Projective Superspace in aid of 5d Chern-Simons

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Why Augmented Superspaces? Projective or otherwise

- Generaly, $\theta, \bar{\theta}$ expansion of superfields have more than needed components. Constraints to make them irreducible multiplets, often leads to the free e.o.m's of the component fields. e.g., $\mathcal{N} = 2$ Fayet-Sohnius matter hypermultiplet: $q^i(x, \theta, \bar{\theta})$.
Impose constraint: $D_\alpha^{(i} q^{j)} = \bar{D}_{\dot{\alpha}}^{(i} q^{j)} = 0$, $\{D_\alpha^i, \bar{D}_{\dot{\alpha} j}\} = i\delta_j^i \sigma^a \partial_a$

$$\textit{leads to} \quad \Box f^i(x) = \sigma^a \partial_a \psi_\alpha(x) = \sigma^a \partial_a \bar{\kappa}^{\dot{\alpha}} = 0 \ ,$$

$$f^i(x) = q^i\Big|_{\theta=\bar{\theta}=0} \ , \ \psi_\alpha(x) = D_\alpha^i q_i\Big|_{\theta=\bar{\theta}=0} \ , \ \bar{\kappa}^{\dot{\alpha}}(x) = \bar{D}_i^{\dot{\alpha}} q^i\Big|_{\theta=\bar{\theta}=0} \qquad (1)$$

- In some $\mathcal{N} \geq 2$ theories it is necessary to introduce infinite number of auxiliaries.

Projective Superspace

The theory possess an SU(2) R-symmetry or a product of it.

Extend the susy group by the coset of its automorphism group. $\mathbb{C}P^1 = \frac{SU(2)}{U(1)}$. Superfields depend holomorphically on $\mathbb{C}P^1$.

Parameterize the spinor covariant derivatives and superfields with the $\mathbb{C}P^1$ coordinate ζ, so that the new projective covariant derivatives would annihilate the projective superfields $\mathcal{P}(x, \theta, \bar{\theta}, \zeta)$:

$$\begin{aligned} \nabla_\alpha \mathcal{P} &\equiv (D_{1\alpha} + \zeta D_{2\alpha})\mathcal{P} = 0 \\ \bar{\nabla}_{\dot{\alpha}} \mathcal{P} &\equiv (\bar{D}^2_{\dot{\alpha}} - \zeta \bar{D}^1_{\dot{\alpha}})\mathcal{P} = 0 \end{aligned} \tag{2}$$

Linearly independent operators to the above serve as the spinor parts of the measure for the construction of actions.

$$\Delta_\alpha \equiv D_{2\alpha} - \frac{1}{\zeta} D_{1\alpha} \,, \;\; \bar{\Delta}_{\dot\alpha} \equiv \bar{D}^1_{\dot\alpha} + \frac{1}{\zeta} \bar{D}^2_{\dot\alpha} \tag{3}$$

One could also parameterize the Grassmann coordinates of $\mathcal{N} = 2$:

$$\Theta^\alpha \equiv \theta^{2\alpha} - \zeta\theta^{1\alpha} \,, \;\; \bar{\Theta}^{\dot\alpha} \equiv \bar{\theta}^{\dot\alpha}_1 + \zeta\bar{\theta}^{\dot\alpha}_2 \tag{4}$$

Then, it is clear that the $\Delta_\alpha \sim \frac{\partial}{\partial\Theta^\alpha}$ and $\bar{\Delta}_{\dot\alpha} \sim \frac{\partial}{\partial\Theta^{\dot\alpha}}$.
A lagrangian is independent of the "ortogonal subspace", e.g., $\nabla\mathcal{L}(\mathcal{P}(\zeta)) = \bar{\nabla}\mathcal{L}(\mathcal{P}(\zeta)) = 0 \longrightarrow$ construct the action with a half measure :

$$S \sim \int d^4x \oint \zeta d\zeta \Delta^2 \bar{\Delta}^2 \mathcal{L}(\eta(\zeta)) \tag{5}$$

Expressing $\Delta, \bar{\Delta}$ in terms of their orthogonal operators $\nabla, \bar{\nabla}$:

$$\Delta = \zeta^{-1}(2D_1 - \nabla)\ ,\ \bar{\Delta} = 2\bar{D}^1 + \zeta^{-1}\bar{\nabla} \tag{6}$$

The action with the projective Lagrangian simplifies to:

$$S = \frac{1}{2\pi i}\int d^4x \oint \frac{d\zeta}{\zeta}(D_1)^2(\bar{D}^1)^2\mathcal{L}(\mathcal{P}(\zeta)) \tag{7}$$

The new derivatives mostly anticommute:

$$\begin{aligned} &\{\nabla,\nabla\} = \{\nabla,\bar{\nabla}\} = \{\Delta,\bar{\Delta}\} = \{\bar{\Delta},\bar{\Delta}\} = \{\nabla,\Delta\} = 0\ , \\ &\{\nabla,\bar{\Delta}\} = -\{\bar{\nabla}_\alpha,\Delta_{\dot{\beta}}\} = 2i\partial_{\alpha\dot{\beta}} \end{aligned} \tag{8}$$

- Conjugation : $\overline{f(\zeta)} \equiv f^*(-\frac{1}{\zeta})$

Classes of projective superfields, annihilated by $\nabla, \bar{\nabla}$ are defined:

- $\mathcal{O}(k)$ multiplets: $\Upsilon = \sum_{n=0}^{k} \Upsilon_n \zeta^n$, $\bar{\Upsilon} = \sum_{n=0}^{k} \bar{\Upsilon}_{-n}(-\frac{1}{\zeta})^n$
- polar multiplets: when $k \longrightarrow \infty$
- tropical multiplet with the reality condition: $\overline{V_+(\zeta)} = V_+(\zeta)$

$$V(\zeta) = \sum_{n=-\infty}^{\infty} v_n \zeta^n = V_-(\zeta^{-1}) + v_0 + V_+(\zeta) \; , \;\; v_{-n} = (-1)^n \bar{v}_n \qquad (9)$$

After employing the projective constraints on these multiplets, their components become related as ($D_{1\alpha} \equiv D_\alpha$; $D_{2\alpha} \equiv Q_\alpha$):

$$D_\alpha \Upsilon_{n+1} = -Q_\alpha \Upsilon_n \; , \; \bar{D}_{\dot{\alpha}} \Upsilon_n = \bar{Q}_{\dot{\alpha}} \Upsilon_{n+1} \qquad (10)$$

That makes the two lowest components in Υ, the two highest in $\bar{\Upsilon}$ constrained in $\mathcal{N} = 1$, and the rest unconstrained auxiliaries.

Yang Mills Theory

Gauge invariance

The idea of unconstrained prepotential can arrive from the gauge coupling to the matter multiplets. Free hypermultiplet action in $\mathcal{N} = 2$ is:

$$S_{freematter} = \int d^4x D^2 \bar{D}^2 \oint \frac{d\zeta}{2\pi i \zeta} \bar{\Upsilon}\Upsilon \quad (11)$$

Under the internal symmetry, hypermultiplets transform locally:

$$\Upsilon' \longrightarrow e^{i\Lambda(x,\theta^i,\bar{\theta}_i,\zeta)}\Upsilon \, , \; \bar{\Upsilon}' \longrightarrow \bar{\Upsilon} e^{-i\bar{\Lambda}(x,\theta^i,\bar{\theta}_i,\zeta)} \quad (12)$$

The action is not invariant under this transformation; need to introduce gauge field with the transformation property:

$$e^{V'} \longrightarrow e^{i\bar{\Lambda}} e^{V} e^{-i\Lambda} \quad (13)$$

This makes the action $S_{int} = \mathrm{Tr} \int d^4x D^2 \bar{D}^2 \oint \frac{d\zeta}{2\pi i \zeta} \bar{\Upsilon} e^V \Upsilon$ invariant.

The infinitesimal Abelian version of the gauge field transformation is:

$$\delta V = i(\bar{\Lambda} - \Lambda) \implies \delta v_0 = i(\bar{\lambda}_0 - \lambda_0)\ ,\ \delta v_i = -i\lambda_i\ ,\ \delta v_{-i} = i\bar{\lambda}_i \quad (14)$$

λ_0 is antichiral, λ_1 antilinear, higher orders are unconstrained $\implies$ put V in a gauge: $v_n = 0, n \neq -1, 0, 1$ and also gauge away all v_1 except $D^2 v_1$, all v_{-1} except $\bar{D}^2 v_{-1}$.
This allows to identify the $\mathcal{N} = 1$ physical fields.

- $-iv_{-1}| = \bar{\psi}$ prepotential for the chiral scalar f.s $\phi = \bar{D}^2\bar{\psi}$
- $iv_1| = \psi$ prepotential for the antichiral $\bar{\phi} = D^2\psi$
- $v_0| = v$ prepotential for the chiral spinor f.s $W_\alpha = i\bar{D}^2 D_\alpha v_0$

Gauge covariantization

Now, we intend to find the projective gauge connection and the off-shell $\mathcal{N} = 2$ Yang-Mills action in terms of the unconstrained tropical superfield $V(x, \theta, \bar{\theta}, \zeta)$.

$$\mathbb{D}_\alpha = D_\alpha + \Gamma^1_\alpha \, , \; \mathbb{Q}_\alpha = Q_\alpha + \Gamma^2_\alpha$$
$$\{\mathbb{D}_\alpha, \mathbb{Q}_\alpha\} = iC_{\alpha\beta}\mathbb{W}^\dagger \, , \; \{\mathbb{D}_\alpha, \bar{\mathbb{D}}_{\dot{\alpha}}\} = \{\mathbb{Q}_\alpha, \bar{\mathbb{Q}}_{\dot{\alpha}}\} = i\nabla_{\alpha\dot{\alpha}} \quad (15)$$

Projective gauge covariantized derivative: $\nabla_\alpha = \mathbb{D}_\alpha + \zeta\mathbb{Q}_\alpha$

$$\left\{\nabla_\alpha(\zeta_1), \nabla_\beta(\zeta_2)\right\} = iC_{\alpha\beta}(\zeta_1 - \zeta_2)\mathbb{W}^\dagger \quad \Longrightarrow \quad \left\{\nabla_\alpha, [\partial_\zeta, \nabla_\beta]\right\} = iC_{\alpha\zeta}\mathbb{W}^\dagger \quad (16)$$

$\mathbb{W}, \mathbb{W}^\dagger$ are the vector representation field strengths.

We can properly define the vector representation the following way:
Split symmetrically: $e^{V} = e^{\bar{U}} e^{U}$

$$e^{U} \longrightarrow e^{iK} e^{U} e^{-i\Lambda} \;,\; e^{\bar{U}} \longrightarrow e^{i\bar{\Lambda}} e^{\bar{U}} e^{-iK} \tag{17}$$

Now redefine the hypermultiplets and let them transform with the ζ-independent real field K.

$$\tilde{\Upsilon}_{vec} \equiv e^{U}\Upsilon \longrightarrow e^{iK}\tilde{\Upsilon} \;,\; \tilde{\bar{\Upsilon}}_{vec} \equiv \bar{\Upsilon} e^{\bar{U}} \longrightarrow \tilde{\bar{\Upsilon}} e^{-iK} \tag{18}$$

They are annihilated by:

$$\begin{aligned} \mathbb{\nabla}_{\alpha} &\equiv e^{U}\nabla_{\alpha}e^{-U} = e^{-\bar{U}}\nabla_{\alpha}e^{\bar{U}} = \nabla_{\alpha} + \Gamma_{\alpha}(\zeta) \\ \overline{\mathbb{\nabla}}_{\dot{\alpha}} &\equiv e^{U}\bar{\nabla}_{\dot{\alpha}}e^{-U} = e^{-\bar{U}}\bar{\nabla}_{\dot{\alpha}}e^{\bar{U}} = \bar{\nabla}_{\dot{\alpha}} + \bar{\Gamma}_{\dot{\alpha}}(\zeta) \end{aligned} \tag{19}$$

The Arctic/Antarctic representations are defined analogously:

$$\tilde{\Upsilon}_A \equiv \Upsilon \longrightarrow e^{i\Lambda}\tilde{\Upsilon} \ , \ \bar{\tilde{\Upsilon}}_A \equiv \bar{\Upsilon}e^V \longrightarrow \bar{\tilde{\Upsilon}}e^{-i\Lambda} \ ; \tag{20}$$

$$\tilde{\Upsilon}_{\bar{A}} \equiv e^V\Upsilon \longrightarrow e^{i\bar{\Lambda}}\tilde{\Upsilon} \ , \ \bar{\tilde{\Upsilon}}_{\bar{A}} \equiv \bar{\Upsilon} \longrightarrow \bar{\tilde{\Upsilon}}e^{-i\bar{\Lambda}} \tag{21}$$

Those polar multiplets are annihilated by ∇_α.
The field strengths in these representations are obtained from the previous anticommutation relations:

$$\{\nabla_\alpha, [e^{-U}\partial_\zeta e^U, \nabla_\beta]\} = iC_{\alpha\beta}e^{-U}\mathbb{W}^\dagger e^U \equiv iC_{\alpha\beta}\mathcal{W}^\dagger(\zeta)$$
$$\{\nabla_\alpha, [e^{\bar{U}}\partial_\zeta e^{-\bar{U}}, \nabla_\beta]\} = iC_{\alpha\beta}e^{\bar{U}}\mathbb{W}^\dagger e^{-\bar{U}} \equiv iC_{\alpha\beta}\widetilde{\mathcal{W}}^\dagger(\zeta) \tag{22}$$

From here we define the projective gauge connection A_ζ:

$$\mathcal{D}_\zeta = \partial_\zeta + A_\zeta = e^{-U}\partial_\zeta e^U \ , \ \widetilde{\mathcal{D}}_\zeta = \partial_\zeta + \tilde{A}_\zeta = e^{\bar{U}}\partial_\zeta e^{-\bar{U}} \tag{23}$$

Field strengths are expressed by the A_ζ-connections:

$$\mathcal{W}^\dagger = -i\nabla^2 A_\zeta \ , \ \widetilde{\mathcal{W}}^\dagger = -i\nabla^2\tilde{A}_\zeta \tag{24}$$

The arctic and antarctic ζ-connections are related as:

$$e^{-V}(\partial_\zeta e^V) = A_\zeta - e^{-V}\tilde{A}_\zeta e^V \tag{25}$$

The ζ-connections explicitly in terms of the prepotential.

$$A_\zeta = \sum_{n=1}^{\infty}(-1)^{n+1}\oint d\zeta_1 \ldots \oint d\zeta_n \frac{(e^V-1)_1 \ldots (e^V-1)_n}{\zeta_{21}\cdots\zeta_{n,n-1}}\frac{1}{\zeta_{10}\zeta_{n0}} \tag{26}$$

$$\tilde{A}_\zeta = \sum_{n=1}^{\infty}(-1)^{n+1}\oint d\zeta_1 \ldots \oint d\zeta_n \frac{(e^V-1)_1 \ldots (e^V-1)_n}{\zeta_{21}\cdots\zeta_{n,n-1}}\frac{1}{\zeta_{01}\zeta_{0n}} \tag{27}$$

notations:

$$\frac{1}{\zeta_{ab}} \equiv \frac{1}{\zeta_a}\sum_{k=0}^{\infty}\left(\frac{\zeta_b}{\zeta_a}\right)^k = \frac{1}{\zeta_a-\zeta_b} \quad if \quad \left|\frac{\zeta_b}{\zeta_a}\right| < 1 \tag{28}$$

$$X^+(\zeta_0) \equiv \oint d\zeta_1 \frac{X_1}{\zeta_{10}} = \sum_{n=0}^{\infty} x_n \zeta_0^n$$

$$X^-(\zeta_0) \equiv \oint d\zeta_1 \frac{X_1}{\zeta_{01}} = \sum_{n=-\infty}^{-1} x_n \zeta_0^n \tag{29}$$

Non-symmetric splitting of e^V

In the symmetric splitting $e^V = e^{\bar{U}} e^U$, the two parts are projectively conjugate to each other and the ζ-independent terms are split equally. But our notations prefer non-symmetric ways in a manner, where all the ζ-independent terms sit on one side.

$$e^V = e^{\bar{U}} e^U \qquad e^V = e^{\bar{\check{U}}} e^{\hat{U}} \qquad e^V = e^{\bar{\hat{U}}} e^{\check{U}}$$

Gauge ζ-connection A_ζ is independent from types of splittings.

$$A_\zeta = e^{-U}(\partial_\zeta e^U) = e^{-\hat{U}}(\partial_\zeta e^{\hat{U}}) = e^{-\check{U}}(\partial_\zeta e^{\check{U}}) \tag{30}$$

Expand the two sides of splittings in powers of $X \equiv e^V - 1$'s:

$$\begin{aligned} e^{\hat{U}} &= 1 + \hat{Y}^{(1)} + \hat{Y}^{(2)} + \dots \\ e^{\check{\bar{U}}} &= 1 + \check{\bar{Y}}^{(1)} + \check{\bar{Y}}^{(2)} + \dots \end{aligned} \tag{31}$$

Solve the equation recursively to find:

$$\begin{aligned} e^{\hat{U}} &= 1 + X^{+} - [X^{-}X]^{+} + [[X^{-}X]^{-}X]^{+} - \dots \\ e^{\check{\bar{U}}} &= 1 + X^{-} - [XX^{+}]^{-} + \left[X[XX^{+}]^{+}\right]^{-} - \dots \end{aligned} \tag{32}$$

This can be written compactly using contour integrals:

$$e^{\hat{U}} = 1 + \sum_{n=1}^{\infty} (-1)^{n+1} \oint d\zeta_1 \dots d\zeta_n \frac{X_1 \dots X_n}{\zeta_{21} \cdots \zeta_{n,n-1}} \frac{1}{\zeta_{n0}} \tag{33}$$

$$e^{\check{U}} = 1 + \sum_{n=1}^{\infty} (-1)^{n+1} \oint d\zeta_1 \dots d\zeta_n \frac{1}{\zeta_{01}} \frac{X_1 \dots X_n}{\zeta_{21} \cdots \zeta_{n,n-1}} \tag{34}$$

The same can be done if the ζ-independent terms are included in the negative power side. $e^V = e^{\hat{\hat{U}}} e^{\check{U}}$. This can be achieved by projections:

$$\oint d\zeta_1 \frac{X_1}{\zeta_{10}} \frac{\zeta_0}{\zeta_1} = \sum_{n=1}^{\infty} x_n \zeta_0^n \ , \quad \oint d\zeta_1 \frac{X_1}{\zeta_{01}} \frac{\zeta_0}{\zeta_1} = \sum_{n=-\infty}^{0} x_n \zeta_0^n \tag{35}$$

Recursively solving in powers of X's, we find the projections to be:

$$e^{\check{U}} = 1 + \left[\frac{X}{\zeta}\right]^{+}\zeta - \left[\left[\frac{X}{\zeta}\right]^{-}X\right]^{+}\zeta + \left[\left[\left[\frac{X}{\zeta}\right]^{-}X\right]^{-}X\right]^{+}\zeta - \dots \tag{36}$$

$$e^{\hat{U}} = 1 + \left[\frac{X}{\zeta}\right]^{-}\zeta - \left[X\left[\frac{X}{\zeta}\right]^{+}\right]^{-}\zeta + \left[X\left[X\left[\frac{X}{\zeta}\right]^{+}\right]^{+}\right]^{-}\zeta - \dots \tag{37}$$

or in contour integrals:

$$e^{\check{U}} = 1 + \sum_{n=1}^{\infty}(-1)^{n+1}\oint d\zeta_1 \dots d\zeta_n \frac{\zeta_0}{\zeta_1}\frac{X_1 \dots X_n}{\zeta_{21}\cdots\zeta_{n,n-1}}\frac{1}{\zeta_{n0}} \tag{38}$$

$$e^{\hat{U}} = 1 + \sum_{n=1}^{\infty}(-1)^{n+1}\oint d\zeta_1 \dots d\zeta_n \frac{1}{\zeta_{01}}\frac{X_1 \dots X_n}{\zeta_{21}\cdots\zeta_{n,n-1}}\frac{\zeta_0}{\zeta_n} \tag{39}$$

Writing the splittings together: $e^V = e^{\bar{U}} e^U = e^{\check{\bar{U}}} e^{\hat{U}} = e^{\hat{\bar{U}}} e^{\check{U}}$, it is convenient to isolate the ζ-independent terms as exp's.

$$e^P = e^{\hat{U}} e^{-U} = e^{-\check{\bar{U}}} e^{\bar{U}} \tag{40}$$

$$e^{\bar{P}} = e^U e^{-\check{U}} = e^{-\bar{U}} e^{\hat{\bar{U}}} \tag{41}$$

$$e^P e^{\bar{P}} = e^{\hat{U}} e^{-\check{U}} = e^{-\check{\bar{U}}} e^{\hat{\bar{U}}} \tag{42}$$

Using the decompositions of above recursively, we find:

$$e^P e^{\bar{P}} = 1 + \sum_{n=1}^{\infty} (-1)^{n+1} \oint d\zeta_1 \dots d\zeta_n \frac{X_1 \dots X_n}{\zeta_{21} \cdots \zeta_{n,n-1}} \frac{1}{\zeta_n} \tag{43}$$

$$e^{-\bar{P}} e^{-P} = 1 + \sum_{n=1}^{\infty} (-1)^n \oint d\zeta_1 \dots d\zeta_n \frac{1}{\zeta_1} \frac{X_1 \dots X_n}{\zeta_{21} \cdots \zeta_{n,n-1}} \tag{44}$$

Field strength in terms of non-symmetric splittings

Recall the f.s in the arctic representation: $i\mathcal{W} = \bar{\nabla}^2 A_\zeta$

$$\bar{\nabla}_0^2 A_0 = \bar{\nabla}_0^2 \sum_{n=1}^{\infty} \frac{(-1)^{n+1} X_1 \dots X_n}{\zeta_{10}\zeta_{21} \cdots \zeta_{n,n-1}\zeta_{n0}} =$$
$$= \sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{2} \sum_{k=1}^{n} \sum_{i=1}^{n} \frac{X_1 \dots \bar{\nabla}_0^\alpha X_k \dots \bar{\nabla}_{0\alpha} X_i \dots X_n}{\zeta_{10}\zeta_{21} \cdots \zeta_{n,n-1}\zeta_{n0}} \tag{45}$$

Using the relations coming from the projectivity of fields:

$$\bar{\nabla}_0 X_1 = (\zeta_1 - \zeta_0)\bar{D} X_1$$
$$\bar{\nabla}_0^2 X_1 = (\zeta_1 - \zeta_0)^2 \bar{D}^2 X_1 \tag{46}$$

when one of $\bar{\nabla}_0$'s hits the X_k, we decompose the $\zeta_k - \zeta_0$ factor by shifting left or right depending on which side the given $\bar{\nabla}$ sits.

$$\begin{aligned} \zeta_k - \zeta_0 &= (\zeta_k - \zeta_{k-1}) + (\zeta_{k-1} - \zeta_{k-2}) + \ldots (\zeta_2 - \zeta_1) + (\zeta_1 - \zeta_0) \\ \zeta_i - \zeta_0 &= (\zeta_i - \zeta_{i+1}) + (\zeta_{i+1} - \zeta_{i+2}) + \ldots (\zeta_{n-1} - \zeta_n) + (\zeta_n - \zeta_0) \end{aligned} \quad (47)$$

Cancelling each terms with the corresponding $\zeta_{a,a+1}$ in the denominator, we can arrange:

$$\bar{\nabla}_0^2 A_0 = \sum_{n=1}^{\infty} \sum_{b=1}^{n} \sum_{a=0}^{b-1} \oint \frac{(-1)^a X_1 \ldots X_a}{\zeta_{10} \cdots \zeta_{a,a-1}} \cdot \frac{(-1)^{b-(a+1)} \bar{D}^2 (X_{a+1} \ldots X_b)}{\zeta_{a+2,a+1} \cdots \zeta_{b,b-1}} \cdot \cdot \frac{(-1)^{n+1-(b+1)} X_{b+1} \ldots X_n}{\zeta_{b+2,b+1} \cdots \zeta_{n0}} \quad (48)$$

The arctic f.s separates into ζ-independant and dependant parts:

$$i\mathcal{W}(\zeta_0) = e^{-\hat{U}_0}\ \bar{D}^2 F\ e^{\hat{U}_0} \tag{49}$$

where, $F = \sum_{n=1}^{\infty}(-1)^{n+1}\oint d\zeta_1 \dots d\zeta_n \frac{X_1 \dots X_n}{\zeta_{21}\cdots\zeta_{n,n-1}}$

We could have equivalently done the above in terms of $\bar{Q}$ derivatives.

$$\bar{\nabla}_0 X_1 = \frac{\zeta_1 - \zeta_0}{\zeta_1}\bar{Q}X_1 \ \ , \ \ \bar{\nabla}_0^2 X_1 = \frac{(\zeta_1 - \zeta_0)^2}{\zeta_1^2}\bar{Q}^2 X_1 \tag{50}$$

in this case, defining: $\bar{F} = \sum_{n=1}^{\infty}(-1)^{n+1}\oint d\zeta_1 \dots d\zeta_n \frac{1}{\zeta_1}\frac{X_1 \dots X_n}{\zeta_{21}\cdots\zeta_{n,n-1}}\frac{1}{\zeta_n}$

$$i\mathcal{W}(\zeta_0) = e^{-\check{U}_0}\ \bar{Q}^2\ \bar{F}e^{\check{U}_0} \tag{51}$$

Summarizing them:

$$i\mathcal{W} = e^{-\hat{U}}\bar{D}^2 F e^{\hat{U}} = e^{-\check{U}}\bar{Q}^2 \bar{F} e^{\check{U}} = e^{-U}e^{-P}\bar{D}^2 F e^{P} e^{U} = e^{-U}e^{\bar{P}}\bar{Q}^2 \bar{F} e^{-\bar{P}} e^{U}$$
$$i\mathcal{W}^\dagger = e^{-\hat{U}} Q^2 F e^{\hat{U}} = e^{-\check{U}} D^2 \bar{F} e^{\check{U}} = e^{-U}e^{-P} Q^2 F e^{P} e^{U} = e^{-U}e^{\bar{P}} D^2 \bar{F} e^{-\bar{P}} e^{U} \tag{52}$$

Then, the field strength in the vector representation can be written as:

$$\begin{aligned} \mathbb{W} =& e^{-P}\bar{D}^2 F e^{P} = e^{\bar{P}}\bar{Q}^2 \bar{F} e^{-\bar{P}} \\ \mathbb{W}^\dagger =& e^{\bar{P}} D^2 \bar{F} e^{-\bar{P}} = e^{-P} Q^2 F e^{P} \end{aligned} \tag{53}$$

5D cases

Superfields in the superspace $\mathbb{R}^{5|8}$ depend on $(x^{\hat{a}}, \theta_i^{\hat{\alpha}})$ coordinates, with $\hat{\alpha}$ as 4-spinor indices of the Lorentz group $SO(4,1)$; i is the index of the $SU(2)$ automorphism group.
Gauged covariant derivatives satisfy the following algebra:

$$\{\mathcal{D}^i_{\hat{\alpha}}, \mathcal{D}^j_{\hat{\beta}}\} = i\epsilon^{ij}\left(\nabla_{\hat{\alpha}\hat{\beta}} + \epsilon_{\hat{\alpha}\hat{\beta}}\mathbb{W}\right) \text{ , } where \ \ \nabla_{\hat{\alpha}\hat{\beta}} = (C\Gamma^{\hat{a}})_{\hat{\alpha}\hat{\beta}}\mathcal{D}_{\hat{a}} \tag{54}$$

Now, we translate this algebra into projective language:

$$\boldsymbol{\nabla}_{\hat{\alpha}}(\zeta) \equiv \mathcal{D}_{1\hat{\alpha}} + \zeta\mathcal{D}_{2\hat{\alpha}} \quad ; \quad \{\boldsymbol{\nabla}_{\hat{\alpha}}(\zeta), \boldsymbol{\nabla}_{\hat{\beta}}(\zeta)\} = 0 \tag{55}$$

As usual, taking the projective gauge covariant derivatives at different points of ζ-coordinates, the f.s's in vector representation can be obtained.

$$\{\boldsymbol{\nabla}_{\hat{\alpha}}(\zeta_1), \boldsymbol{\nabla}_{\hat{\beta}}(\zeta_2)\} = i(\zeta_1 - \zeta_2)\left(\nabla_{\hat{\alpha}\hat{\beta}} + \epsilon_{\hat{\alpha}\hat{\beta}}\mathbb{W}\right) \tag{56}$$

Then, do the same trick as in 4D case, the additional term won't be a problem, since: $\epsilon^{\hat{\alpha}\hat{\beta}}(C\Gamma^{\hat{a}})_{\hat{\alpha}\hat{\beta}} = 0$.
Then, the f.s in the arctic representation $\mathcal{W} = e^{-U}\mathbb{W}e^{U}$ is:

$$\mathcal{W} = \frac{1}{2}i\nabla^2 A \tag{57}$$

Employing the non-symmetric splittings as in the 4D case, we find the f.s in convenient ζ-separated forms again.

$$\mathcal{W} = e^{-\check{U}}(D_1^{\hat{\alpha}} D_{1\hat{\alpha}} \bar{F}) e^{\check{U}} \tag{58}$$

$$\mathcal{W} = e^{-\hat{U}}(D_2^{\hat{\alpha}} D_{2\hat{\alpha}} F) e^{\hat{U}} \tag{59}$$

An important relation connecting these two representations is:

$$D_1^{\hat{\alpha}} D_{1\hat{\alpha}} \bar{F} = e^{-\bar{P}} e^{-P} (D_2^{\hat{\alpha}} D_{2\hat{\alpha}} F) e^{P} e^{\bar{P}} \tag{60}$$

Chern-Simons action

5D supersymmetric Chern-Simons action has been constructed at the Abelian level, but the non-Abelian action has only been defined as a variation with respect to the infinitesimal deformation of the prepotential field. In the projective superspace setting:

$$\begin{aligned}\delta S_{CS} =& k_5 \mathrm{Tr} \int \mathrm{d}^5 \mathrm{x} \mathrm{d}^8 \theta \oint \mathrm{d}\zeta \mathrm{e}^{-\mathrm{V}} \delta \mathrm{e}^{\mathrm{V}} \{\mathrm{A}\,,\, \mathcal{W}\} = \\ =& - k_5 \mathrm{Tr} \int \mathrm{d}^5 \mathrm{x} \mathrm{d}^8 \theta \oint \mathrm{d}\zeta \mathrm{e}^{-\mathrm{V}} \delta \mathrm{e}^{\mathrm{V}} \nabla^{\hat{\alpha}} \mathrm{A} \nabla_{\hat{\alpha}} \mathrm{A} \end{aligned} \quad (61)$$

This variation is integrable and gauge invariant under the transformations:

$$\delta e^V = i\bar{\Lambda} e^V - e^V i\Lambda \;\;,\;\; \delta A = -i\partial_\zeta \Lambda + [i\Lambda, A] \;\;,\;\; \delta W = [i\Lambda, \mathcal{W}] \quad (62)$$

The abelian SCS action can easily be obtained from the above generic form of variation:

$$S_{CS}^{Ab} = \frac{1}{3} k_5 \int d^5x d^8\theta \oint d\zeta_0 V_0 A_{\zeta_0} \nabla_0^2 A_{\zeta_0} \tag{63}$$

Naturally, integrating the non-abelian form requires more technical exercises and involves non-symmetric splittings of the f.s.
First, switch from the full measure to the projective half-measure:

$$\mathrm{d}^8\theta \sim \Delta^4 \nabla^4 \quad \longrightarrow \quad \frac{\left(D_2^{\hat{\alpha}} D_{1\hat{\alpha}}\right)^2}{\zeta^2} \tag{64}$$

Then, the non-symmetric splittings (both represenations) will be utilised to pull out the variation in front of the whole expression.

After a long technical details, we find the non-Abelian SCS action in terms of component derivatives.

$$S_{CS} = \frac{1}{5}k_5 \mathrm{Tr}\int \mathrm{d}^5\mathrm{x}\,(\mathrm{D}_2\mathrm{D}_2)\,(\mathrm{D}_1\mathrm{D}_1)\left\{(D_1D_1)\left[\bar{F}\cdot(D_1D_1\bar{F})^2 - \frac{1}{2}\bar{F}^2\cdot(D_1D_1)^2\bar{F}\right] + \right.$$
$$\left. +(D_2D_2)\left[F\cdot(D_2D_2F)^2 - \frac{1}{2}F^2\cdot(D_2D_2)^2F\right]\right\} \tag{65}$$

Applications

- Mirror symmetry of 3D $\mathcal{N} = 4$ susy theories as a generalized Fourier transform in the path integral. The following two partition functions are equal in the low energy limit:

$$Z[\hat{V}_R] = \int \mathcal{D}\Upsilon_L \mathcal{D}V_L e^{iS[\Upsilon,\bar{\Upsilon},V_L]+iS_{BF}[V_L,\hat{V}_R]}$$
$$Z[\hat{V}_R] = \int \mathcal{D}\hat{\Upsilon}_R e^{iS[\hat{\Upsilon}_R,\hat{\bar{\Upsilon}}_R,\hat{V}_R]}$$

- Application of the non-symmetric splittings into the $\mathcal{N} = 2$ supersymmetrization of the 4D *ModMAx* theory.