rest:

$$\mathcal{C}^2_{2(13)} = \underset{\substack{\| \\ 0}}{\mathcal{C}^2_{12}} + \underset{\substack{\| \\ 1}}{\mathcal{C}^2_{23}} + \tau \qquad \mathcal{C}^2_{3(12)} = \underset{\substack{\| \\ 0}}{\mathcal{C}^2_{13}} + \underset{\substack{\| \\ 1}}{\mathcal{C}^2_{23}} + \tau$$

[KS, M.Spannowsky 2310.01477]

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$$\mathcal{L}_{\rm int} = [\bar{\psi}_1(c_S + ic_A\gamma_5)\psi_0][\bar{\psi}_3(d_S + id_A\gamma_5)\psi_2]$$

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independent of final state momenta θ_2, θ_3

$$= \left[c e^{i\phi} s_{\frac{\theta}{2}} |-\rangle_1 + c^* c_{\frac{\theta}{2}} |+\rangle_1 \right] \otimes \frac{1}{\sqrt{2}} \left[d|--\rangle_{23} - d^*|++\rangle_{23} \right]$$

bi-separable

$$\Rightarrow F_3 = 0$$

- ✤ **1** is **not entangled** with **2** and **3** in any way: $\mathcal{C}_{12} = \mathcal{C}_{13} = \mathcal{C}_{1(23)} = 0$
- ✤ **2** and **3** are **maximally entangled**: $\mathcal{C}_{23} = 1$
- ✤ Due to **monogamy**, **2** and **3** must be **maximally entangled** with the rest:

$$\mathcal{C}^2_{2(13)} = \underset{\substack{\| \\ 0}}{\mathcal{C}^2_{12}} + \underset{\substack{\| \\ 1}}{\mathcal{C}^2_{23}} + \tau \qquad \mathcal{C}^2_{3(12)} = \underset{\substack{\| \\ 0}}{\mathcal{C}^2_{13}} + \underset{\substack{\| \\ 1}}{\mathcal{C}^2_{23}} + \tau$$

$$\Rightarrow \begin{cases} \mathcal{C}_{2(13)} = \mathcal{C}_{3(12)} = 1 \\ \tau = 0 \end{cases}$$

[KS, M.Spannowsky
2310.01477]

Vector

$$P_{R/L} = \frac{1 \pm \gamma^5}{2}$$

$$\mathcal{L}_{\text{int}} = [\bar{\psi}_1 \gamma_\mu (c_L P_L + c_R P_R) \psi_0][\bar{\psi}_3 \gamma^\mu (d_L P_L + d_R P_R) \psi_2] \qquad c_L, c_R, d_L, d_R \in \mathbb{R}$$

[KS, M.Spannowsky 2310.01477]

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$$\Rightarrow \quad |\Psi\rangle = M_{LL}|-+-\rangle + M_{LR}|--+\rangle + M_{RL}|++-\rangle + M_{RR}|+-+\rangle$$

Vector

[KS, M.Spannowsky 2310.01477]

$$P_{R/L} = \frac{1 \pm \gamma^5}{2}$$

$$\mathcal{L}_{\rm int} = [\bar{\psi}_1 \gamma_\mu (c_L P_L + c_R P_R) \psi_0][\bar{\psi}_3 \gamma^\mu (d_L P_L + d_R P_R) \psi_2] \qquad c_L, c_R, d_L, d_R \in \mathbb{R}$$

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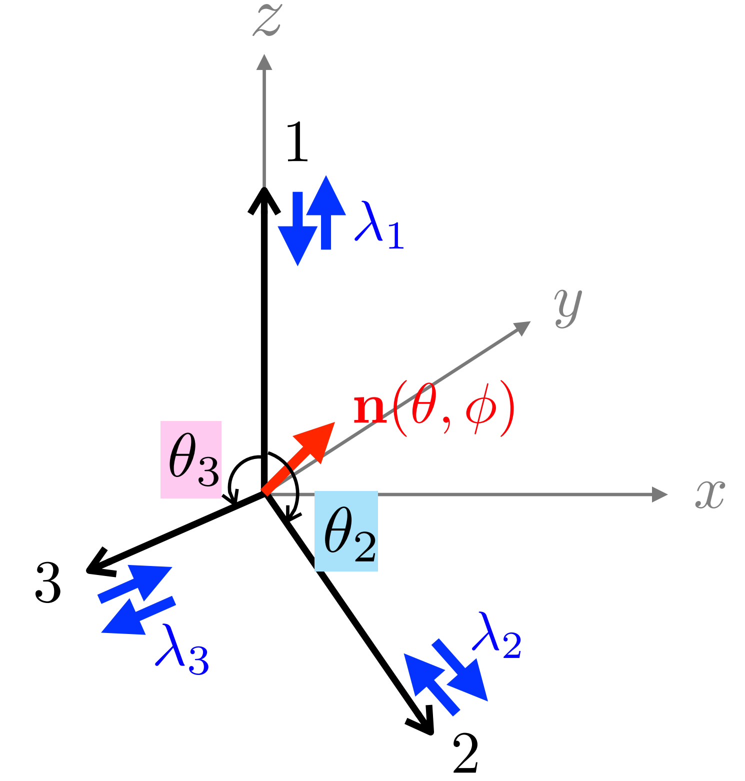
$$\propto \; c_L d_L s_{\frac{\theta_3}{2}} \left[c_{\frac{\theta}{2}} c_{\frac{\theta_2}{2}} + e^{i\phi} s_{\frac{\theta}{2}} s_{\frac{\theta_2}{2}}\right] |-+-\rangle + c_L d_R s_{\frac{\theta_2}{2}} \left[c_{\frac{\theta}{2}} c_{\frac{\theta_3}{2}} + e^{i\phi} s_{\frac{\theta}{2}} s_{\frac{\theta_3}{2}}\right] |--+\rangle$$

$$+ \; c_R d_L s_{\frac{\theta_2}{2}} \left[c_{\frac{\theta}{2}} s_{\frac{\theta_3}{2}} - e^{i\phi} s_{\frac{\theta}{2}} c_{\frac{\theta_3}{2}}\right] |++-\rangle + c_R d_R s_{\frac{\theta_3}{2}} \left[c_{\frac{\theta}{2}} s_{\frac{\theta_2}{2}} - e^{i\phi} s_{\frac{\theta}{2}} c_{\frac{\theta_2}{2}}\right] |+-+\rangle$$



[KS, M.Spannowsky 2310.01477]

Vector

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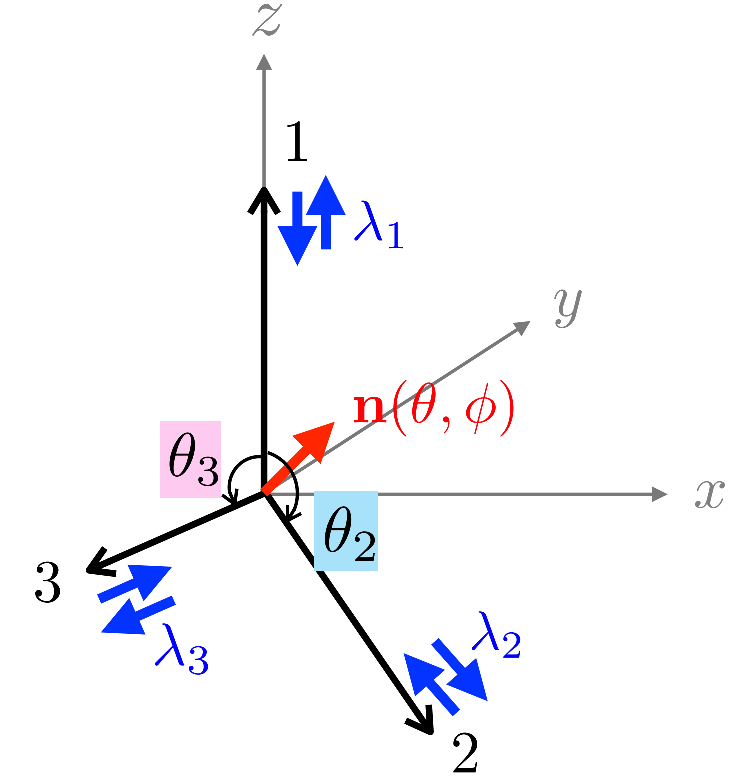
$$\Rightarrow \quad |\Psi\rangle = M_{LL}|-+-\rangle + M_{LR}|--+\rangle + M_{RL}|++-\rangle + M_{RR}|+-+\rangle$$

$$\propto \quad c_L d_L s_{\frac{\theta_3}{2}} \left[c_{\frac{\theta}{2}} c_{\frac{\theta_2}{2}} + e^{i\phi} s_{\frac{\theta}{2}} s_{\frac{\theta_2}{2}}\right] |-+-\rangle \ + \ c_L d_R s_{\frac{\theta_2}{2}} \left[c_{\frac{\theta}{2}} c_{\frac{\theta_3}{2}} + e^{i\phi} s_{\frac{\theta}{2}} s_{\frac{\theta_3}{2}}\right] |--+\rangle$$

$$+ \quad c_R d_L s_{\frac{\theta_2}{2}} \left[c_{\frac{\theta}{2}} s_{\frac{\theta_3}{2}} - e^{i\phi} s_{\frac{\theta}{2}} c_{\frac{\theta_3}{2}}\right] |++-\rangle \ + \ c_R d_R s_{\frac{\theta_3}{2}} \left[c_{\frac{\theta}{2}} s_{\frac{\theta_2}{2}} - e^{i\phi} s_{\frac{\theta}{2}} c_{\frac{\theta_2}{2}}\right] |+-+\rangle$$

✤ Individual 2-party entanglement:

$$\mathcal{C}_{12} = \mathcal{C}_{13} = 0\,, \quad \mathcal{C}_{23} = 2|M_{LL}M_{LR}^* + M_{RL}M_{RR}^*|$$



[KS, M.Spannowsky 2310.01477]

Vector

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$$\Rightarrow |\Psi\rangle = M_{LL}|-+-\rangle + M_{LR}|--+\rangle + M_{RL}|++-\rangle + M_{RR}|+-+\rangle$$

$$\propto c_L d_L s_{\frac{\theta_3}{2}} \left[c_{\frac{\theta}{2}} c_{\frac{\theta_2}{2}} + e^{i\phi} s_{\frac{\theta}{2}} s_{\frac{\theta_2}{2}}\right] |-+-\rangle + c_L d_R s_{\frac{\theta_2}{2}} \left[c_{\frac{\theta}{2}} c_{\frac{\theta_3}{2}} + e^{i\phi} s_{\frac{\theta}{2}} s_{\frac{\theta_3}{2}}\right] |--+\rangle$$

$$+ c_R d_L s_{\frac{\theta_2}{2}} \left[c_{\frac{\theta}{2}} s_{\frac{\theta_3}{2}} - e^{i\phi} s_{\frac{\theta}{2}} c_{\frac{\theta_3}{2}}\right] |++-\rangle + c_R d_R s_{\frac{\theta_3}{2}} \left[c_{\frac{\theta}{2}} s_{\frac{\theta_2}{2}} - e^{i\phi} s_{\frac{\theta}{2}} c_{\frac{\theta_2}{2}}\right] |+-+\rangle$$

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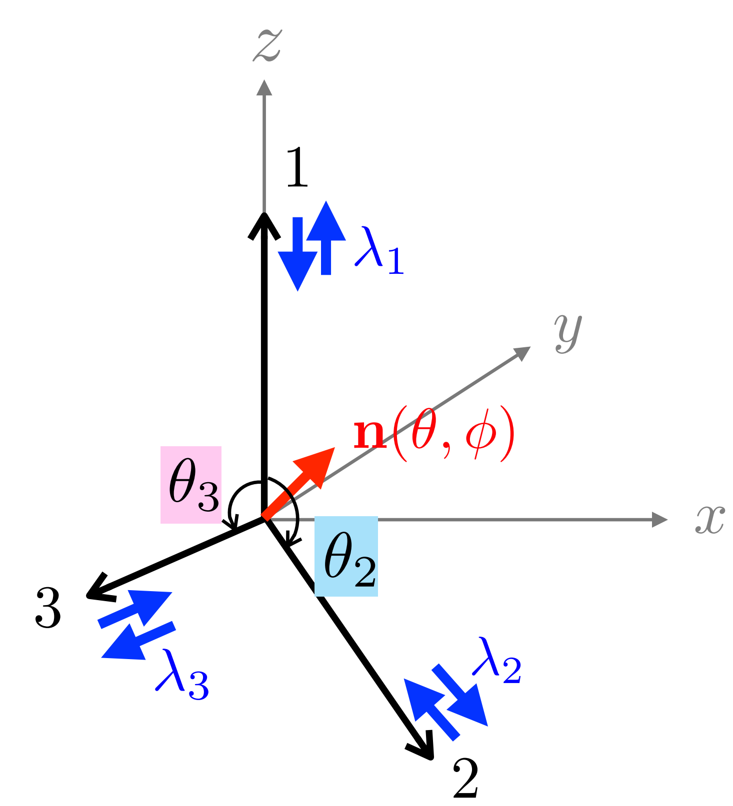
$$\mathcal{C}_{12} = \mathcal{C}_{13} = 0\,, \quad \mathcal{C}_{23} = 2|M_{LL}M_{LR}^* + M_{RL}M_{RR}^*|$$

✤ one-to-other entanglement:

$$\mathcal{C}_{2(13)} = \mathcal{C}_{3(12)} = 2\sqrt{\left(|M_{LL}|^2 + |M_{RL}|^2\right)\left(|M_{LR}|^2 + |M_{RR}|^2\right)}$$

$$\mathcal{C}_{1(23)} = 2\big|M_{RR}M_{LL} - M_{LR}M_{RL}\big|$$



[KS, M.Spannowsky 2310.01477]

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$$\Rightarrow \; |\Psi\rangle = M_{LL}|-+-\rangle + M_{LR}|--+\rangle + M_{RL}|++-\rangle + M_{RR}|+-+\rangle$$

$$\propto \; c_L d_L s_{\frac{\theta_3}{2}} \left[c_{\frac{\theta}{2}} c_{\frac{\theta_2}{2}} + e^{i\phi} s_{\frac{\theta}{2}} s_{\frac{\theta_2}{2}} \right] |-+-\rangle \; + \; c_L d_R s_{\frac{\theta_2}{2}} \left[c_{\frac{\theta}{2}} c_{\frac{\theta_3}{2}} + e^{i\phi} s_{\frac{\theta}{2}} s_{\frac{\theta_3}{2}} \right] |--+\rangle$$

$$+ \; c_R d_L s_{\frac{\theta_2}{2}} \left[c_{\frac{\theta}{2}} s_{\frac{\theta_3}{2}} - e^{i\phi} s_{\frac{\theta}{2}} c_{\frac{\theta_3}{2}} \right] |++-\rangle \; + \; c_R d_R s_{\frac{\theta_3}{2}} \left[c_{\frac{\theta}{2}} s_{\frac{\theta_2}{2}} - e^{i\phi} s_{\frac{\theta}{2}} c_{\frac{\theta_2}{2}} \right] |+-+\rangle$$

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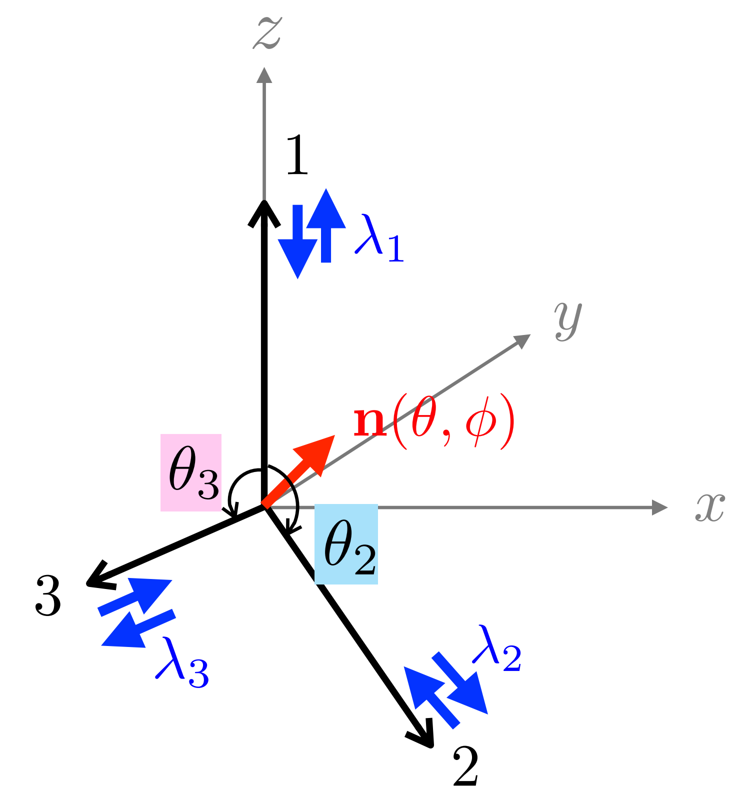
$$\mathcal{C}_{2(13)} = \mathcal{C}_{3(12)} = 2\sqrt{\left(|M_{LL}|^2 + |M_{RL}|^2\right)\left(|M_{LR}|^2 + |M_{RR}|^2\right)}$$

$$\mathcal{C}_{1(23)} = 2\big|M_{RR}M_{LL} - M_{LR}M_{RL}\big|$$

✤ 3-tangle

$$\tau = \mathcal{C}^2_{1(23)} - [\underbrace{\mathcal{C}^2_{12}}_{=0} + \underbrace{\mathcal{C}^2_{13}}_{=0}] = \mathcal{C}^2_{1(23)}$$



[KS, M.Spannowsky 2310.01477]

Vector

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$$\mathcal{L}_{\rm int} = [\bar{\psi}_1 \gamma_\mu (c_L P_L + c_R P_R) \psi_0][\bar{\psi}_3 \gamma^\mu (d_L P_L + d_R P_R) \psi_2] \qquad c_L, c_R, d_L, d_R \in \mathbb{R}$$

$$\Rightarrow \; |\Psi\rangle = M_{LL}|{-}{+}{-}\rangle + M_{LR}|{-}{-}{+}\rangle + M_{RL}|{+}{+}{-}\rangle + M_{RR}|{+}{-}{+}\rangle$$

$$\propto \; c_L d_L s_{\frac{\theta_3}{2}} \left[c_{\frac{\theta}{2}} c_{\frac{\theta_2}{2}} + e^{i\phi} s_{\frac{\theta}{2}} s_{\frac{\theta_2}{2}}\right] |{-}{+}{-}\rangle + c_L d_R s_{\frac{\theta_2}{2}} \left[c_{\frac{\theta}{2}} c_{\frac{\theta_3}{2}} + e^{i\phi} s_{\frac{\theta}{2}} s_{\frac{\theta_3}{2}}\right] |{-}{-}{+}\rangle$$

$$+ \; c_R d_L s_{\frac{\theta_2}{2}} \left[c_{\frac{\theta}{2}} s_{\frac{\theta_3}{2}} - e^{i\phi} s_{\frac{\theta}{2}} c_{\frac{\theta_3}{2}}\right] |{+}{+}{-}\rangle + c_R d_R s_{\frac{\theta_3}{2}} \left[c_{\frac{\theta}{2}} s_{\frac{\theta_2}{2}} - e^{i\phi} s_{\frac{\theta}{2}} c_{\frac{\theta_2}{2}}\right] |{+}{-}{+}\rangle$$

- **Individual 2-party entanglement:**

$$\mathcal{C}_{12} = \mathcal{C}_{13} = 0\,, \quad \mathcal{C}_{23} = 2|M_{LL} M^*_{LR} + M_{RL} M^*_{RR}| \quad \longleftarrow \text{vanish if } d_L d_R = 0$$

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$$\mathcal{C}_{2(13)} = \mathcal{C}_{3(12)} = 2\sqrt{(|M_{LL}|^2 + |M_{RL}|^2)\,(|M_{LR}|^2 + |M_{RR}|^2)} \quad \longleftarrow \text{vanish if } c_L c_R = d_L d_R = 0$$

$$\mathcal{C}_{1(23)} = 2\big|M_{RR} M_{LL} - M_{LR} M_{RL}\big| \quad \longleftarrow \text{vanish if } c_L c_R d_L d_R = 0$$

- **3-tangle**

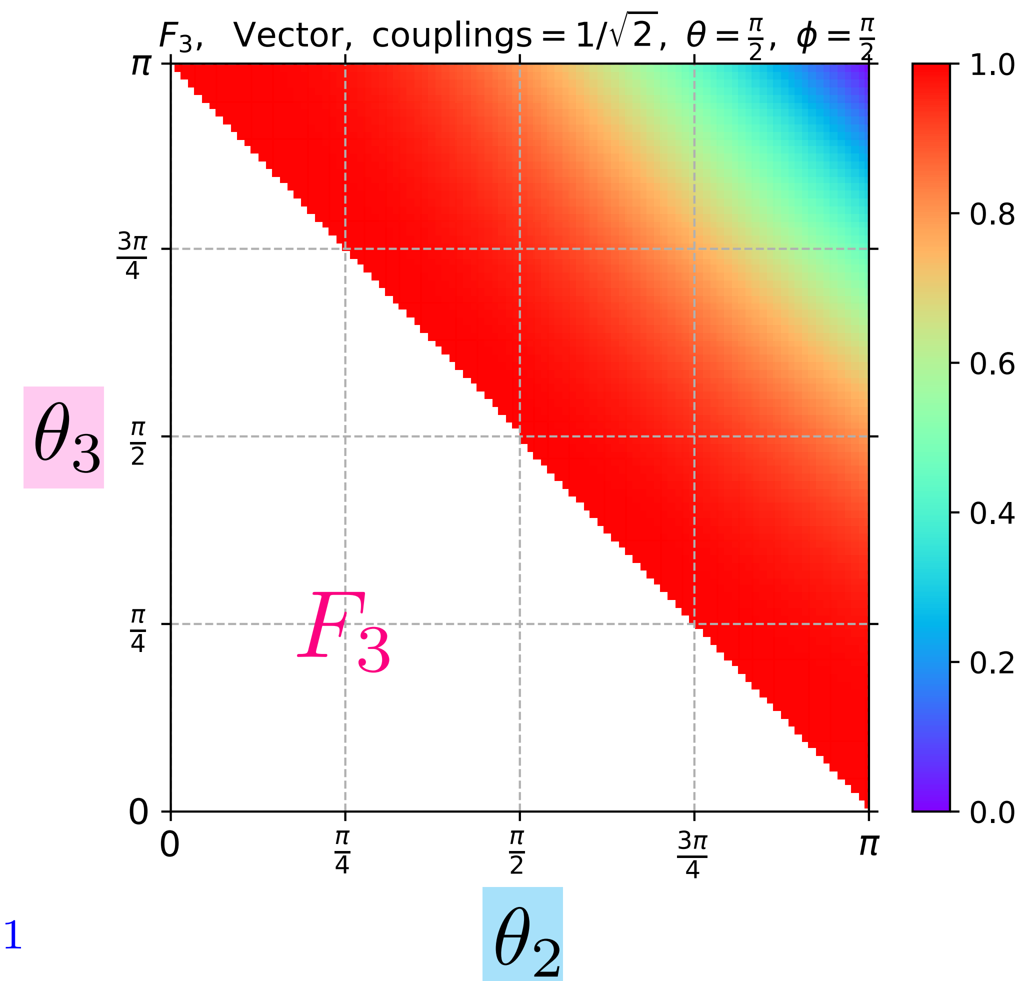
$$\tau = \mathcal{C}^2_{1(23)} - [\underbrace{\mathcal{C}^2_{12}}_{=0} + \underbrace{\mathcal{C}^2_{13}}_{=0}] = \mathcal{C}^2_{1(23)}$$

➡ **All** entanglements **vanish** for **weak decays**

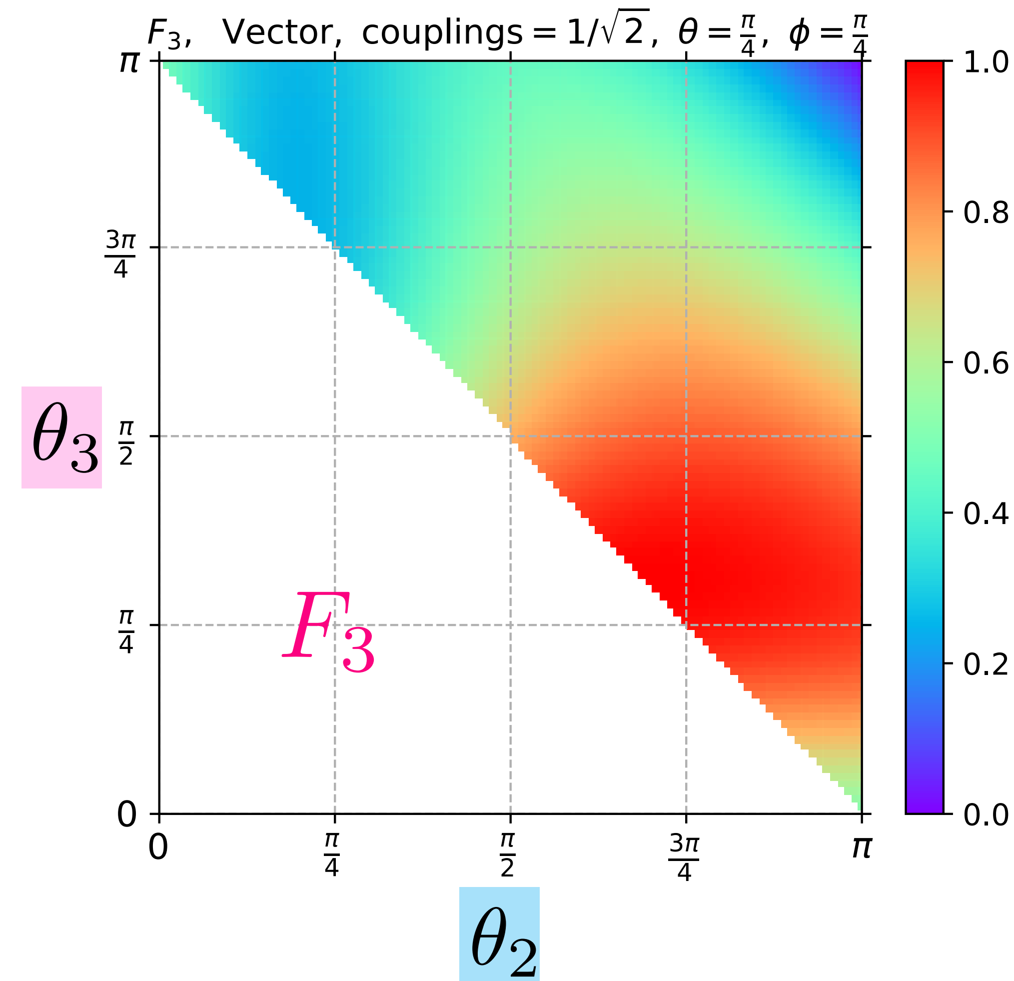
$$c_R = d_R = 0$$

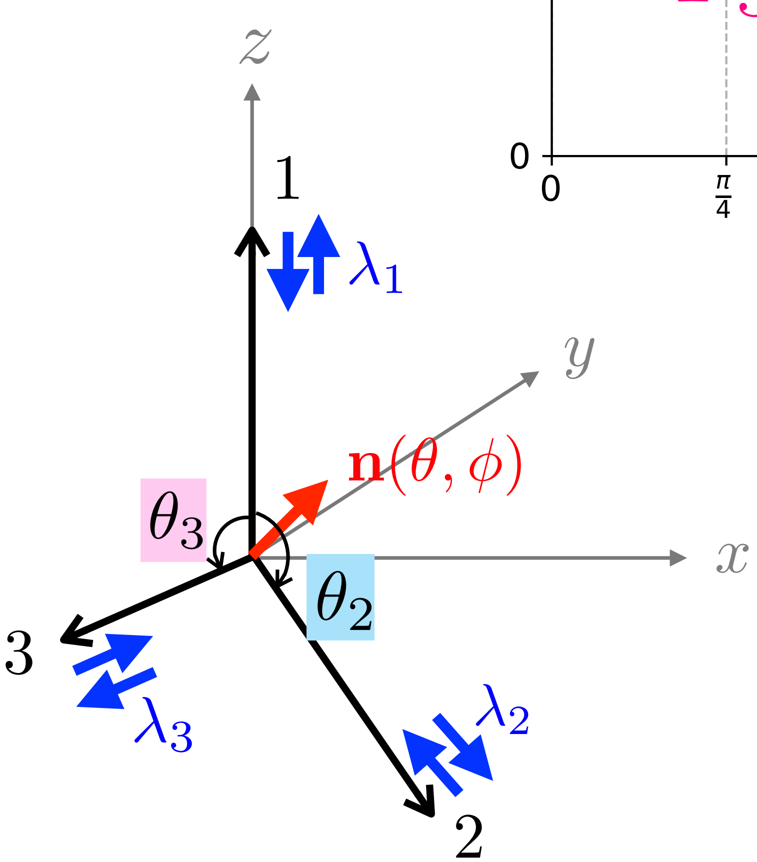
F_3 for Vector

$$\mathbf{n} = \mathbf{e}_y$$

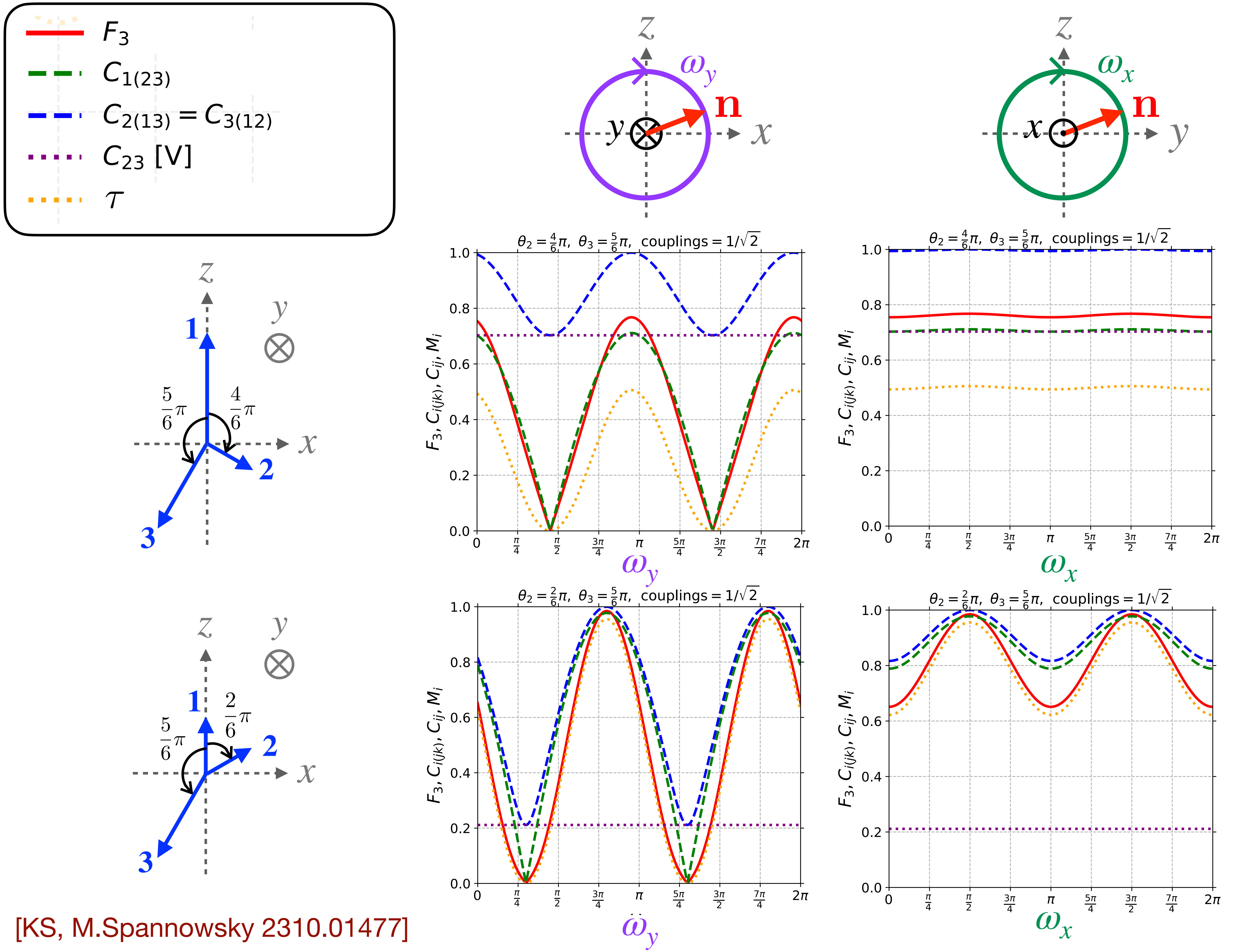


$$\mathbf{n} = \frac{1}{\sqrt{3}}(\mathbf{e}_x + \mathbf{e}_y + \mathbf{e}_z)$$





[KS, M.Spannowsky 2310.01477]



F_3
$C_{1(23)}$
$C_{2(13)} = C_{3(12)}$
C_{23} [V]
τ
z
x
y
ω_y
ω_x
n
1
2
3
$\frac{5}{6}\pi$
$\frac{4}{6}\pi$
$\frac{2}{6}\pi$
$\theta_2 = \frac{4}{6}\pi,\ \theta_3 = \frac{5}{6}\pi,$ couplings $= 1/\sqrt{2}$
$\theta_2 = \frac{2}{6}\pi,\ \theta_3 = \frac{5}{6}\pi,$ couplings $= 1/\sqrt{2}$
$F_3, C_{i(jk)}, C_{ij}, M_i$
0.0
0.2
0.4
0.6
0.8
1.0
0
$\frac{\pi}{4}$
$\frac{\pi}{2}$
$\frac{3\pi}{4}$
π
$\frac{5\pi}{4}$
$\frac{3\pi}{2}$
$\frac{7\pi}{4}$
2π
[KS, M.Spannowsky 2310.01477]

[KS, M.Spannowsky
2310.01477]

Tensor

$$\mathcal{L}_{\rm int} = [\bar{\psi}_1(c_M + ic_E\gamma_5)\sigma^{\mu\nu}\psi_0][\bar{\psi}_3(d_M + id_E\gamma_5)\sigma_{\mu\nu}\psi_2]$$

$$c \equiv c_M + ic_E = e^{i\omega_1}$$
$$d \equiv d_M + id_E = e^{i\omega_2}$$

[KS, M.Spannowsky 2310.01477]

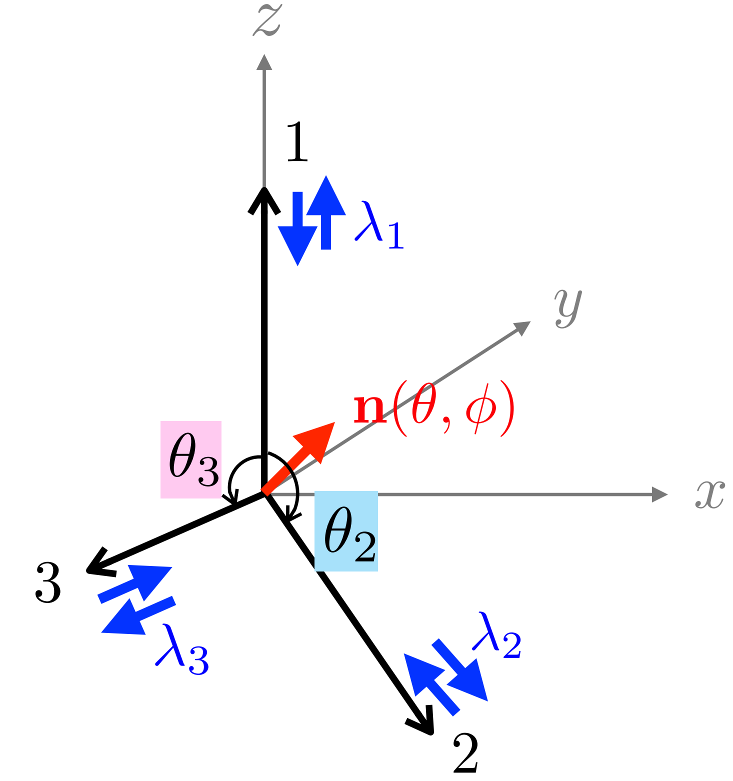
Tensor

$$\mathcal{L}_{\rm int} = [\bar{\psi}_1(c_M + ic_E\gamma_5)\sigma^{\mu\nu}\psi_0][\bar{\psi}_3(d_M + id_E\gamma_5)\sigma_{\mu\nu}\psi_2]$$

$$c \equiv c_M + ic_E = e^{i\omega_1}$$
$$d \equiv d_M + id_E = e^{i\omega_2}$$

$$\Rightarrow \quad |\Psi\rangle = M_R|+++\rangle + M_L|---\rangle$$

$$\propto c^* d^* [2e^{i\phi}s_{\frac{\theta}{2}}s_{\frac{\theta_2}{2}}s_{\frac{\theta_3}{2}} + c_{\frac{\theta}{2}}s_{\frac{\theta_3-\theta_2}{2}}]|+++\rangle + cd\,[-e^{i\phi}s_{\frac{\theta}{2}}s_{\frac{\theta_3-\theta_2}{2}} + 2c_{\frac{\theta}{2}}s_{\frac{\theta_2}{2}}s_{\frac{\theta_3}{2}}]|---\rangle$$



[KS, M.Spannowsky 2310.01477]

Tensor

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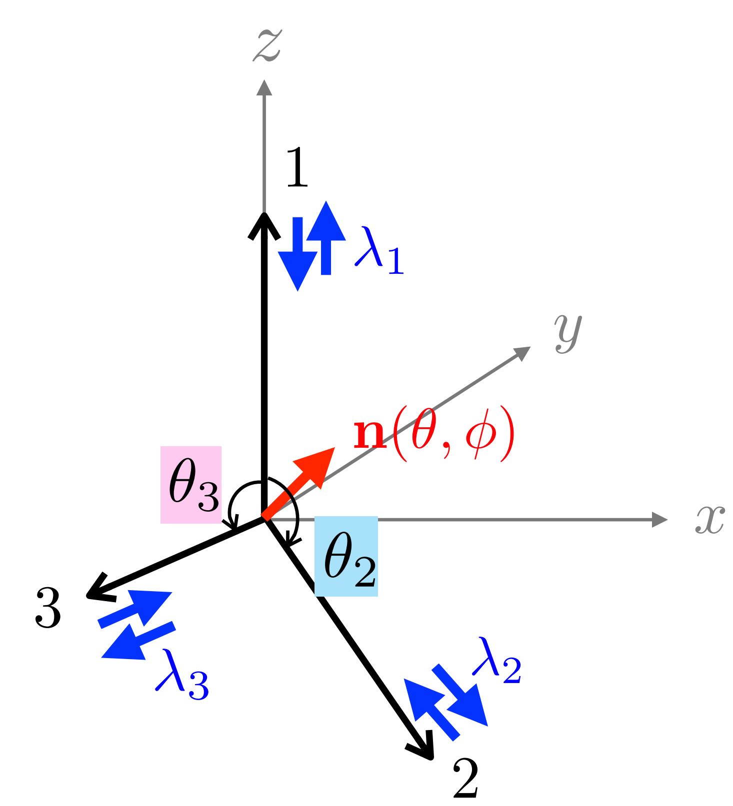
$$\propto c^* d^* \left[2e^{i\phi} s_{\frac{\theta}{2}} s_{\frac{\theta_2}{2}} s_{\frac{\theta_3}{2}} + c_{\frac{\theta}{2}} s_{\frac{\theta_3-\theta_2}{2}}\right]|+++\rangle + cd\left[-e^{i\phi} s_{\frac{\theta}{2}} s_{\frac{\theta_3-\theta_2}{2}} + 2c_{\frac{\theta}{2}} s_{\frac{\theta_2}{2}} s_{\frac{\theta_3}{2}}\right]|---\rangle$$

✤ $|\Psi\rangle$ *interpolates* **product states** and the **maximally entangled** state:

$(M_R M_L = 0)$ $\quad |\pm\pm\pm\rangle \longleftrightarrow |{\rm GHZ}\rangle = \frac{1}{\sqrt{2}}(|+++\rangle + |---\rangle)$ $\quad (M_R = M_L)$



[KS, M.Spannowsky 2310.01477]

Tensor

$$\mathcal{L}_{\rm int} = [\bar{\psi}_1(c_M + ic_E\gamma_5)\sigma^{\mu\nu}\psi_0][\bar{\psi}_3(d_M + id_E\gamma_5)\sigma_{\mu\nu}\psi_2]$$

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$$(M_R M_L = 0) \qquad |\pm\pm\pm\rangle \longleftrightarrow |{\rm GHZ}\rangle = \frac{1}{\sqrt{2}}(|+++\rangle + |---\rangle) \qquad (M_R = M_L)$$

✤ **No** individual 2-party entanglements:

$$\mathcal{C}_{12} = \mathcal{C}_{13} = \mathcal{C}_{23} = 0$$

[KS, M.Spannowsky 2310.01477]

Tensor

$$\mathcal{L}_{\rm int} = [\bar{\psi}_1(c_M + ic_E\gamma_5)\sigma^{\mu\nu}\psi_0][\bar{\psi}_3(d_M + id_E\gamma_5)\sigma_{\mu\nu}\psi_2]$$

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$$\propto \; c^* d^* \left[2e^{i\phi} s_{\frac{\theta}{2}} s_{\frac{\theta_2}{2}} s_{\frac{\theta_3}{2}} + c_{\frac{\theta}{2}} s_{\frac{\theta_3-\theta_2}{2}}\right] |+++\rangle + cd\left[-e^{i\phi} s_{\frac{\theta}{2}} s_{\frac{\theta_3-\theta_2}{2}} + 2c_{\frac{\theta}{2}} s_{\frac{\theta_2}{2}} s_{\frac{\theta_3}{2}}\right] |---\rangle$$

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✤ **No** individual 2-party entanglements:

3-tangle

$$\mathcal{C}_{12} = \mathcal{C}_{13} = \mathcal{C}_{23} = 0 \longrightarrow \tau = \mathcal{C}^2_{i(jk)} - [\underbrace{\mathcal{C}^2_{ij}}_{=0} + \underbrace{\mathcal{C}^2_{ik}}_{=0}] = \mathcal{C}^2_{i(jk)}$$

[KS, M.Spannowsky 2310.01477]

Tensor

$$\mathcal{L}_{\rm int} = [\bar{\psi}_1 (c_M + i c_E \gamma_5) \sigma^{\mu\nu} \psi_0][\bar{\psi}_3 (d_M + i d_E \gamma_5) \sigma_{\mu\nu} \psi_2]$$

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✤ **No** individual 2-party entanglements:

3-tangle

$$\mathcal{C}_{12} = \mathcal{C}_{13} = \mathcal{C}_{23} = 0 \longrightarrow \tau = \mathcal{C}^2_{i(jk)} - [\mathcal{C}^2_{ij} + \mathcal{C}^2_{ik}] = \mathcal{C}^2_{i(jk)}$$

$$\mathcal{C}^2_{ij} = 0, \quad \mathcal{C}^2_{ik} = 0$$

✤ one-to-other entanglements are **universal**:

$$\mathcal{C}_{1(23)} = \mathcal{C}_{2(13)} = \mathcal{C}_{3(12)} = 2|M_R M_L|$$

$$F_3 = 4|M_R M_L|^2$$

[KS, M.Spannowsky 2310.01477]

Tensor

$$\mathcal{L}_{\rm int} = [\bar{\psi}_1 (c_M + i c_E \gamma_5) \sigma^{\mu\nu} \psi_0][\bar{\psi}_3 (d_M + i d_E \gamma_5) \sigma_{\mu\nu} \psi_2]$$

$$c \equiv c_M + i c_E = e^{i\omega_1}$$
$$d \equiv d_M + i d_E = e^{i\omega_2}$$

$$\Rightarrow \quad |\Psi\rangle = M_R |+++\rangle + M_L |---\rangle$$

$$\propto \; c^* d^* \, [2 e^{i\phi} s_{\frac{\theta}{2}} s_{\frac{\theta_2}{2}} s_{\frac{\theta_3}{2}} + c_{\frac{\theta}{2}} s_{\frac{\theta_3 - \theta_2}{2}}] \, |+++\rangle + cd \, [-e^{i\phi} s_{\frac{\theta}{2}} s_{\frac{\theta_3 - \theta_2}{2}} + 2 c_{\frac{\theta}{2}} s_{\frac{\theta_2}{2}} s_{\frac{\theta_3}{2}}] \, |---\rangle$$

✤ $|\Psi\rangle$ *interpolates* **product states** and the **maximally entangled** state:

$$(M_R M_L = 0) \quad |\pm\pm\pm\rangle \longleftrightarrow |{\rm GHZ}\rangle = \frac{1}{\sqrt{2}}(|+++\rangle + |---\rangle) \quad (M_R = M_L)$$

✤ **No** individual 2-party entanglements:

$$\mathcal{C}_{12} = \mathcal{C}_{13} = \mathcal{C}_{23} = 0 \longrightarrow$$

3-tangle

$$\tau = \mathcal{C}^2_{i(jk)} - [\mathcal{C}^2_{ij} + \mathcal{C}^2_{ik}] = \mathcal{C}^2_{i(jk)}, \quad \mathcal{C}^2_{ij} = 0, \; \mathcal{C}^2_{ik} = 0$$

✤ one-to-other entanglements are **universal**:

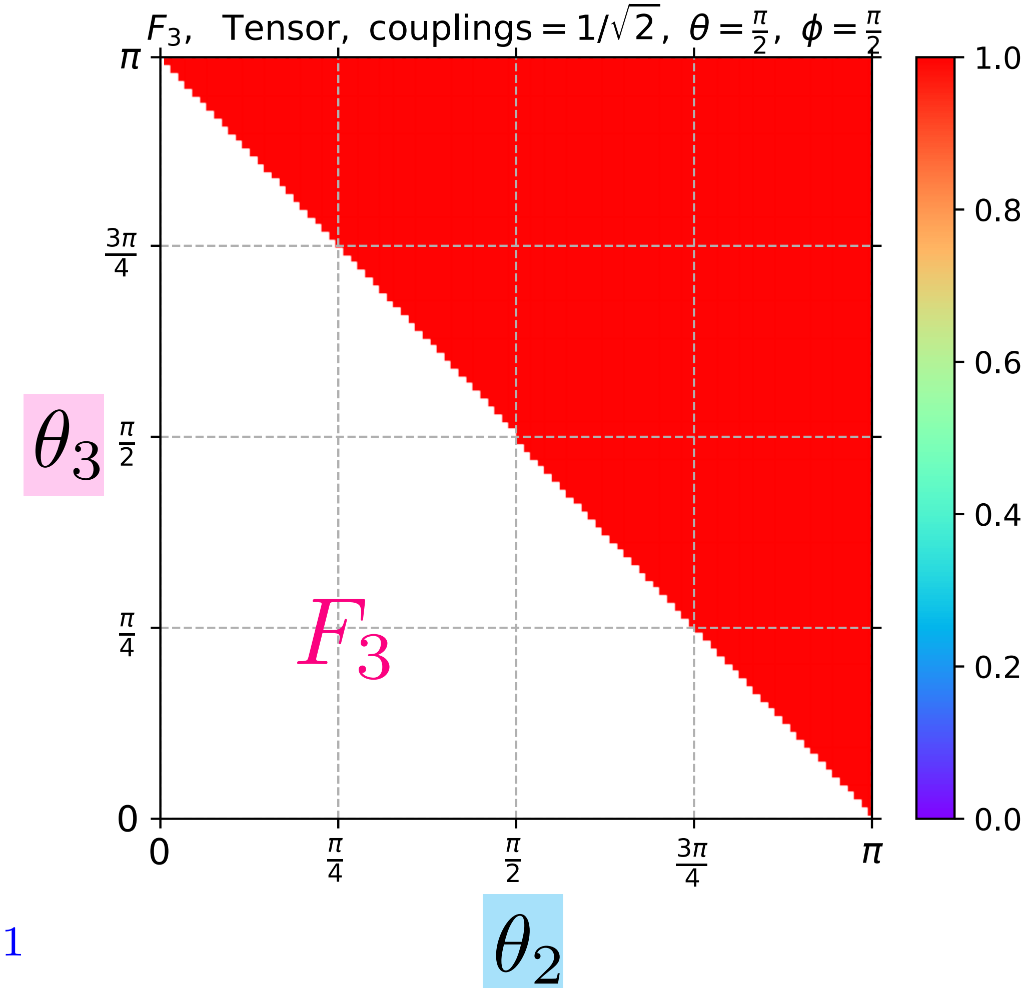
$$\mathcal{C}_{1(23)} = \mathcal{C}_{2(13)} = \mathcal{C}_{3(12)} = 2|M_R M_L|$$

$$F_3 = 4|M_R M_L|^2$$

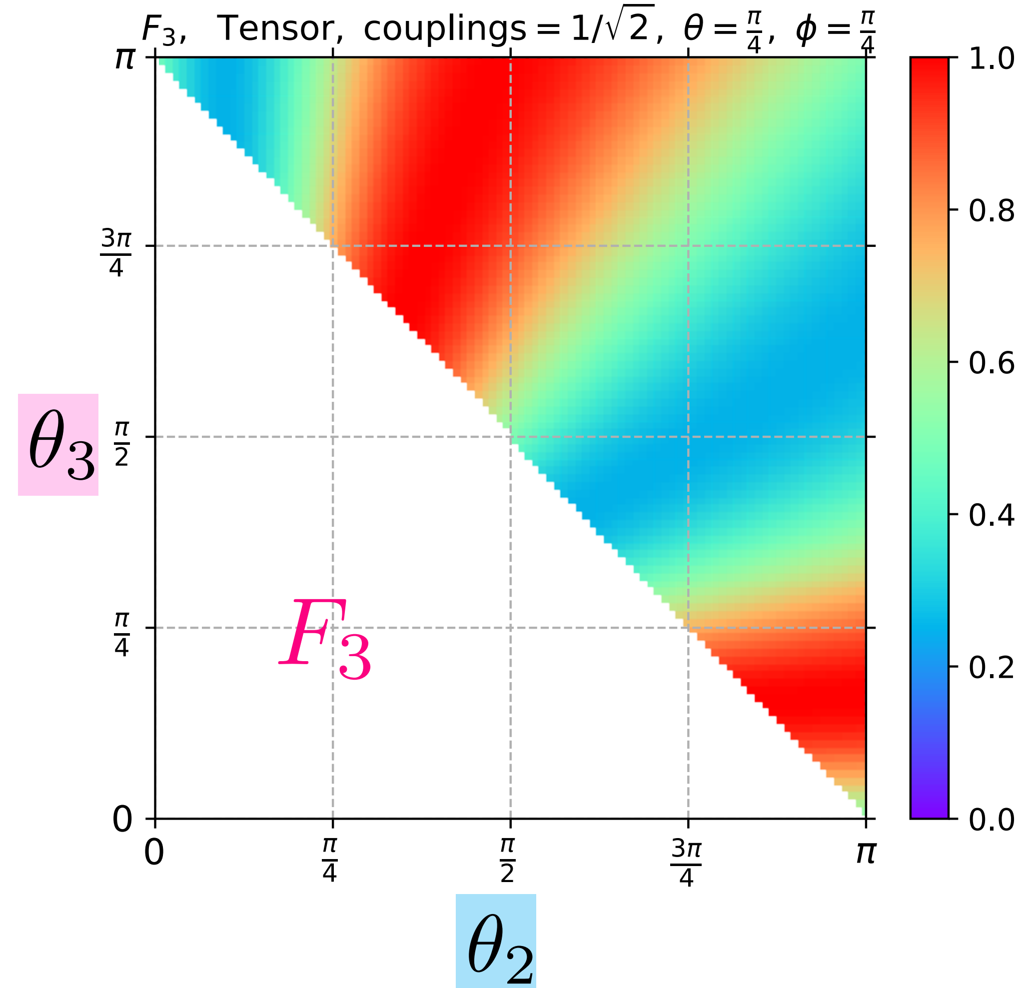
independent of the coupling structure (CP phases) ω_1, ω_2

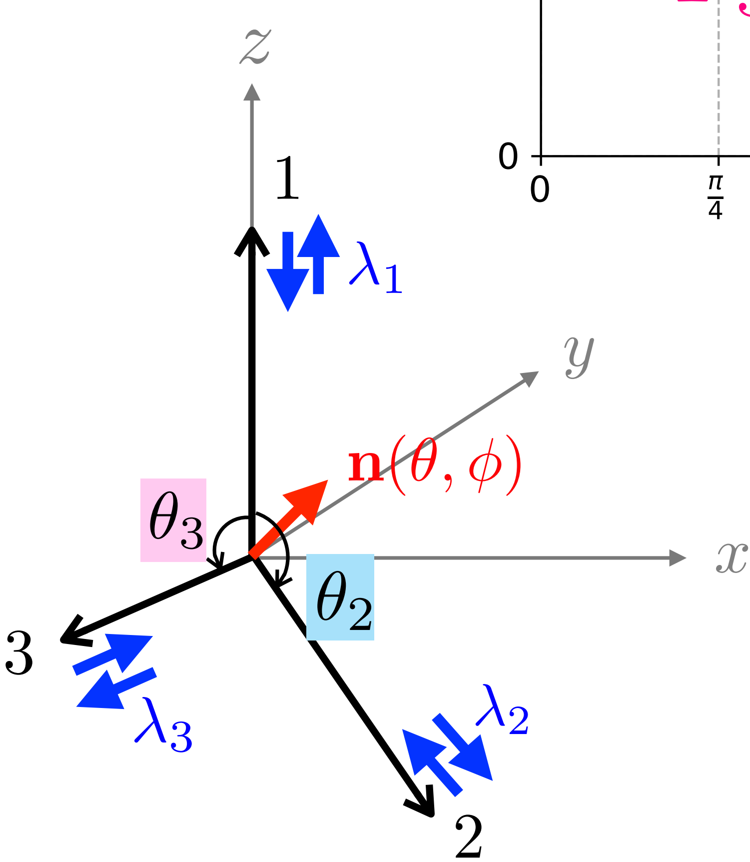
F_3 for Tensor

$\mathbf{n} = \mathbf{e}_y$



$\mathbf{n} = \frac{1}{\sqrt{3}}(\mathbf{e}_x + \mathbf{e}_y + \mathbf{e}_z)$





[KS, M.Spannowsky 2310.01477]

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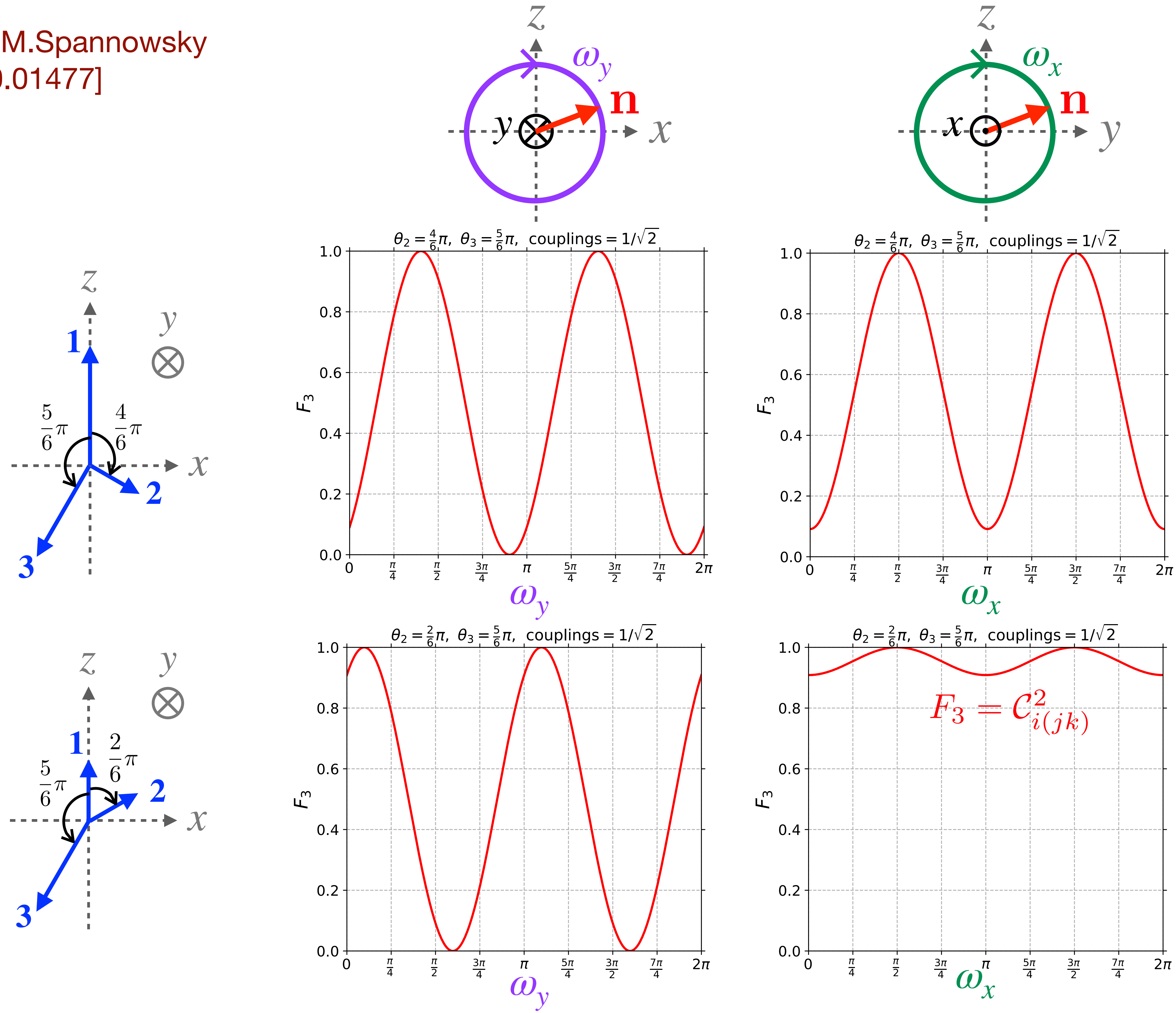
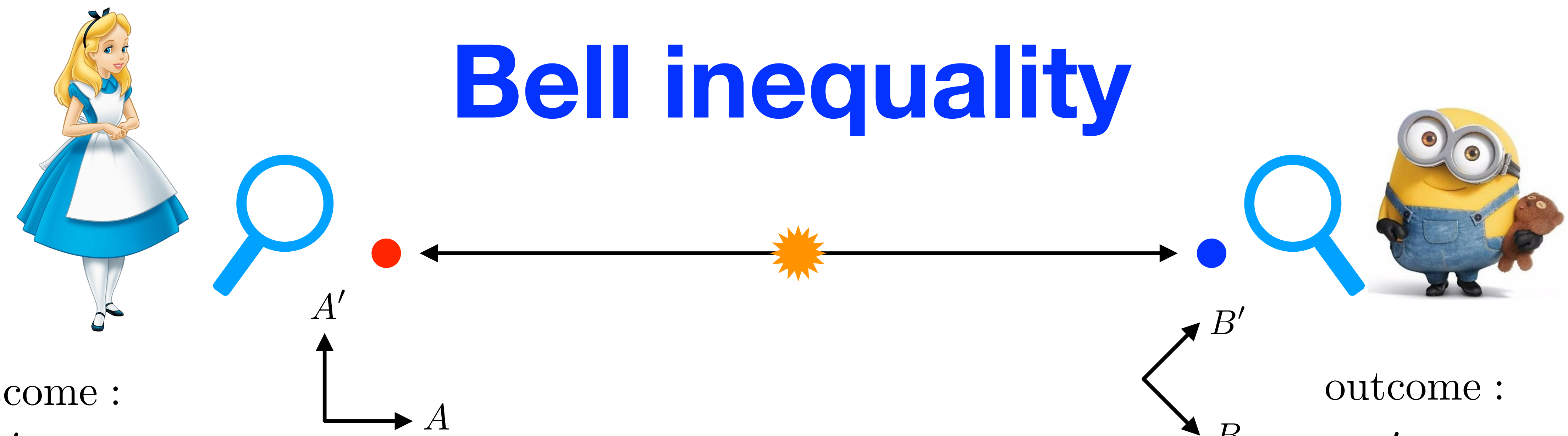
2310.01477]



Bell inequality



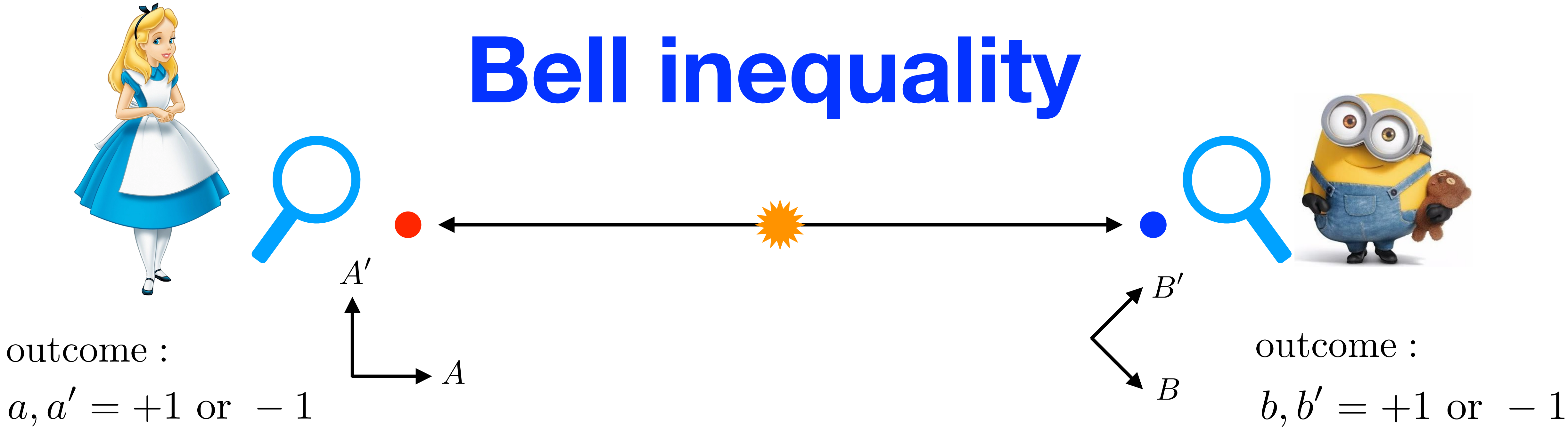
outcome :
$a, a' = +1$ or -1

outcome :
$b, b' = +1$ or -1

- Alice and Bob measure one of their two directions at a time and compute the correlation:

$$\langle \mathcal{B} \rangle = (\langle AB \rangle + \langle A'B \rangle) + (\langle AB' \rangle - \langle A'B' \rangle)$$

Bell inequality



- Alice and Bob measure one of their two directions at a time and compute the correlation:

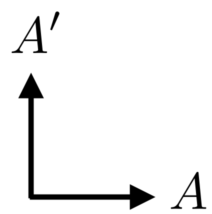
$$\langle \mathcal{B} \rangle = (\langle AB \rangle + \langle A'B \rangle) + (\langle AB' \rangle - \langle A'B' \rangle)$$

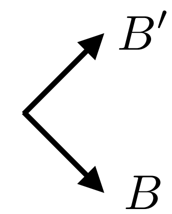
- If nature is local and real, $b = b'$ or $b = -b'$ independently of a and a'

$$\Rightarrow \quad \langle \mathcal{B} \rangle_{\rm LR} \leq 2$$

- If this bound is violated, local-real theories, e.g. Hidden Variable Theories, will be excluded.

Bell inequality





outcome :
$a, a' = +1 \text{ or } -1$

outcome :
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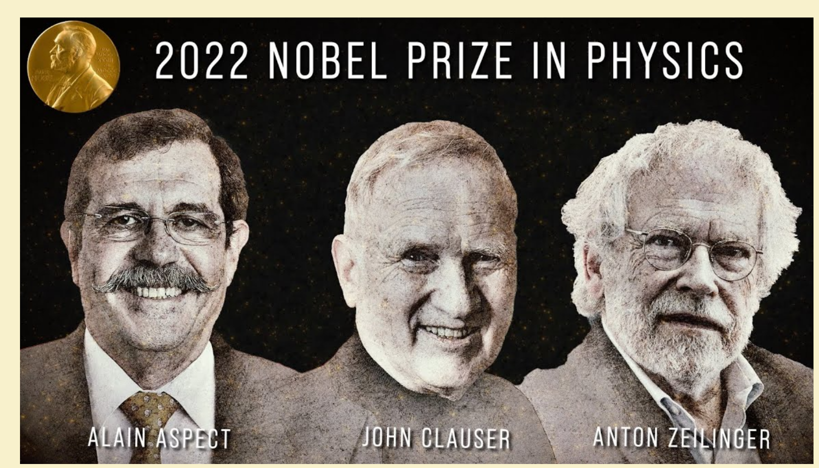
$$\Rightarrow \quad \langle \mathcal{B} \rangle_{\rm LR} \leq 2$$

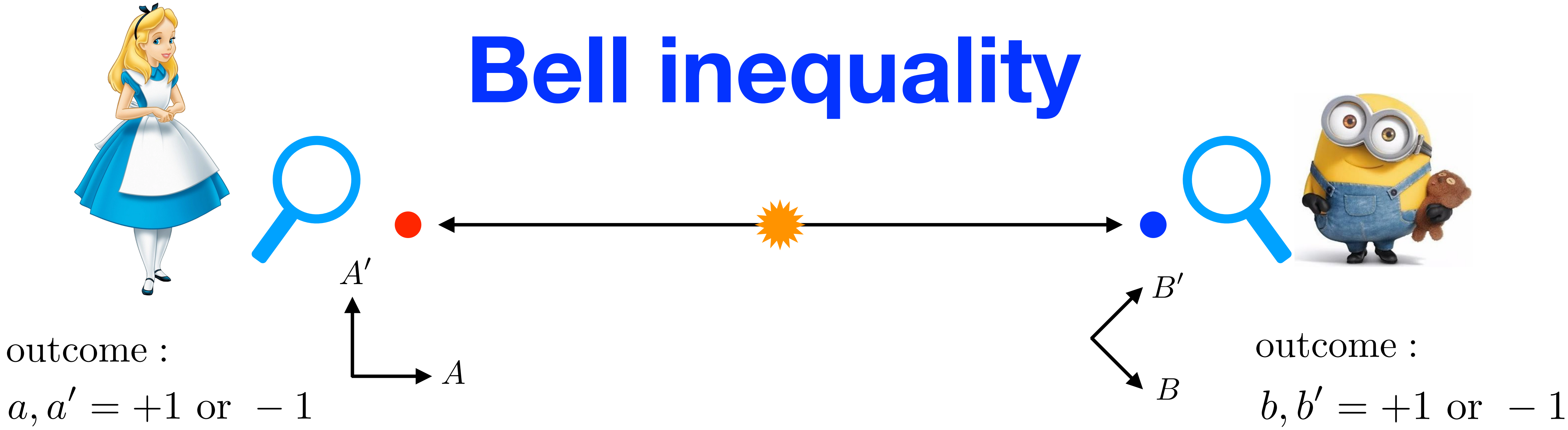
- If this bound is violated, local-real theories, e.g. Hidden Variable Theories, will be excluded.

$$\langle AB \rangle_{\rm HV} = \int p(\lambda) a(\lambda) b(\lambda) d\lambda$$

λ : hidden variable
$p(\lambda)$: probability of λ



Bell inequality



- Alice and Bob measure one of their two directions at a time and compute the correlation:

$$\langle \mathcal{B} \rangle = (\langle AB \rangle + \langle A'B \rangle) + (\langle AB' \rangle - \langle A'B' \rangle)$$

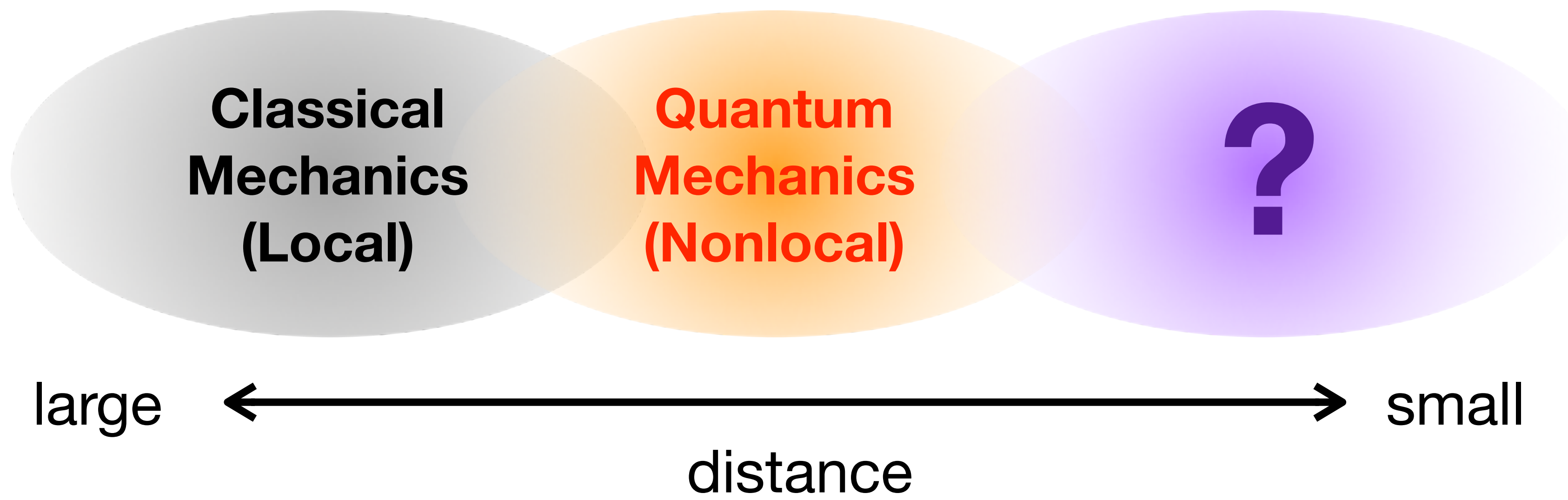
- In QM,

$$\hat{\mathcal{B}}^2 = 4 - [\hat{A}, \hat{A}'][\hat{B}, \hat{B}'] \qquad |[\hat{A}, \hat{A}']|, |[\hat{B}, \hat{B}']| \le 2 \quad \longleftarrow \quad [\sigma_x, \sigma_y] = 2i\sigma_z$$

$$\Rightarrow \; \langle \mathcal{B}^2 \rangle_{\rm QM} \le 8 \quad \Rightarrow \; \boxed{\langle \mathcal{B} \rangle_{\rm QM} \le 2\sqrt{2}}$$

[Tsirelson ‘87]

- If this bound is violated, QM will be excluded.



- Alice and Bob measure one of their two directions at a time and compute the correlation:

$$\langle \mathcal{B} \rangle = (\langle AB \rangle + \langle A'B \rangle) + (\langle AB' \rangle - \langle A'B' \rangle)$$

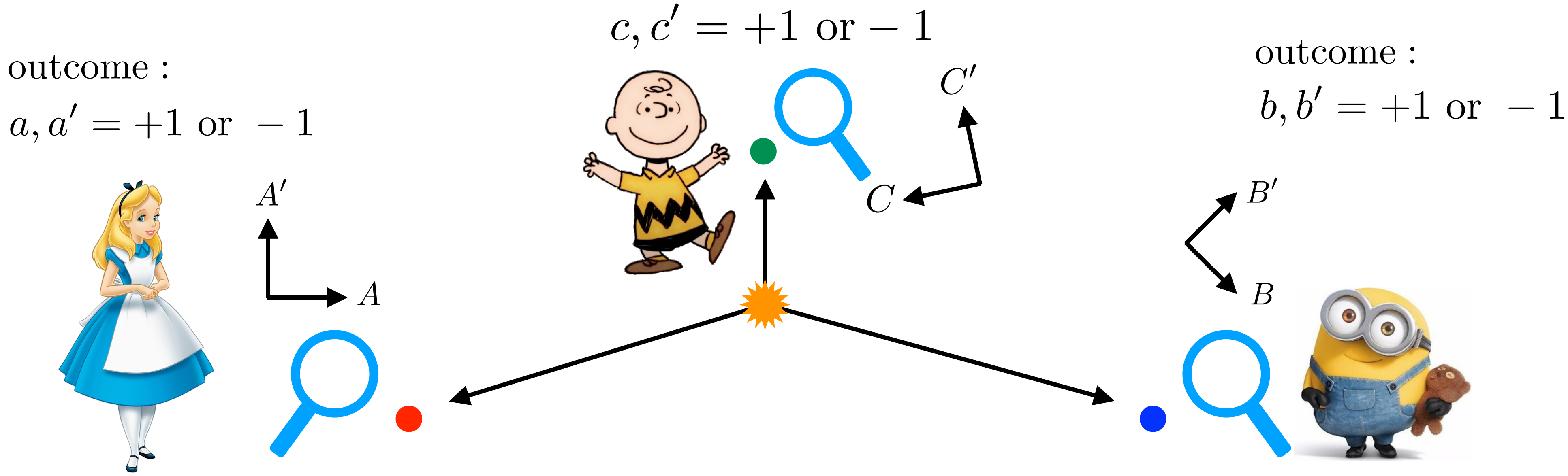
- In QM,

$$\hat{\mathcal{B}}^2 = 4 - [\hat{A}, \hat{A}'][\hat{B}, \hat{B}'] \qquad |[\hat{A}, \hat{A}']|, |[\hat{B}, \hat{B}']| \leq 2 \quad \longleftarrow \quad [\sigma_x, \sigma_y] = 2i\sigma_z$$

$$\Longrightarrow \quad \langle \mathcal{B}^2 \rangle_{\rm QM} \leq 8 \quad \Longrightarrow \quad \boxed{\langle \mathcal{B} \rangle_{\rm QM} \leq 2\sqrt{2}}$$

[Tsirelson '87]

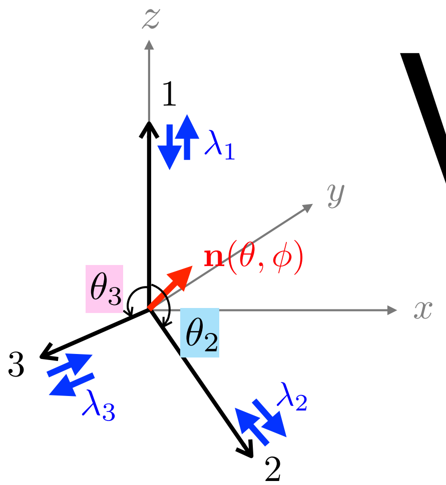
- If this bound is violated, QM will be excluded.
- **High-energy Bell inequality test is important to probe beyond QM at short distances.**



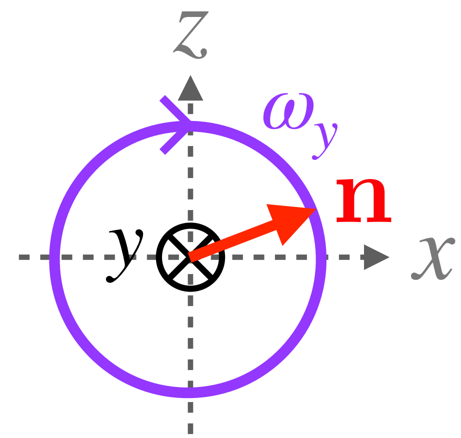
- Completely Hidden Variable: $\langle ABC \rangle_{\mathrm{CHV}} = \int p(\lambda)[a(\lambda)b(\lambda)c(\lambda)]d\lambda$
- Mermin correlation: $\mathcal{B}_{\mathrm{M}} = ABC' + AB'C + A'BC - A'B'C'$
- Mermin inequalities: $\langle \mathcal{B}_{\mathrm{M}} \rangle_{\mathrm{CHV}} \leq 2$ $\langle \mathcal{B}_{\mathrm{M}} \rangle_{\mathrm{QM}} \leq 4$ [Mermin '90]

- Partially Hidden Variable: $\langle ABC \rangle_{\mathrm{PHV}} = \int p(\lambda)[ab(\lambda)c(\lambda)]d\lambda\ , \int p(\lambda)[a(\lambda)bc(\lambda)]d\lambda, \cdots$
- Svetlichny correlation: $$\mathcal{B}_{\mathrm{S}} = ABC + ABC' + AB'C + A'BC$$ $$-A'B'C' - A'B'C - A'BC' - AB'C'$$
- Svetlichny inequalities: $\langle \mathcal{B}_{\mathrm{S}} \rangle_{\mathrm{PHV}} \leq 4$ $\langle \mathcal{B}_{\mathrm{S}} \rangle_{\mathrm{QM}} \leq 4\sqrt{2}$ [Svetlichny '87]

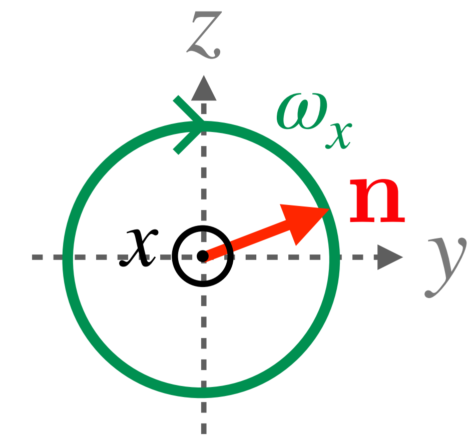
Vector



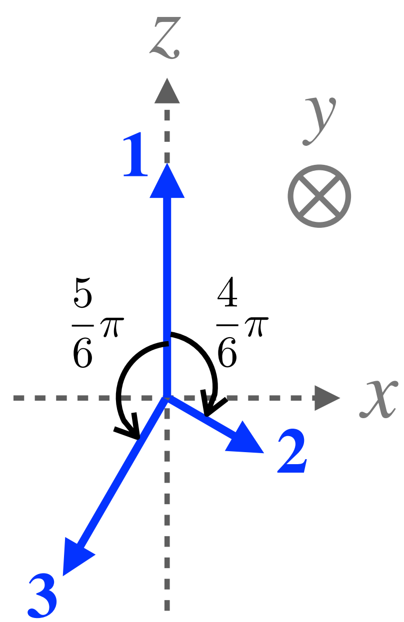
z
1
λ_1
y
$\mathbf{n}(\theta, \phi)$
θ_3
x
θ_2
3
λ_3
λ_2
2



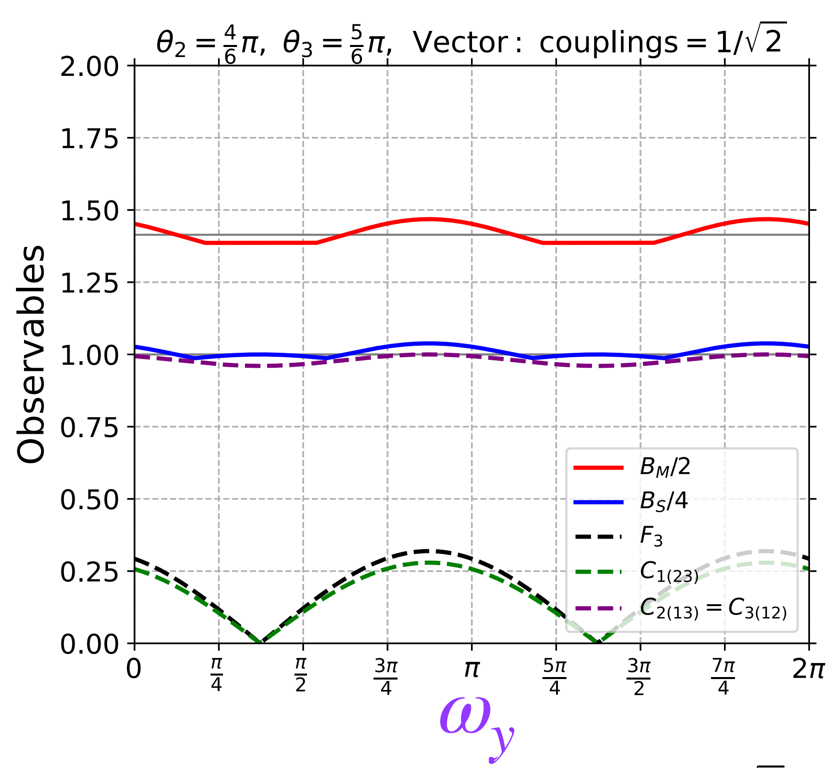
z
ω_y
n
y
x



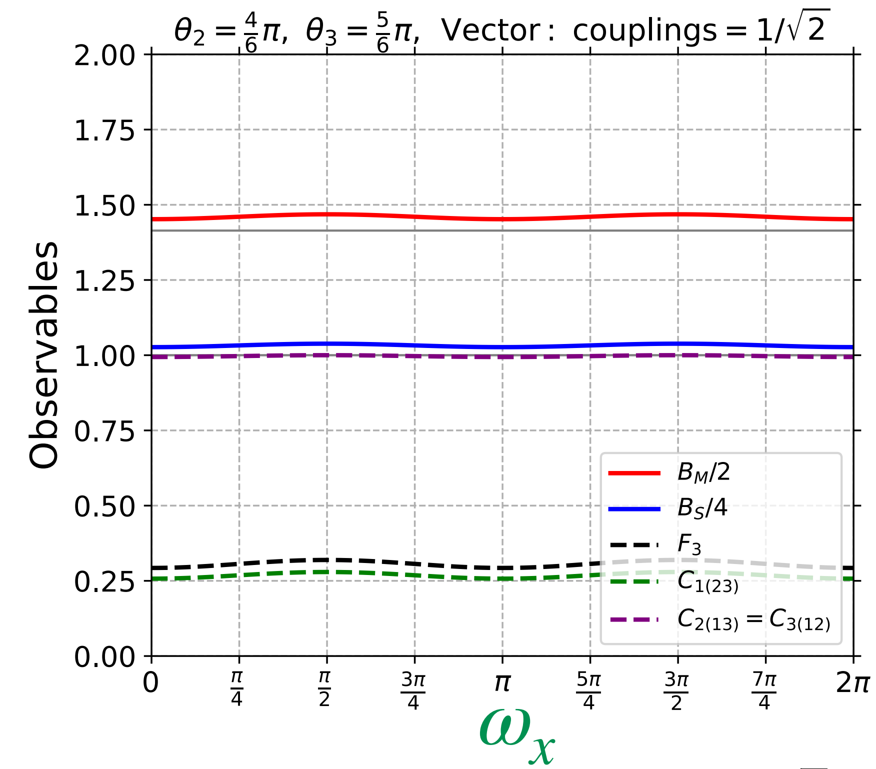
z
ω_x
n
x
y



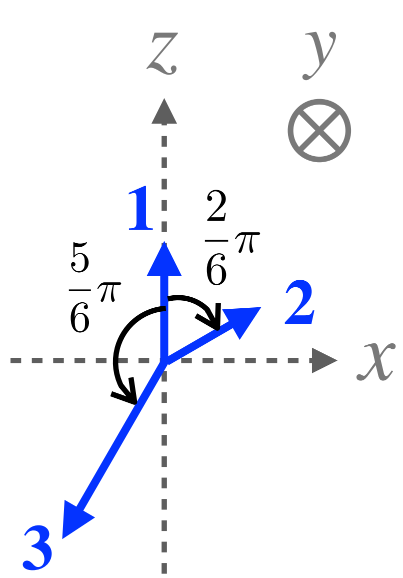
z
y
1
$\frac{5}{6}\pi$
$\frac{4}{6}\pi$
x
2
3



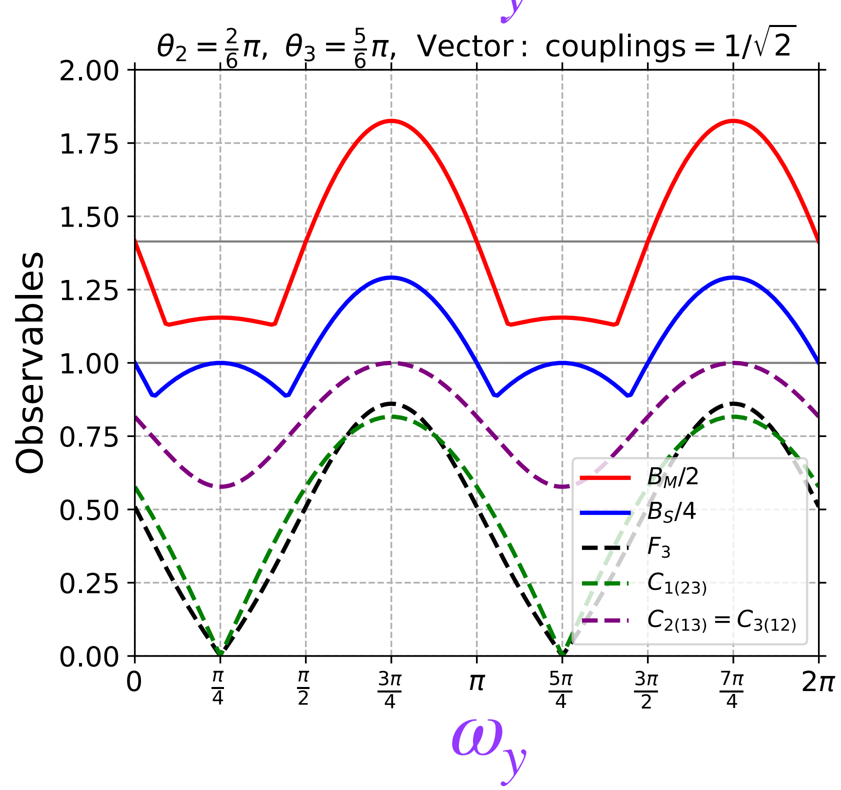
$\theta_2 = \frac{4}{6}\pi,\ \theta_3 = \frac{5}{6}\pi,$ Vector : couplings $= 1/\sqrt{2}$
Observables
ω_y
$B_M/2$
$B_S/4$
F_3
$C_{1(23)}$
$C_{2(13)} = C_{3(12)}$



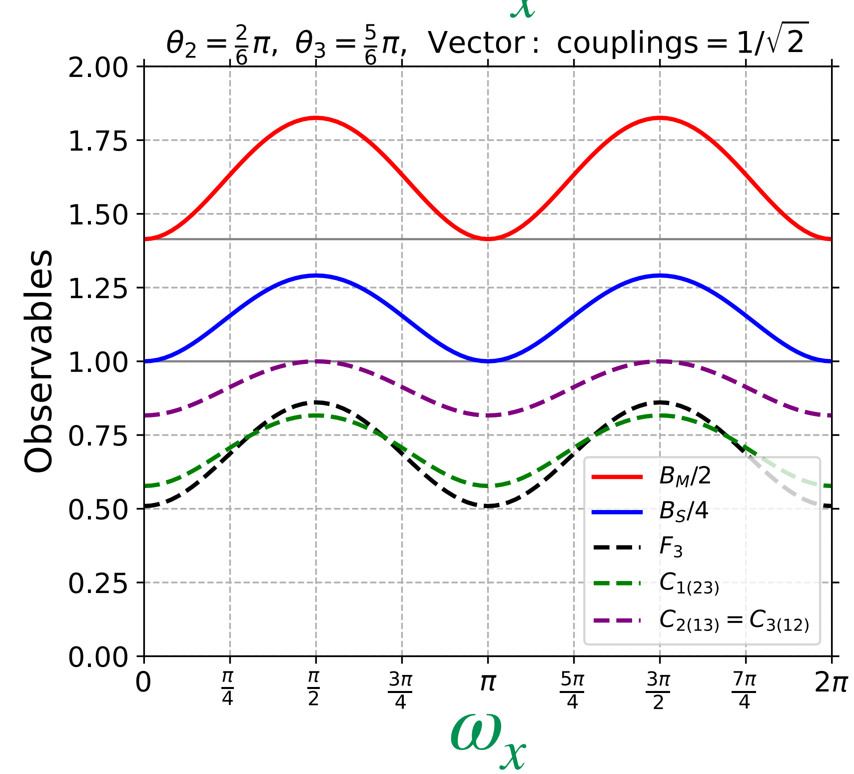
$\theta_2 = \frac{4}{6}\pi,\ \theta_3 = \frac{5}{6}\pi,$ Vector : couplings $= 1/\sqrt{2}$
Observables
ω_x
$B_M/2$
$B_S/4$
F_3
$C_{1(23)}$
$C_{2(13)} = C_{3(12)}$



z
y
1
$\frac{2}{6}\pi$
$\frac{5}{6}\pi$
2
x
3

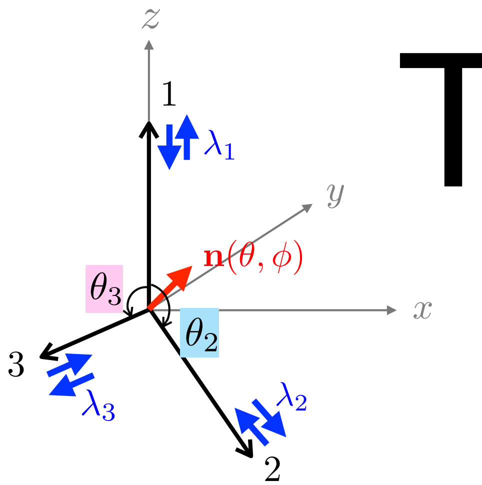


$\theta_2 = \frac{2}{6}\pi,\ \theta_3 = \frac{5}{6}\pi,$ Vector : couplings $= 1/\sqrt{2}$
Observables
ω_y
$B_M/2$
$B_S/4$
F_3
$C_{1(23)}$
$C_{2(13)} = C_{3(12)}$

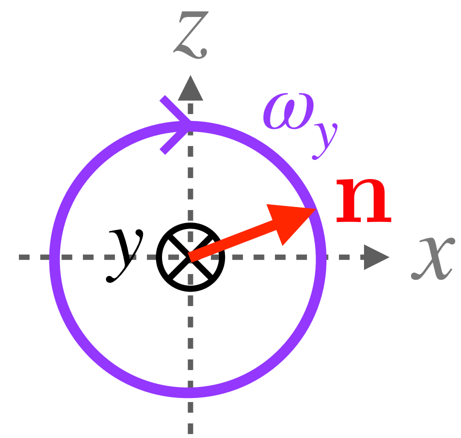


$\theta_2 = \frac{2}{6}\pi,\ \theta_3 = \frac{5}{6}\pi,$ Vector : couplings $= 1/\sqrt{2}$
Observables
ω_x
$B_M/2$
$B_S/4$
F_3
$C_{1(23)}$
$C_{2(13)} = C_{3(12)}$

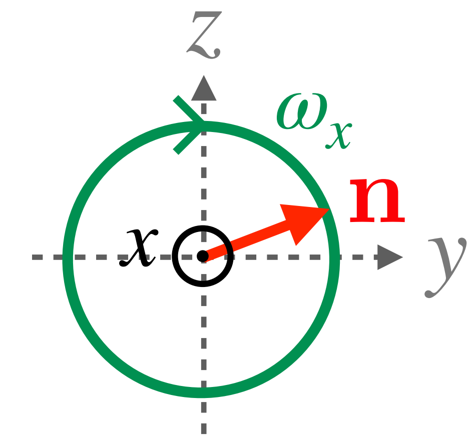
Tensor



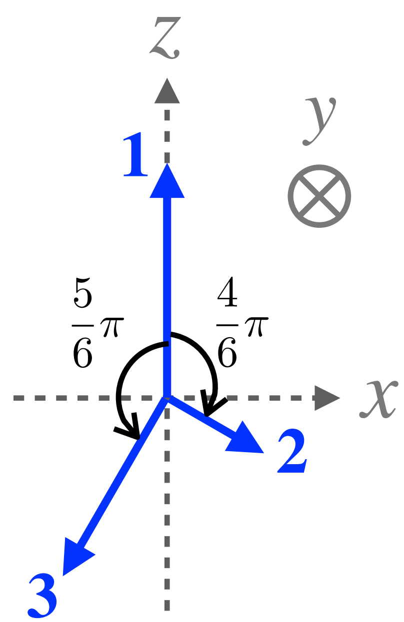
z
1
λ_1
y
$\mathbf{n}(\theta, \phi)$
θ_3
x
θ_2
3
λ_3
λ_2
2



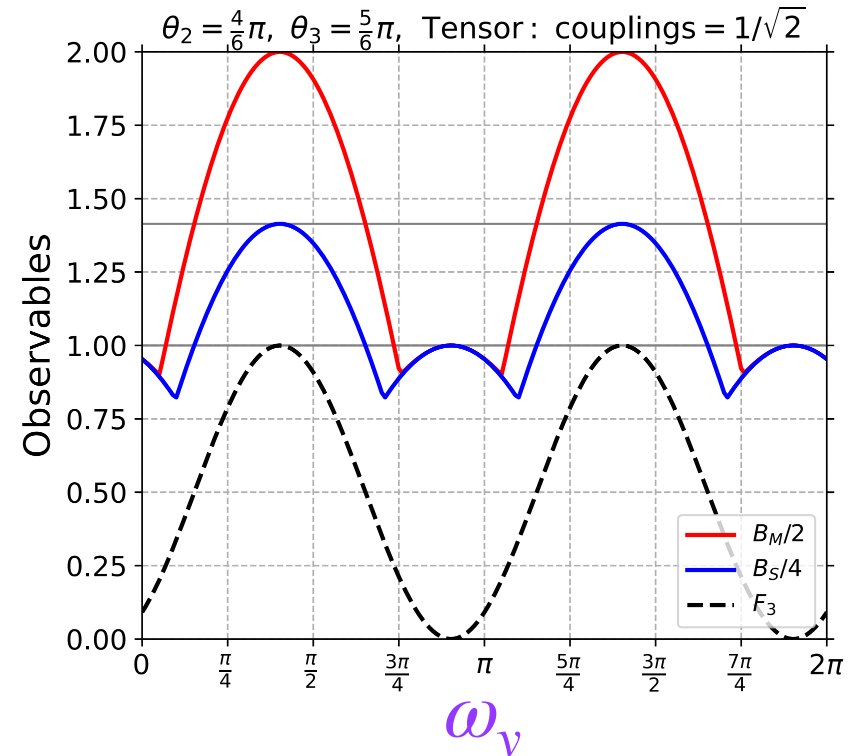
z
ω_y
n
y
x



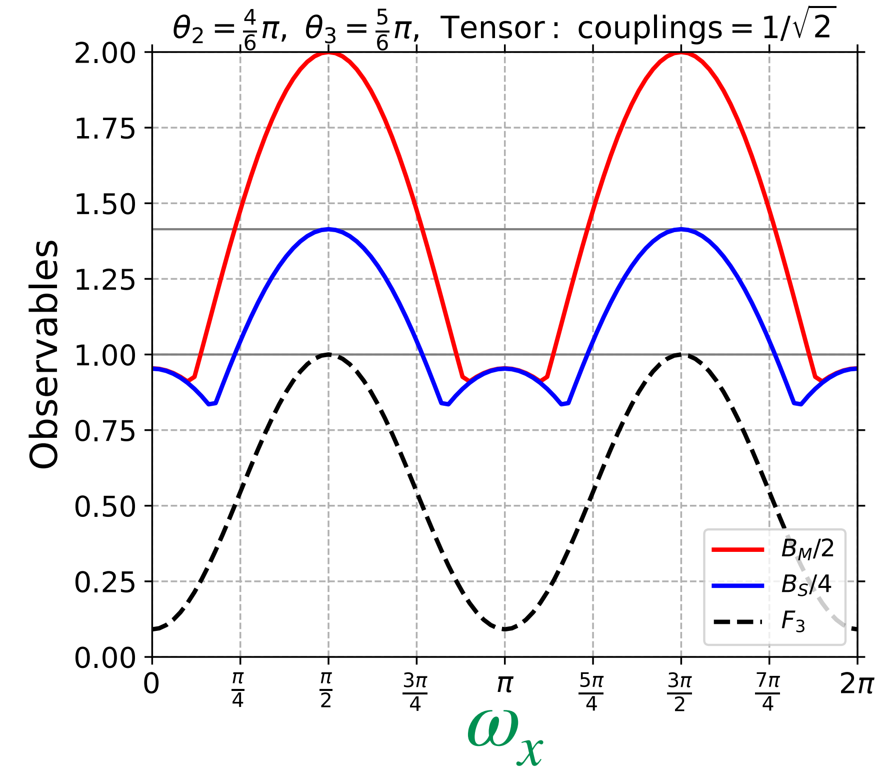
z
ω_x
n
x
y



z
y
1
$\frac{5}{6}\pi$
$\frac{4}{6}\pi$
x
2
3



$\theta_2 = \frac{4}{6}\pi,\ \theta_3 = \frac{5}{6}\pi,\ \mathrm{Tensor:\ couplings} = 1/\sqrt{2}$
Observables
$B_M/2$
$B_S/4$
F_3
ω_y



$\theta_2 = \frac{4}{6}\pi,\ \theta_3 = \frac{5}{6}\pi,\ \mathrm{Tensor:\ couplings} = 1/\sqrt{2}$
Observables
$B_M/2$
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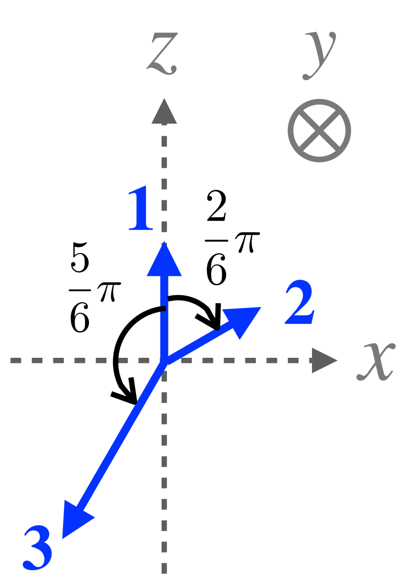
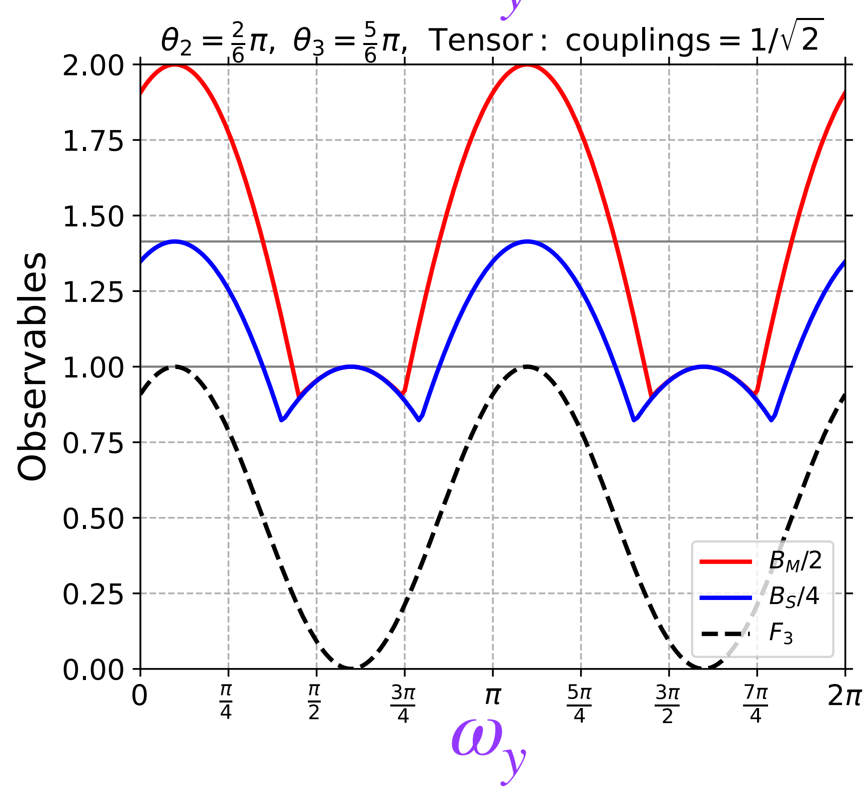
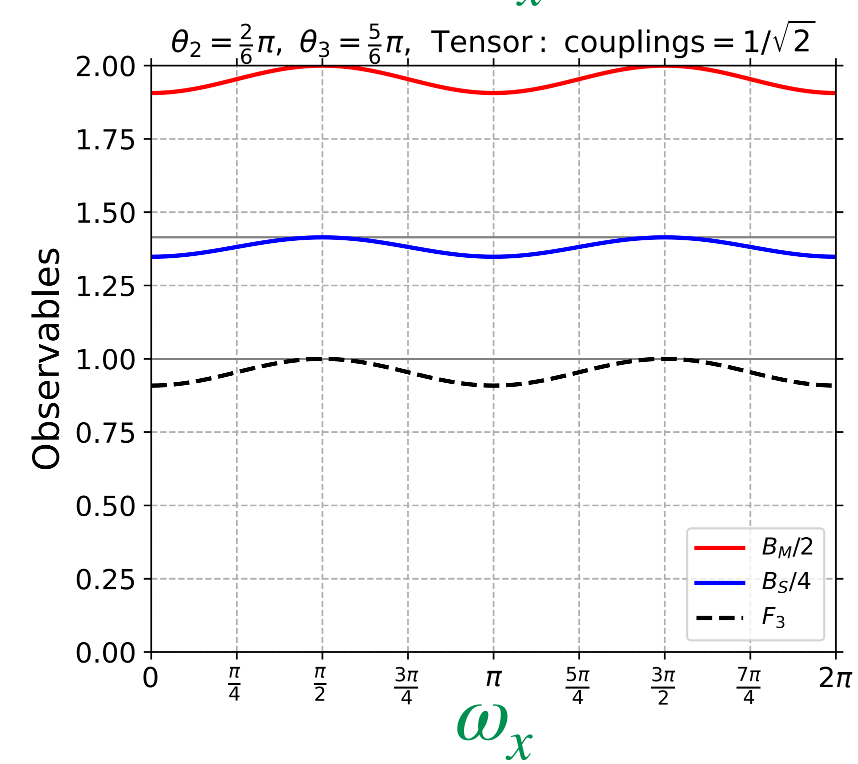
F_3
ω_x



z
y
1
$\frac{5}{6}\pi$
$\frac{2}{6}\pi$
2
x
3



$\theta_2 = \frac{2}{6}\pi,\ \theta_3 = \frac{5}{6}\pi,\ \mathrm{Tensor:\ couplings} = 1/\sqrt{2}$
Observables
$B_M/2$
$B_S/4$
F_3
ω_y



$\theta_2 = \frac{2}{6}\pi,\ \theta_3 = \frac{5}{6}\pi,\ \mathrm{Tensor:\ couplings} = 1/\sqrt{2}$
Observables
$B_M/2$
$B_S/4$
F_3
ω_x

Summary

- The exploration of QFT and Particle Physics using the Quantum Information Theory (QIT) framework has seen rapid progress recently.
- QIT concepts with entanglement and Bell non-locality are useful both in experimentally testing the SM and QM at high-energies and in re-thinking QFT in the QIT language.
- Entanglement and Non-Locality in the 3-qubit system from a 3-body decay are studied in this talk.

Future works and directions

- Effect of **masses** in the final particles
- More **spin structures**: $SFFV, VVFF, SFVF_{3/2}, SVVT \cdots$
- How entanglement is created and evolves in the particle physics interactions/measurements?
- Can we constrain new physics from the general properties of entanglement?

Thank you for listening!