CORFU 2023 SUMMER INST.
(18.09-24.09.2023)

JERZY LUKIERSKI
WROCŁAW

DOUBLY κ-DEFORMED YANG MODELS

1. Introduction
2. Snyder and κ-deformed Snyder model
3. Yang models by Born-dual extensions of Snyder models
4. Doubly κ-deformed Yang models as Born-selfdual extension of κ-deformed Snyder model
5. Algebraic description of doubly κ-deformed Yang models by particular choice of nonstandard $\hat{o}(5,1)$ basis
6. Outlook

(paper by J.L, S. Meljanac, S. Mignemi, A. Pachoł, M. Woronowicz, to appear soon)

1. Introduction

Our aim: provide new noncommutative (NC) model of quantum-deformed relativistic phase space, selfdual under the generalization of standard Born map $(\hat{x}_\mu \to \hat{p}_\mu, \hat{p}_\mu \to -\hat{x}_\mu)$

$$
\begin{array}{ccc}
\begin{array}{c}\text{Snyder model}\\ (\hat{x}_\mu, \hat{M}_{\mu\nu}) \quad (\mu,\nu = 0,1,2,3)\\ (o(4,1)\ \text{algebra basis}\ \hat{M}_{AB})\end{array} & \xrightarrow[\text{map}]{\text{Born}} & \begin{array}{c}\text{Yang model}\\ (\hat{x}_\mu, \hat{p}_\mu, \hat{M}_{\mu\nu}), \hat{I} \sim \hat{o}(2)\\ (o(5,1)\ \text{algebra}\ \hat{M}_{KL})\end{array} \\
\Big\downarrow \begin{array}{c}\kappa\text{-extension}\\ (2020)\end{array} & & \Big\downarrow \begin{array}{c}\text{double } \kappa\text{-extension}\\ (\text{new})\end{array} \\
\begin{array}{c}\kappa\text{-deformation of}\\ \text{Snyder model}\\ (\text{modified } o(4,1)\\ \text{algebra basis } \hat{M}_{AB})\\ (\text{A,B}=0,1,2,3,4)\end{array} & \xrightarrow[\text{Born map}]{\text{generalized}} & \begin{array}{c}\text{doubly } \kappa\text{-deformed}\\ \text{Yang model}\\ (\text{modified } o(5,1)\\ \text{algebra basis } \hat{M}_{KL})\\ (\text{K,L}=0,1,2,3,4,5)\end{array}
\end{array}
$$

The appearance of deformation parameters ($c = \hbar = 1$)

Snyder model		Yang model
one deformation parameter	$\longrightarrow$	two deformation parameters
$M \quad ([M] = L^{-1})$		$M, R \quad ([R] = L)$
$\downarrow$		$\downarrow$
κ-deformed Snyder model		doubly κ-deformed Yang model
$M, \kappa \quad ([\kappa] = L^{-1})$	$\longrightarrow$	$M, R, \kappa, \tilde{\kappa} \quad ([\tilde{\kappa}] = L^{-1})$
normalized fourvector a_μ		two independent normalized fourvectors a_μ, b_μ
$a_\mu a^\mu = \epsilon \qquad \epsilon = 0, \pm 1$		$b_\mu b^\mu = \tilde{\epsilon} \qquad \tilde{\epsilon} = 0, \pm 1$

$\kappa \leftarrow$ standard κ-deformation of curved quantum space-time

$\tilde{\kappa} \leftarrow \tilde{\kappa}$-deformation of curved quantum fourmomentum space

(both related with Born duality, not Hopf-algebraic duality)

2. Snyder and κ-deformed Snyder model

i) Snyder model $(\hat{x}_\mu, \hat{M}_{\mu\nu})$

$$[\hat{M}_{\mu\nu}, \hat{M}_{\rho\tau}] = i(\eta_{\mu\rho}\hat{M}_{\nu\tau} - \eta_{\mu\tau}\hat{M}_{\nu\rho} + \eta_{\nu\tau}\hat{M}_{\mu\rho} - \eta_{\nu\rho}\hat{M}_{\mu\tau})$$

$$[\hat{M}_{\mu\nu}, \hat{x}_\rho] = i(\eta_{\mu\rho}\hat{x}_\nu - \eta_{\nu\rho}\hat{x}_\mu)$$

$$[\hat{x}_\mu, \hat{x}_\nu] = \frac{i}{M^2}\hat{M}_{\mu\nu}$$

These relations are described by classical $\hat{o}(4,1)$ algebra $M_{AB} = (\hat{M}_{\mu\nu}, \hat{M}_{4\nu})$ if

$$\hat{M}_{4\mu} = M\hat{x}_\mu$$

and we get $(A, B = 0, 1, 2, 3, 4)$

$$[\hat{M}_{AB}, \hat{M}_{CD}] = i(\eta_{AC}\hat{M}_{BD} - \eta_{AD}\hat{M}_{BC} + \eta_{BD}\hat{M}_{AC} - \eta_{BC}\hat{M}_{AD})$$

Snyder phase space $(\hat{x}_\mu, p_\nu)$ (not Born-selfdual)

$$[p_\mu, p_\nu] = 0$$

$$[\hat{M}_{\mu\nu}, p_\rho] = i(\eta_{\mu\rho} p_\nu - \eta_{\nu\rho} p_\mu)$$

$$[\hat{x}_\mu, p_\nu] = \eta_{\mu\nu}\left[F\left(\frac{p^2}{M^2}\right) - \frac{p_\mu p_\nu}{M^2} G\left(\frac{p^2}{M^2}\right)\right] \qquad (F(0) = 1)$$

From Jacobi identity one can derive the linear differential equation linking functions F and G (Battisti, Meljanac 2007).

ii) κ-deformed Snyder model (Meljanac, Mignemi 2020)

$$[\hat{M}_{\mu\nu}, \hat{M}_{\rho\tau}] = i(\eta_{\mu\rho}\hat{M}_{\nu\tau} - \eta_{\mu\tau}\hat{M}_{\nu\rho} + \eta_{\nu\tau}\hat{M}_{\mu\rho} - \eta_{\nu\rho}\hat{M}_{\mu\tau})$$

$$[\hat{M}_{\mu\nu}, \hat{x}_\rho] = i(\eta_{\mu\rho}\hat{x}_\nu - \eta_{\nu\rho}\hat{x}_\mu) + \frac{i}{\kappa}(a_\mu \hat{M}_{\rho\nu} - a_\nu \hat{M}_{\rho\mu})$$

$$[\hat{x}_\mu, \hat{x}_\nu] = \frac{i}{M^2}\hat{M}_{\mu\nu} + \frac{i}{\kappa}(a_\mu \hat{x}_\nu - a_\nu \hat{x}_\mu)$$

- **If $M \to \infty$, κ finite** $\longrightarrow$ **κ-deformed Minkowski space-time $\hat{x}_\mu$**
- **If $\kappa \to \infty$, M finite** $\longrightarrow$ **standard Snyder model**

3. Yang models by Born-selfdual extensions of Snyder models

Born duality implies adding the second Born-dual parameter R and selfdual generator $\hat{I}$

$$[\hat{x}_\mu, \hat{x}_\nu] = \frac{i}{M^2}\hat{M}_{\mu\nu} \qquad \overset{B}{\longleftrightarrow} \qquad [\hat{p}_\mu, \hat{p}_\nu] = \frac{i}{R^2}\hat{M}_{\mu\nu}$$

$$[\hat{x}_\mu, \hat{p}_\nu] = \frac{i}{MR}\eta_{\mu\nu}\hat{I} \qquad (\hat{I}\text{-internal } \hat{o}(2))$$

$$[\hat{I}, \hat{x}_\mu] = \frac{i}{M^2}\hat{p}_\mu \qquad \overset{B}{\longleftrightarrow} \qquad [\hat{I}, \hat{p}_\mu] = -\frac{i}{R^2}\hat{x}_\mu.$$

If

$$\hat{M}_{\mu 4} = M\hat{x}_\mu, \qquad \hat{M}_{\mu 5} = R\hat{p}_\mu, \qquad \hat{M}_{45} = MR\hat{I}$$

the five relations defining Yang model are described as $\hat{o}(5,1)$ algebra $(K, L = 0, 1, \ldots, 5)$

$$[\hat{M}_{KL}, \hat{M}_{PR}] = i(\eta_{KP}\hat{M}_{LR} - \eta_{KR}\hat{M}_{PL} + \eta_{LR}\hat{M}_{KP} - \eta_{PL}\hat{M}_{KR})$$

4. Doubly κ-deformed Yang models as the Born-selfdual extension of κ-deformed Snyder models

We define generalized Born map by the relations $(\hat{M}_{\mu\nu}, \hat{I})$ - Born selfdual

$$\hat{x}_\mu \overset{B}{\leftrightarrow} \hat{p}_\mu, \quad \hat{p}_\mu \overset{B}{\leftrightarrow} -\hat{x}_\mu, \quad \hat{M}_{\mu\nu} \overset{B}{\leftrightarrow} \hat{M}_{\mu\nu}, \quad \hat{I} \overset{B}{\leftrightarrow} \hat{I}$$

$$M \overset{B}{\leftrightarrow} R, \quad a_\mu \overset{B}{\leftrightarrow} b_\mu \quad \kappa \overset{B}{\leftrightarrow} \frac{1}{\tilde{\kappa}}, \quad [L] \overset{B}{\leftrightarrow} [L^{-1}]$$

where $(b_\mu, \tilde{\kappa})$ determine new $\tilde{\kappa}$-deformation in curved fourmomenta sector. We get the following set of additional commutators (besides the ones defining κ-deformed Snyder model)

$$[\hat{M}_{\mu\nu}, \hat{p}_\rho] = i\left[\eta_{\mu\rho}\hat{p}_\nu - \eta_{\nu\rho}\hat{p}_\mu + \tilde{\kappa}(b_\mu \hat{M}_{\rho\nu} - b_\nu \hat{M}_{\rho\mu})\right]$$

$$[\hat{p}_\mu, \hat{p}_\nu] = i\left[\frac{1}{R^2} M_{\mu\nu} + \tilde{\kappa}(b_\mu \hat{p}_\nu - b_\nu \hat{p}_\mu)\right]$$

and using Jacobi identities one gets the set of the relations

$$[\hat{x}_\mu, \hat{p}_\nu] = i\left(\eta_{\mu\nu}\hat{I} + \tilde{\kappa} b_\mu \hat{x}_\nu - \frac{a_\nu}{\kappa}\hat{p}_\mu + \frac{\rho}{MR}\hat{M}_{\mu\nu}\right)$$

$$[\hat{I}, \hat{x}_\mu] = i\left(\frac{1}{M^2}\hat{p}_\mu - \frac{1}{MR}\rho\hat{x}_\mu - \frac{a_\mu}{\kappa}\hat{I}\right) \qquad [\hat{I}, \hat{p}_\mu] = i\left(-\frac{1}{R^2}\hat{x}_\mu + \frac{1}{MR}\rho\hat{p}_\mu - \kappa b_\mu \hat{I}\right)$$

$$[\hat{I}, \hat{M}_{\mu\nu}] = i\left[\tilde{\kappa} b_\mu \hat{x}_\nu - \tilde{\kappa} b_\nu \hat{x}_\mu - \frac{a_\mu}{\kappa}\hat{p}_\nu + \frac{a_\nu}{\kappa}\hat{p}_\mu\right]$$

Our algebra has five parameters

$$\boldsymbol{M, R, \kappa, \tilde{\kappa}, \rho} \quad ([M] = L^{-1}, [R] = L, [\kappa] = L^{-1}, [\tilde{\kappa}] = L^{-1}, [\rho] = L^{0})$$

and two normalized dimensionless fourvectors a_μ, b_μ
($a_\mu a^\mu = 1$ - standard, $a_\mu a^\mu = -1$ - tachyonic, $a_\mu a^\mu = 0$- light cone κ-deformations).
The additional dimensionless parameter ρ permits to introduce the following two cases:

i) $\rho = 0 \quad \Rightarrow \quad$ standard doubly κ-deformed Yang model

ii) $\rho \neq 0 \quad \Rightarrow \quad$ one getsdoubly κ-deformed TSR model (Triply Special Relativity)
(Kowalski-Glikman, Smolin 2004)

One gets the following assignement of coset and subalgebra generators

$$\frac{\hat{o}(5,1)}{\hat{o}(3,1) \otimes \hat{o}(2)} \to (\hat{x}_\mu, \hat{p}_\mu) \qquad \hat{o}(3,1) \otimes \hat{o}(2) \to (M_{\mu\nu}, I).$$

If we generalize $\hat{o}(5,1) \to \hat{o}(3+2n,1)$, one can obtain $(i = 1, 2 \ldots n)$

$$\frac{\hat{o}(3+2n,1)}{\hat{o}(3,1) \otimes \hat{o}(n)} \to (\hat{x}_{\mu;i}, \hat{p}_{\mu;i}) \qquad \hat{o}(3,1) \otimes \hat{o}(n) \to (M_{\mu\nu}, I_{ij}).$$

where $\hat{o}(n)$ describes the algebra of internal symmetries (this is Kaluza-Klein extension of standard Yang model).

5. Algebraic descriptions of doubly κ-deformed Yang models by particular choice of nonstandard $o(5,1)$ basis

i) κ-deformed Snyder model described by $\hat{o}(4,1;g_{AB})$

$$\hat{o}(4,1;g_{AB}): \qquad [\hat{M}_{AB},\hat{M}_{CD}] = i(g_{AC}\hat{M}_{BD} - g_{AD}\hat{M}_{BC} + g_{BD}\hat{M}_{AC} - g_{BC}\hat{M}_{AD})$$

Because $\hat{M}_{AB} = L^0, g_{AB}$ are also zero-dimensional ($[g_{AB}] = L^0$).

One gets the description of κ-deformed Snyder algebra by $\hat{o}(4,1;g_{AB})$ if we choose $\hat{M}_{4\mu} = M\hat{x}_\mu$ and

$$\mathbb{G}^{(5)} = g_{AB} = \begin{pmatrix} \eta_{\mu\nu} & \frac{M}{\kappa}a_\mu \\ \frac{M}{\kappa}a_\nu & 1 \end{pmatrix}$$

The 5×5 metric tensor $\mathbb{G}^{(5)}$ can be described by the formula

$$\mathbb{G}^{(5)} = \mathbb{S}^{(5)}\eta(\mathbb{S}^{(5)})^\dagger$$

where (Meljanac, Mignemi 2021)

$$\mathbb{S}^{(5)} = g_{AB} = \begin{pmatrix} \delta_{\mu\nu} & \frac{M}{\kappa}a_\mu \\ 0 & 1 \end{pmatrix}$$

The upper-triangular matrix $\mathbb{S}^{(5)}$ maps the algebra $\hat{o}(4,1;\eta_{AB})$ into $\hat{o}(4,1;g_{AB})$.

ii) **Doubly κ-deformed Yang model described by $\hat{o}(5,1;g_{KL})$ Lie algebra**

is obtained from $\hat{o}(5,1) = \hat{o}(5,1;\eta_{KL})$ if we replace the metric tensor $\eta_{KL} = diag(-1,1,1,1,1,1)$ by the symmetric 6×6-dimensional metric tensor g_{KL}

$$\hat{o}(5;1;g_{KL}): \qquad [\hat{M}_{KL}, \hat{M}_{PR}] = i(g_{KP}\hat{M}_{LR} - g_{KR}\hat{M}_{LP} + g_{LR}\hat{M}_{KP} - g_{LP}\hat{M}_{KR})$$

where g_{KL} are the zero-dimensional ($[g_{KL}] = L^0$) functions of the parameters $M, R, \kappa, \tilde{\kappa}, a_\mu, b_\mu$ and ρ. In order to get the algebraic structure of doubly κ-deformed Yang models the basic operators $(\hat{M}_{\mu\nu}, \hat{x}_\mu, \hat{p}_\mu, \hat{I})$ we relate with $\hat{M}_{KL}$ as follows

$$\hat{M}_{KL} = \left(\hat{M}_{\mu\nu},\ \hat{M}_{4\mu} = M\hat{x}_\mu,\ \hat{M}_{5\mu} = R\hat{q}_\mu,\ \hat{M}_{45} = RM\hat{I}\ \right)$$

It appears that we should choose

$$\mathbb{G}^{(6)} = g_{KL} = \begin{pmatrix} \eta_{\mu\nu} & \frac{M}{\kappa}a_\mu & R\tilde{\kappa}b_\mu \\ \frac{M}{\kappa}a_\nu & 1 & \rho \\ R\tilde{\kappa}b_\nu & \rho & 1 \end{pmatrix}$$

For any nondegenerate symmetric matrix $\mathbb{G}^{(6)}$, with signature described by diagonal matrix $\eta = \eta_{KL}$ one can find 6×6-dimensional matrix $\mathbb{S}^{(6)}$ which satisfies the relation $\mathbb{G}^{(6)} = \mathbb{S}^{(6)}\eta^{(6)}(\mathbb{S}^{(6)})^T$ and maps $\hat{M}^{(g)}_{KL}$ into $M^{(\eta)}_{KL}$ in the following way

$$\hat{M}^{(g)}_{KL} = (\mathbb{S}\hat{\mathbb{M}}^{(\eta)}\mathbb{S}^T)_{KL} \qquad \longleftrightarrow \qquad \hat{M}^{(\eta)}_{KL} = (\mathbb{S}^{-1}\hat{\mathbb{M}}^{(g)}(\mathbb{S}^T)^{-1})_{KL}.$$

where $\hat{M}^{(g)}_{KL}$ describes the generators of $\hat{o}(5,1;g_{KL})$ algebra.

The choice of 6×6-dimensional matrix $\mathbb{S}^{(6)}$ is valid modulo matrix $\mathbb{R}$ describing arbitrary pseudo-orthogonal 6×6 dim. rotations ($\mathbb{R}\eta\mathbb{R}^T = \eta$). We choose

$$\mathbb{S}^{(6)}_{KL} = \begin{pmatrix} \eta_{\mu\nu} & 0 & 0 \\ \frac{M}{\kappa}a_\mu & a & d \\ R\tilde{\kappa}b_\mu & c & b \end{pmatrix}$$

where a, b, c, d satisfy the equations

$$a^2 + d^2 = 1 - \frac{M^2}{\kappa^2}a_\mu a^\mu$$

$$b^2 + c^2 = 1 - R^2\tilde{\kappa}^2 b_\mu b^\mu$$

$$ac + bd = \rho - \frac{MR\tilde{\kappa}}{\kappa}a_\mu b^\mu$$

If we choose d=0 we get for Yang model the matrix $\mathbb{S}^{(6)}_{KL}$ which is lower triangular.

6. Outlook

i) **Spontaneous symmetry breaking (SSB) in Snyder and Yang type models**
The idea of SSB in the models described algebraically by pseudoorthogonal Lie algebras was discussed by our group in 2022, see e.g.

1) **J.L, S. Meljanac, S. Mignemi, A. Pachoł**, ***Quantum perturbative solutions of extended Snyder and Yang models with SSB***, **arXiv:2212.02316[hep-th] (v2 September 2023)**
2) **J.L, A. Pachoł**, ***$\hbar$-perturbative solutions of quantum Snyder and Yang models wiyh parameters describing SSB***, **Proceedings of CORFU2022; arXiv:2307.12379[hep-th]**

Quantum Snyder/Yang models: if we introduce explicit $\hbar$-dependence and consider perturbative solutions as $\hbar$-power series.

Perturbative solutions of Snyder model as $\hbar$-power expansions look as follows:

$$M_{AB} = M_{AB}^{(0)} + \hbar M_{AB}^{(1)} + \hbar^2 M_{AB}^{(2)} + \cdots$$

where $(A, B = 0, 1 \ldots 4)$

$$M_{AB}^{(0)} = X_{AB} = -X_{BA} \qquad \text{(zero order term in } \hbar\text{-expansion)}$$

Main observation: X_{AB} are the Nambu-Goldstone (NG) parameters which describe SSB of ten one-dimensional $o(2)$ $(o(1,1))$ symmetries in ten planes (A,B) of $D = 4$ dS space, with symmetries generated by single generators $\hat{M}_{AB}$.

If we supplement tensorial momenta P_{AB} canonically dual to X_{AB}

$$[X_{AB}, P_{CD}] = i\hbar(\eta_{AC}\eta_{BD} - \eta_{AD}\eta_{BC})$$

one can solve perturbatively the coefficients $\hat{M}_{AB}^{(n)}$ as functions of X_{AB}, P_{CD}, which are n-linear in P_{CD}. In iterative solutions o $\hat{M}_{AB}^{(n)}$ the variables X_{AB} describe the free parameters (input) in calculation of the $\hbar$-perturbative solutions of Snyder model.

Our plan is to show that one can introduce also the SSB and NG parameters by using the $\hbar$-perturbative solutions of κ-deformed Snyder models and doubly κ-deformed Yang models.

ii) Quantum-deformed Snyder and Yang models

The algebraic structure of Snyder model is the following

$$[M, M] \sim M, \qquad [M, X] \sim X, \qquad [X, X] \sim M \qquad\qquad (A)$$

where M are Lorentz generators and X describe quantum dS space-time.

In standard Snyder model one can introduce primitive coproducts $\Delta_0(X)$ and $\Delta_0(M)$, satisfying homomorphic coalgebraic realization of the relations (A).
One can pass however to quantum-deformed Snyder models described by quantum $\hat{o}^q(4,1)$ Hopf algebra, with nonprimitive coproducts $\Delta(X)$ and $\Delta(M)$. First task is to consider M as the Hopf subalgebra, i.e assume that

$$[\Delta(M), \Delta(M)] \sim \Delta(M).$$

Further we should extend Snyder algebraic structure (see (A)) by adding the commutators of nonprimitive coproducts $\Delta(X)$.

Interestingly, for D=4 dS quantum algebras the Lorentz Hopf subalgebras have been recently classified, see
A. Ballesteros, I. Gutierrez-Sagredo, F. Herranz, *Noncommutative (A)dS and Minkowski spacetimes from quantum Lorentz subgroups*, arXiv: 2108.02683 [math-ph]
One gets that

$$\Delta(M) = F^{-1}\Delta_0(M)F$$

where F is the twist generated by explicitly provided triplet of classical triangular r-matrices describing the deformations of D=4 Lorentz algebra.
Further goal is to calculate $\Delta(X)$ and see how the relations

$$[\Delta_0(X), \Delta_0(X)] \sim \Delta(M), \qquad [\Delta_0(M), \Delta_0(X)] \sim \Delta(X)$$

are modified for the possible pairs of D=4 dS quantum algebras and their D=4 Lorentz quantum subalgebras.

Passing to quantum-deformed Yang models requires analogous considerations of quantum $\hat{o}(5,1)$ algebras containing Hopf subalgebras which are spanned by Lorentz and $\hat{o}(2)$ internal symmetry generators $(\hat{M}_{\mu\nu}, \hat{I})$.

THANK YOU