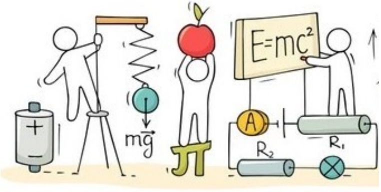
Quantum groups and quantum reference frames: the twisted Poincaré case



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Partially joint work with
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Based on:

[FL24] G.F., F. Lizzi, forthcoming paper + work in progress

[F10] G.F., JPA **43** (2010), 155401

Plan

Introduction

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Noncommutative (NC) spacetimes

NC Moyal spaces; QFT attempts on them

The Hopf algebra H, its NC deformation $\hat{H}$, and QRFs

The Hopf algebra H

The Hopf algebra $\hat{H}$, and QRFs

Regular representation of $\hat{H}$; coherent states for QRFs

Introduction

A RF $\mathfrak{R}$ is usually built using a macroscopic body with $N \gg 1$ atoms. Then "collective" observables like the position $\mathbf{X}$ of its CM and its total momentum $\mathbf{P}$ tipically are not significantly affected by the observation of a system S or another RF $\mathfrak{R}'$ with respect to (wrt) $\mathfrak{R}$. That's why RFs are usually idealized as classical.
But the ultimate quantum nature of these bodies will spoil their classical (i.e. idealized) properties, via UR, etc; particularly manifest if $N \sim O(1)$.

Can we formulate a consistent theory of QRFs?

The idea of QRFs was first proposed by Aharonov & Susskind 1967, Aharonov & Kaufherr 1984. Ever since many hundreds papers.

[arXiv:1712.07207] "Quantum mechanics and the covariance of physical laws in quantum reference frames", by F. Giacomini, E. Castro-Ruiz, Č. Brukner, Nat. Commun. **10**, 494 (2019), is particularly significant.

They all use "Relational Quantum Mechanics" (C. Rovelli, ...): $\nexists$ unique "absolute" state of a system S; rather, one state relative to each observer. Thus, a composite system can be in an entangled state wrt QRF $\mathfrak{R}$, a factorized state wrt QRF $\mathfrak{R}'$.

Use of spacetime observables relative to QRFs can heal QFT divergences:

[arXiv:2403.11973], "Quantum reference frames, measurement schemes and the type of local algebras in QFT" (by C. Fewster, D. Janssen, L. Loveridge, K. Rejzner, J. Waldron), proposes a operational framework for local measurements of QFs on a symmetric background wrt a QRF: under suitable assumptions the algebra of (relative) observables is a type II factor (instead of type III_1), i.e. has a semifinite, or even finite, (instead of an infinite) trace, which allows e.g. computing entropy.

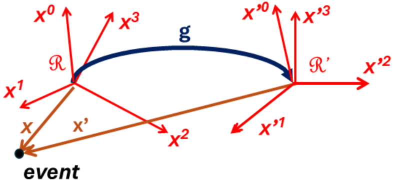
This paper builds on ideas+results of "An Algebra of Observables for de Sitter Space" by V. Chandrasekaran, R. Longo, G. Penington, E. Witten, JHEP02(2023)082, [arXiv:2206.10780].

The approach to investigate properties of a QRF can be:

1. bottom-up: start from quantum properties of its microscopic constituents, operationally measuring spacetime coords wrt it.
2. top-down: study which classical properties of RFs are compatible with their quantum nature, or must be generalized, and how

Here we adopt 2., focusing on group structure of changes of RFs.

Preliminaries, paradoxes for CRFs. Need generalized groups



Changes of classical reference frames (RF) $g:\mathfrak{R}\mapsto\mathfrak{R}'$ in space(time) make up a Lie group G:
the product gg' is the composition of g,g';
the unit is $\mathbf{1}:\mathscr{R}\mapsto\mathscr{R}$;
the inverse of g is $g^{-1}:\mathfrak{R}'\mapsto\mathfrak{R}$.

g sharply specifies how $\mathfrak{R}$ moves wrt $\mathfrak{R}'$.

Let x,x' resp. be the (sets of) spacetime coordinates of a generic event wrt $\mathfrak{R},\mathfrak{R}'$; g determines the 1-to-1 map $x\mapsto x'$. The latter induces a map (*passive transf.*) between the dynamical variables used by $\mathfrak{R},\mathfrak{R}'$ to describe a physical system S; e.g. for scalar fields the map

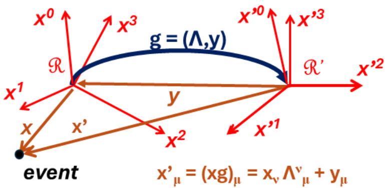
$$\gamma(g):\varphi\mapsto\varphi'$$

is determined by the eq. $\varphi(x)\stackrel{!}{=}\varphi'[x'(x)]$.

Enforcing these maps assumes $\mathfrak{R}'$ has: i) got information about the description of S by $\mathfrak{R}$; ii) sharply determined g, i.e. how $\mathfrak{R}$ moves wrt $\mathfrak{R}'$.

Cartesian coords wrt inertial RFs on $X \equiv$NR/ Minkowski spacetime: $x'_\mu = (xg)_\mu \equiv x_\nu \Lambda^\nu_\mu + y_\mu$, $g \equiv (\Lambda, y) \in G \equiv$ Galilei/Poincaré group,

$$\begin{aligned} &\gamma(g) : \varphi \mapsto \varphi' \equiv \varphi \triangleleft g, \\ &\varphi'(x') \equiv \varphi\left(x' g^{-1}\right). \end{aligned} \qquad (1)$$

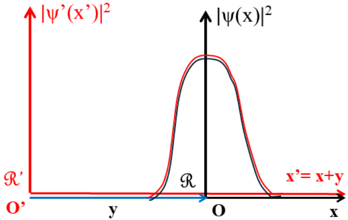


These maps apply also if S is quantum, e.g. a 0-spin elementary particle:

$$\hat{x} \mapsto \hat{x}' = \hat{x}\Lambda + y, \qquad \hat{p} \mapsto \hat{p}' = \hat{p}\Lambda, \qquad \rho \mapsto \rho', \qquad \psi \mapsto \psi'. \qquad (2)$$

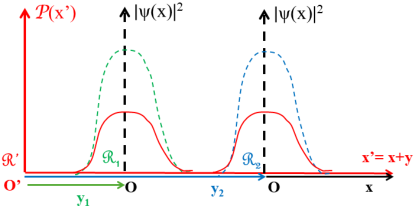
All pure (resp. mixed) states ρ ($\equiv$density operator) wrt $\mathfrak{R}$ are mapped into pure (resp. mixed) states ρ' wrt $\mathfrak{R}'$.

The wavefunctions $\psi(x) = {}_{\mathfrak{R}}\langle x|\Psi\rangle_{\mathfrak{R}}$, $\psi'(x') = {}_{\mathfrak{R}'}\langle x'|\Psi'\rangle_{\mathfrak{R}'}$ of S wrt resp. $\mathfrak{R}, \mathfrak{R}'$ fulfill $|\psi'(x')|^2 = |\psi(x)|^2$, and by Wigner Thm can be chosen so that $\psi'(x') = \psi(x)$.



However, if $\mathfrak{R}'$ has a coarse (i.e., probabilistic) knowledge about $\mathscr{R}$, then a pure state $\rho = |\Psi\rangle_{\mathfrak{R}\mathfrak{R}}\langle\Psi|$ is mapped into a mixed state ρ'.

For instance, if $\mathfrak{R}'$ knows exactly Λ, i.e. the (relative) orientation and velocity of $\mathfrak{R}$, and that the origins' displacement is y_1, y_2 with probabilities $1/2$, then



$$\rho' = \tfrac{1}{2}\,|\Psi'_{y_1}\rangle_{\mathfrak{R}'\mathfrak{R}'}\langle\Psi'_{y_1}| + \tfrac{1}{2}\,|\Psi'_{y_2}\rangle_{\mathfrak{R}'\mathfrak{R}'}\langle\Psi'_{y_2}$$

$$\mathscr{P}(x') = \tfrac{1}{2}\left|\psi'_{y_1}(x')\right|^2 + \tfrac{1}{2}\left|\psi'_{y_2}(x')\right|^2;$$

here $\psi'_y(x') = \psi(x'-y)$, $\mathscr{P}(x') = \mathrm{Tr}\big(|x'\rangle_{\mathfrak{R}'\mathfrak{R}'}\langle x'|\rho'\big)$ is the probability density to find the particle at position x' wrt $\mathfrak{R}'$.

More generally, if $\mathfrak{R}'$ knows that the origins' displacement is y with probability density $\widetilde{\rho}(y)$, then the state of S wrt $\mathfrak{R}'$ will be

$$\rho' = \int d^4y\, \widetilde{\rho}(y)\, |\Psi'_y\rangle_{\mathfrak{R}'\mathfrak{R}'}\langle\Psi'_y|; \tag{4}$$

this is pure iff $\widetilde{\rho}(y) = \delta_a(y) \equiv \delta(y-a)$, for some $a \in \mathbb{R}^4$.

Thus, **purity of states is a frame-dependent notion!**

To explain the paradox: $\tilde{\rho}$ is a classical state (probability distribution) of $\mathfrak{R}$ wrt $\mathfrak{R}'$; it is mixed, and so is the state of $S \cup \mathfrak{R}$ wrt $\mathfrak{R}'$, iff $\widetilde{\rho} \neq \delta_a$.

More formally, regard y_μ, Λ^ν_μ as coordinate functions on G; by def. they take the values a_μ, λ^ν_μ when evaluated at the point $g \equiv (\lambda, a) \in G$. Associate to each $g \in G$ the projector $\mathscr{P}_g$ (on a suitable $\mathscr{H}^{ext}_{\mathfrak{R}}$) such that

$$y_\mu \mathscr{P}_g = a_\mu \mathscr{P}_g = \mathscr{P}_g y_\mu, \qquad \Lambda^\nu_\mu \mathscr{P}_g = \lambda^\nu_\mu \mathscr{P}_g = \mathscr{P}_g \Lambda^\nu_\mu.$$

We can write the most general state of $\mathfrak{R}$ (w.r.t. $\mathfrak{R}'$) in the form

$$\rho_{\mathfrak{R}} = \int_G dg\, \tilde{\rho}(g) \mathscr{P}_g, \tag{5}$$

where dg is the (left and right) G-invariant (Haar) measure on G.

We postulate that the state ρ'_S of S w.r.t. $\mathfrak{R}'$ is obtained from $\rho_{\mathfrak{R}\cup S}$ via

$$\rho'_S = \mathrm{tr}_{\mathscr{H}_{\mathfrak{R}}} \left[\mathscr{U}(\Lambda, y) \rho_{\mathfrak{R}\cup S} \mathscr{U}^\dagger(\Lambda, y) \right] = \int da\, \tilde{\rho}(g)\, U(g) \rho_S U^\dagger(g), \tag{6}$$

where $\mathscr{U}(\Lambda, y) \equiv e^{i\left[y_\mu \otimes p^\mu + (\ln\Lambda)^\nu_\mu \eta^{\mu\rho} L_{\nu\rho}\right]}$, $U(g) \equiv e^{i\left[a_\mu p^\mu + (\ln\lambda)^\nu_\mu \eta^{\mu\rho} L_{\nu\rho}\right]}$.

In particular, if $\rho_{\mathfrak{R}\cup S} = \rho_{\mathfrak{R}} \otimes \mathscr{P}_\Psi$, $\mathscr{P}_\Psi =$ pure state of S w.r.t. $\mathfrak{R}$, then

$$\rho'_S = \int dg\, \tilde{\rho}(g)\, \mathscr{P}_{U(g)\Psi}. \tag{7}$$

ρ'_S is mixed unless $\rho_{\mathfrak{R}}$ is pure ($\tilde{\rho} = \delta_{\bar{g}}$ for some $\bar{g} \in G$; $\Rightarrow \rho'_S = \mathscr{P}_{U(\bar{g})\Psi}$).

The set of (classical) states of $\mathfrak{R}$ wrt $\mathfrak{R}'$ becomes a semigroup if we define the product by convolution. Sticking to translations,

$$(\widetilde{\rho}_1 * \widetilde{\rho}_2)(y) = \int d^4 b\, \widetilde{\rho}_1(b)\, \widetilde{\rho}_2(y-b); \tag{8}$$

δ_0 plays the role of unit element. A mixed state $\widetilde{\rho}$ (e.g. $\widetilde{\rho} = \frac{1}{2}\delta_{a_1} + \frac{1}{2}\delta_{a_2}$) has no inverse. Only pure states have: the inverse of δ_a is δ_{-a}.
Hence the group G can be identified with the set of pure states.

Instead of endowing the set of states with the structure of a (semi)group, one can encode the group structure of G in the *Hopf algebra structure* of *Fun*(G). This is more convenient, because it allows to replace *Fun*(G) by a noncommutative algebra, as we need for dealing with Quantum Reference Frames (QRFs; i.e. RF whose ultimate quantum nature cannot be ignored) and for describing symmetries of a NC spacetime.

Below I consider some NC deformations of $X =$ Minkowski space and $G =$ Poincaré group P, ideally relating inertial QRFs; $-\eta =$diag$(-1,1,1,1)$.

Why NC spacetime?

The idea of noncommutative (NC) spacetime is rather old [Heisenberg]. Possible motivations:

1. framework where to reconcile the principles of QM and GR;
2. inthrinsic regularization mechanism of UV divergences in QFT (Heisenberg's motivation);
3. due to the quantum nature of RFs **(new!)**;
4. effective description of string theory in some low energy regime (e.g. D3-brane with a large B-field).

1. In usual QFT no universal minimum for the localization Δx of events: $\Delta x \sim \hbar/\Delta p$ can be reduced at will by increasing the energy of the probe. On the other hand (argument due to [Mead,Wheeler,Bronstein]), by GR the energy concentration should not cause the formation of a black hole

$$\Rightarrow \qquad \Delta x \gtrsim l_p \quad \text{(Plack length).} \tag{9}$$

Doplicher, Fredenhagen & Roberts [DFR95] propose more sophisticated bounds, and noncommuting x_i that could naturally imply such bounds.

NC Moyal spaces; QFT attempts on them

Simplest NC spacetime: **constant commutators**

$$[\hat{x}_\mu, \hat{x}_\nu] = i\mathbf{1}\theta_{\mu\nu} \tag{10}$$

with $\theta_{\mu\nu} = -\theta_{\nu\mu}$ (Grönewold-Moyal-Weyl, briefly "Moyal"). Theoretical laboratory to investigate "noncommutative" QM, QFT. Note that (10) are translation invariant, not Lorentz-invariant.

Algebra $\widehat{\mathscr{X}}$ of functions on Moyal space: generated by $\mathbf{1}, \hat{x}_\mu$ fulfilling (10), with $\theta_{\mu\nu} \in \mathbb{R}$ (suitably extended). In [DFR95] $\theta_{\mu\nu} \in \mathscr{Z}(\widehat{\mathscr{X}})$ is dynamical.

Various inequivalent approaches to QFT on Moyal spaces. I would divide them according to: quantization approach, spacetime symmetries.

1. Path-integral quantization on Moyal-Euclidean spacetime: T. Filk , M. R. Douglas, A.S. Schwarz, N. A. Nekrasov, N. Seiberg, E. Witten, S. Minwalla, M. Van Raamsdonk, J. Gomis, T. Mehen, L. Alvarez-Gaume, M.A. Vazquez-Mozo, ..., H. Grosse, R. Wulkenhaar,...,R.Oeckl, R.J. Szabo, M. Dimitrijevic,...
2. Field=operator-valued, Moyal-Minkowski spacetime. Quantization: canonical; or á la Wightman; ... DFR, Bahns, Piacitelli, Chaichian, Balachandran et al, Aschieri, Lizzi,Vitale, Abe, Zahn, GF & Wess,...

Various problems, some interesting features.
E.g. in 1: causality violation, non-unitarity (for $\theta_{0i} \neq 0$), UV-IR mixing of divergences, non-renormalizability, claimed changes of statistics, etc.
Some problems may arise because naively deformed Euclidean Feynman rules are not justified by a Wick rotation.

Standard or deformed Poincaré covariance? ...?

Doplicher-Fredenhagen-Roberts, *et* Bahns, Piacitelli,...: since 1995:
First canonical quantization of the free fields. $\theta_{\mu\nu} \mapsto Q_{\mu\nu}$ central Lorentz tensor (obeying some conditions), becoming on each irrep a set of fixed constants $\theta_{\mu\nu}$ (joint spectrum of $Q_{\mu\nu}$). $\Rightarrow$ Poincaré-covariant.
But with interacting fields Lorentz covariance is sooner or later lost.
Doplicher's speculations: $Q_{\mu\nu}$ finally related to v.e.v. of $R_{\mu\nu}$, in turn influenced by matter quantum fields through quantum eq.s of motion.

Chaichian *et al* 2004, Wess 2004, Koch *et al* 2004, Oeckl 2000:
(10) are not Poincaré -invariant; *but "twisted Poincaré"* invariant.

Then attempts to construct twisted Poincaré covariant quantum fields started: Chaichian *et al*, Balachandran *et al*, Lizzi-Vitale, Abe, Zahn, F.-Wess, F. Our framework.

The Hopf algebra $\left(H \equiv Fun\ (\mathrm{P}), \varepsilon, \Delta, S\right)$

$$x_\mu \mapsto x'_\mu = (xg)_\mu = x_\nu \Lambda^\nu_\mu + y_\mu \equiv x_\nu \otimes \Lambda^\nu_\mu + \mathbf{1} \otimes y_\mu =: \Delta^r(x_\mu). \tag{11}$$

Regard: $\mathbf{1}, x_\mu$ as generators of $\mathscr{X} := Fun(X)$; $\mathbf{1}_H, \Lambda^\nu_\mu, y_\mu$ as generators of the algebra $H \equiv Fun\ (\mathrm{P})$ of functions on P. The transf. rule (11) is extended to all of $\mathscr{X}$ an algebra map (i.e. $\Delta^r(fg) = \Delta^r(f)\Delta^r(g)$, etc.), the *coaction* $\Delta^r : \mathscr{X} \to \mathscr{X} \otimes H$, $f(x) \mapsto f(x') =: \left[\Delta^r(f)\right](x)$.
The group structure of P is encoded in the *counit* $\varepsilon : H \to \mathbb{C}$, *coproduct* $\Delta : H \to H \otimes H$, *antipode* $S : H \to H$, defined on the generators by

$$\begin{aligned} &\varepsilon(\Lambda^\nu_\mu) = \delta^\nu_\mu, \quad \Delta(\Lambda^\nu_\mu) = \Lambda^\nu_\rho \otimes \Lambda^\rho_\mu, \quad S(\Lambda^\nu_\mu) = (\eta \Lambda^T \eta)^\nu_\mu \equiv {\Lambda^{-1}}^\nu_\mu, \\ &\varepsilon(y_\mu) = 0, \quad \Delta(y_\mu) = y_\nu \otimes \Lambda^\nu_\mu + \mathbf{1}_H \otimes y_\mu, \quad S(y_\mu) = -y_\nu {\Lambda^{-1}}^\nu_\mu, \end{aligned} \tag{12}$$

which resp. give the identical, (twice) repeated, inverse change of frame. ε, Δ, S are extended as (anti-)algebra maps; fulfill many properties, e.g.

$$(\mathrm{id} \otimes \varepsilon) \circ \Delta^r = \mathrm{id}\,, \quad (\Delta \otimes \mathrm{id}\,) \circ \Delta^r = (\mathrm{id} \otimes \Delta^r) \circ \Delta^r. \tag{13}$$

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Transf. (11) preserves $[x_\mu, x_\nu] = 0$. Does it preserve $[\hat{x}_\mu, \hat{x}_\nu] = i\mathbf{1}\theta_{\mu\nu}$?
Yes, if we "quantize" H, i.e. make it a NC Hopf algebra $\hat{H}$, such that $[\hat{x}'_\mu, \hat{x}'_\nu] = i\mathbf{1}\theta_{\mu\nu}$ holds as well $\Rightarrow$ **all inertial QRFs are equivalent!**

The Hopf algebra $\left(\hat{H} \equiv Fun_\theta(\mathrm{P}), \varepsilon, \Delta, S\right)$

$$\hat{x}_\mu \mapsto \hat{x}'_\mu = \hat{x}_\nu \Lambda^\nu_\mu + \hat{y}_\mu \equiv \hat{x}_\nu \otimes \Lambda^\nu_\mu + \mathbf{1} \otimes \hat{y}_\mu =: \Delta^r(\hat{x}_\mu). \tag{11}$$

Regard: $\mathbf{1}, \hat{x}_\mu$ as generators of $\widehat{\mathscr{X}}$; $\mathbf{1}_H, \Lambda^\nu_\mu, \hat{y}_\mu$ as generators of the algebra $\widehat{H} = Fun_\theta(\mathrm{P})$. The transf. rule (11) is extended to all of $\widehat{\mathscr{X}}$ an algebra map (i.e. $\Delta^r(fg) = \Delta^r(f)\Delta^r(g)$, etc.), the *coaction* $\Delta^r : \widehat{\mathscr{X}} \to \widehat{\mathscr{X}} \otimes \widehat{H}$, $\quad f(\hat{x}) \mapsto f(\hat{x}')$. The *counit* $\varepsilon : \widehat{H} \to \mathbb{C}$, *coproduct* $\Delta : \widehat{H} \to \widehat{H} \otimes \widehat{H}$, *antipode* $S : \widehat{H} \to \widehat{H}$, defined on the generators by

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Transf. (11) preserves $\quad [\hat{x}_\mu, \hat{x}_\nu] = i\mathbf{1}\theta_{\mu\nu} \quad$ if [Oeckl 2000]:

$$[\Lambda^\rho_\mu, \cdot] = 0, \quad \Lambda^\mu_\rho \Lambda^\nu_\sigma \eta^{\rho\sigma} = \eta^{\mu\nu} \mathbf{1}_H, \quad [\hat{y}_\mu, \hat{y}_\nu] = i(\theta_{\mu\nu} \mathbf{1}_H - \theta_{\rho\sigma} \Lambda^\rho_\mu \Lambda^\sigma_\nu). \tag{14}$$

Regular representation of $\hat{H}$; coherent states for QRFs

Abbreviating $\chi := \mathbf{1}_H\theta - \Lambda^T\theta\Lambda$, $\hat{H}$ is generated by $\Lambda^\nu_\mu, \hat{y}_\mu$ fulfilling

$$[\Lambda^\nu_\mu, \cdot\,] = 0, \qquad \Lambda\eta\Lambda^T = \mathbf{1}_H\eta, \qquad [\hat{y}_\mu, \hat{y}_\nu] = i\,\chi_{\mu\nu}. \tag{15}$$

It can be faithfully represented on the space V of functions $f(y,\lambda)$ of real commuting variables y_μ, λ^ν_μ fulfilling $\lambda\eta\lambda^T = \eta$, e.g. by

$$\Lambda^\nu_\mu f(y,\lambda) = \lambda^\nu_\mu f(y,\lambda), \qquad \hat{y}_\mu f(y,\lambda) = \left(y_\mu + \frac{i}{2}\chi_{\mu\rho}\frac{\partial}{\partial y_\rho}\right) f(y,\lambda); \tag{16}$$

reducible representation of $\hat{H}$ with the correct commutative limit $\theta \to 0$.
In fact, rhs(16b) is the star-product $y_\mu \star f(y,\Lambda)$ induced by the twist $\mathscr{F} = \exp[\frac{i}{2}\theta_{\mu\rho}p^\mu \otimes p^\rho]$, leading to $\hat{y}_\mu \star \hat{y}_\nu - \hat{y}_\nu \star \hat{y}_\mu = i\chi_{\mu\nu}$.
By a suitable orthogonal transf. $\hat{y}_\mu \mapsto \hat{\mathrm{y}}_\mu = \hat{y}_\rho Q^\rho_\mu(\lambda)$ (15c) become

$$\left[\hat{\mathrm{y}}_0, \hat{\mathrm{y}}_1\right] = -\left[\hat{\mathrm{y}}_1, \hat{\mathrm{y}}_0\right] = i\beta, \qquad \left[\hat{\mathrm{y}}_2, \hat{\mathrm{y}}_3\right] = -\left[\hat{\mathrm{y}}_3, \hat{\mathrm{y}}_2\right] = i\gamma, \tag{17}$$

where β, γ are λ-dependent linear combinations of the $\theta_{\mu\nu}$; (16) becomes

$$\hat{\mathrm{y}}_0 = \mathrm{y}_0 + \frac{i\beta}{2}\frac{\partial}{\partial \mathrm{y}_1}, \quad \hat{\mathrm{y}}_1 = \mathrm{y}_1 - \frac{i\beta}{2}\frac{\partial}{\partial \mathrm{y}_0}, \quad \hat{\mathrm{y}}_2 = \mathrm{y}_2 + \frac{i\gamma}{2}\frac{\partial}{\partial \mathrm{y}_3}, \quad \hat{\mathrm{y}}_3 = \mathrm{y}_3 - \frac{i\gamma}{2}\frac{\partial}{\partial \mathrm{y}_2}.$$

$$b_1 = \frac{\hat{y}_0 + i\hat{y}_1}{\sqrt{2\beta}}, \quad b_1^\dagger = \frac{\hat{y}_0 - i\hat{y}_1}{\sqrt{2\beta}}, \quad b_2 = \frac{\hat{y}_2 + i\hat{y}_3}{\sqrt{2\gamma}}, \quad b_2^\dagger = \frac{\hat{y}_2 - i\hat{y}_3}{\sqrt{2\gamma}} \qquad (18)$$

are ladder operators fulfilling the CCR

$$[b_1, b_2] = [b_1^\dagger, b_2^\dagger] = 0, \qquad [b_a, b_b^\dagger] = \delta_{ab}. \qquad (19)$$

One can easily show that for all $\alpha \equiv (\alpha_0, \alpha_1, \alpha_2, \alpha_3) \in \mathbb{R}^4$ the

$$\psi_\alpha(y, \lambda) = \sqrt{\frac{4}{\pi_2 \beta \gamma}} \exp\left[-\frac{(y_0 - \alpha_0)^2 + (y_1 - \alpha_1)^2}{\beta(\lambda)} - \frac{(y_2 - \alpha_2)^2 + (y_3 - \alpha_3)^2}{\gamma(\lambda)} \right] \qquad (20)$$

are normalized coherent states, eigenvectors of b_1, b_2 with eigenvalues $z_1 \equiv (\alpha_0 + i\alpha_1)/\sqrt{\beta}$, $z_2 \equiv (\alpha_2 + i\alpha_3)/\sqrt{\gamma}$ respectively; $\boldsymbol{z} \equiv (z_1, z_2) \in \mathbb{C}_2$. They make up a family of gaussian states centered around (and parametrized by) the 4 real parameters $\alpha \in \mathbb{R}^4$, which label all possible classical translations.

The corresponding expectation values, uncertainties and saturated Heisenberg uncertainty relations read

$$\langle \hat{y}_\mu \rangle = \alpha_\mu, \qquad \Delta\hat{y}_2 = \Delta\hat{y}_3 = \sqrt{\frac{\gamma}{2}}, \qquad \Delta\hat{y}_0 = \Delta\hat{y}_1 = \sqrt{\frac{\beta}{2}}, \tag{21}$$

$$\Delta\hat{y}_0 \Delta\hat{y}_1 = \frac{\beta}{2}, \qquad \Delta\hat{y}_2 \Delta\hat{y}_3 = \frac{\gamma}{2}. \tag{22}$$

As $\lambda \to I$ (or as $\theta \to 0$) we have $\beta, \gamma \to 0$, and

$$\psi_\alpha(y, \lambda) \to \delta^{(4)}(y - \alpha). \tag{23}$$

In other words, in this limit the change of reference frame becomes "classical" (i.e. commutative). This is welcome.
A consequence is the relation, invariant under orthogonal transformations,

$$\sum_\mu \left(\Delta\hat{y}_\mu\right)^2 = \sqrt{2\mathrm{Pf}(\chi) - \tfrac{1}{2}\mathrm{tr}(\chi^2)} \tag{24}$$

$$= \sqrt{2\boldsymbol{e}\cdot(\boldsymbol{b}'-\boldsymbol{b}) + 2(\boldsymbol{e}'-\boldsymbol{e})\cdot\boldsymbol{b} + (\boldsymbol{e}-\boldsymbol{e}')^2 + (\boldsymbol{b}-\boldsymbol{b}')^2}. \tag{25}$$

where $\boldsymbol{e}', \boldsymbol{b}'$ are the 3-vectors of components $e'^i \equiv \theta'_{0i}$, $b'^i \equiv \frac{1}{2}\varepsilon^{ijk}\theta'_{jk}$.

Rhs(25) depends on the specific Lorentz transformation λ. Sticking to $\lambda =$rotations, one can easily show the rotation-independent bound

$$\sum_{\mu} \left(\Delta \hat{y}_{\mu}\right)^2 \leq 2(e+b) \tag{26}$$

(here e, b are the norms of $\boldsymbol{e}, \boldsymbol{b}$). More generally, one can show the bound

$$\sum_{\mu} \left(\Delta \hat{y}_{\mu}\right)^2 \leq 4\lambda_0^0 \sqrt{2(e^2+b^2)}. \tag{27}$$


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