Current status of ε_K in lattice QCD

Weonjong Lee
(LANL-SWME Collaboration)

Lattice Gauge Theory Research Center
Department of Physics and Astronomy
Seoul National University, Seoul 08826, South Korea

Corfu 2022, 2022-08-30

Contents I

Contents II

Contents III

LANL-SWME Collaboration

LANL–SWME Collaboration I

- Seoul National University (SWME):
  Prof. Weonjong Lee
  6 graduate students $\leftarrow$ [data analysis]
- Yonsei University (SWME):
  Prof. Jon Bailey
- University of Washington (SWME):
  Prof. Stephen Sharpe
- Columbia University (SWME)
  Dr. Yong-Chull Jang (Postdoc) $\leftarrow$ [data analysis]
- Brookhaven National Laboratory (SWME):
  Dr. Chulwoo Jung (Staff Scientist)

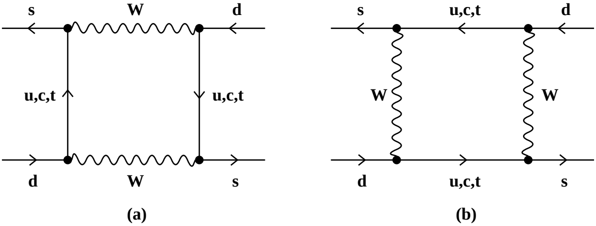
LANL–SWME Collaboration II

- Los Alamos National Laboratory (LANL):
  Dr. Rajan Gupta (Lab Fellow)
  Dr. Tanmoy Bhattacharya (Staff Scientist)
  Dr. Boram Yoon (Staff Scientist)
- Jefferson Laboratory (SWME):
  Dr. Sungwoo Park (Postdoc) ← [data analysis]
- Forschungszentrum (=Research Center) Jülich, Germany (SWME):
  Dr. Jangho Kim (Postdoc) ← [NPR]
- Korea Institute for Advanced Study, KIAS (SWME):
  Dr. Jaehoon Leem (Postdoc) ← [C.I. Theory]

CP Violation in Neutral Kaons

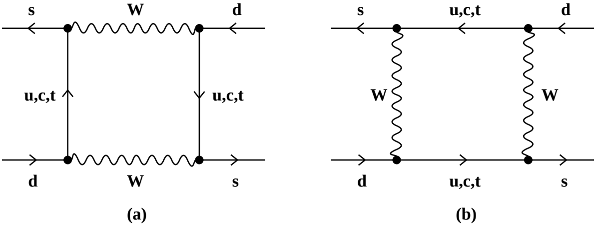
Kaon Eigenstates and ε

- Flavor eigenstates, $K^0 = (\bar{s}d)$ and $\bar{K}^0 = (s\bar{d})$ mix via box diagrams.



Kaon Eigenstates and ε

- Flavor eigenstates, $K^0 = (\bar{s}d)$ and $\bar{K}^0 = (s\bar{d})$ mix via box diagrams.

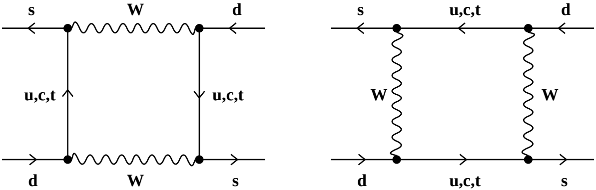


- CP eigenstates K_1(even) and K_2(odd).

$$K_1 = \frac{1}{\sqrt{2}}(K^0 - \bar{K}^0) \qquad K_2 = \frac{1}{\sqrt{2}}(K^0 + \bar{K}^0)$$

Kaon Eigenstates and ε

- Flavor eigenstates, $K^0 = (\bar{s}d)$ and $\bar{K}^0 = (s\bar{d})$ mix via box diagrams.



(a) (b)

- CP eigenstates K_1(even) and K_2(odd).

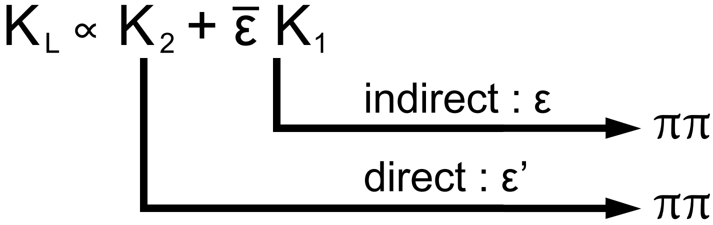
$$K_1 = \frac{1}{\sqrt{2}}(K^0 - \bar{K}^0) \qquad K_2 = \frac{1}{\sqrt{2}}(K^0 + \bar{K}^0)$$

- Neutral Kaon eigenstates K_S and K_L.

$$K_S = \frac{1}{\sqrt{1+|\bar{\varepsilon}|^2}}(K_1 + \bar{\varepsilon}K_2) \qquad K_L = \frac{1}{\sqrt{1+|\bar{\varepsilon}|^2}}(K_2 + \bar{\varepsilon}K_1)$$

Indirect CP violation and direct CP violation

- $\Gamma_{K_L} \cong 500 \times \Gamma_{K_S} \rightarrow$ only for neutral Kaons.
- It is possible to produce a high quality beam of K_L.



- $|\varepsilon_K| = |\varepsilon| \cong 2.2 \times 10^{-3}$.
- $|\varepsilon'/\varepsilon| \cong 1.7 \times 10^{-3}$.

ε_K and $\hat{B}_K$, $|V_{cb}|$ I

- Definition of ε_K

$$\varepsilon_K \equiv \frac{A[K_L \to (\pi\pi)_{I=0}]}{A[K_S \to (\pi\pi)_{I=0}]}, \quad |\varepsilon_K| = 2.228(11) \times 10^{-3}$$

- Master formula for ε_K in the Standard Model.

$$\varepsilon_K = \exp(i\theta)\,\sqrt{2}\,\sin(\theta)\left(C_\varepsilon\, X_{\mathrm{SD}}\, \hat{B}_K + \frac{\xi_0}{\sqrt{2}} + \xi_{\mathrm{LD}}\right) + \mathcal{O}(\omega\varepsilon') + \mathcal{O}(\xi_0\Gamma_2/\Gamma_1)$$

$$X_{\mathrm{SD}} = \mathrm{Im}\lambda_t\Big[\mathrm{Re}\,\lambda_c\,\eta_{cc}\,S_0(x_c) - \mathrm{Re}\,\lambda_t\,\eta_{tt}\,S_0(x_t) - (\mathrm{Re}\,\lambda_c - \mathrm{Re}\,\lambda_t)\,\eta_{ct}\,S_0(x_c, x_t)\Big]$$

ε_K and $\hat{B}_K$, $|V_{cb}|$ II

$$\lambda_i = V_{is}^* V_{id}, \qquad x_i = m_i^2 / M_W^2, \qquad C_\varepsilon = \frac{G_F^2 F_K^2 m_K M_W^2}{6\sqrt{2}\,\pi^2 \Delta M_K}$$

$$\frac{\xi_0}{\sqrt{2}} = \frac{1}{\sqrt{2}} \frac{\mathrm{Im} A_0}{\mathrm{Re} A_0} \approx -5\% \quad \rightarrow \quad \text{Absorptive Long Distance Effect}$$

$$\xi_{\text{LD}} = \text{Dispersive Long Distance Effect} \approx 2\% \ \rightarrow \ \text{explain it later.}$$

- Inami-Lim functions:

$$S_0(x_i) = x_i \left[\frac{1}{4} + \frac{9}{4(1-x_i)} - \frac{3}{2(1-x_i)^2} - \frac{3x_i^2 \ln x_i}{2(1-x_i)^3} \right],$$

$$S_0(x_i, x_j) = \left\{ \frac{x_i x_j}{x_i - x_j} \left[\frac{1}{4} + \frac{3}{2(1-x_i)} - \frac{3}{4(1-x_i)^2} \right] \ln x_i - (i \leftrightarrow j) \right\} - \frac{3 x_i x_j}{4(1-x_i)(1-x_j)}$$

ε_K and $\hat{B}_K$, $|V_{cb}|$ III

$$
\begin{aligned}
t - t \longrightarrow \quad & S_0(x_t) \longrightarrow \quad & +72.4\% \\
c - t \longrightarrow \quad & S_0(x_c, x_t) \longrightarrow \quad & +45.4\% \\
c - c \longrightarrow \quad & S_0(x_c) \longrightarrow \quad & -17.8\%
\end{aligned}
$$

- Dominant contribution ($\approx$72%) comes with $|V_{cb}|^4$.

$$
\begin{aligned}
&\lambda_i \equiv V_{is}^* V_{id} \\
&\mathrm{Im}\lambda_t \cdot \mathrm{Re}\lambda_t = \bar{\eta}\lambda^2 |V_{cb}|^4 (1 - \bar{\rho}) \\
&\mathrm{Re}\lambda_c = -\lambda(1 - \frac{\lambda^2}{2}) + \mathcal{O}(\lambda^5) \\
&\mathrm{Re}\lambda_t = -(1 - \frac{\lambda^2}{2}) A^2 \lambda^5 (1 - \bar{\rho}) + \mathcal{O}(\lambda^7) \\
&\mathrm{Im}\lambda_t = \eta A^2 \lambda^5 + \mathcal{O}(\lambda^7)
\end{aligned}
$$

ε_K and $\hat{B}_K$, $|V_{cb}|$ IV

$$\mathrm{Im}\lambda_c = -\mathrm{Im}\lambda_t$$

- Definition of $\hat{B}_K$ in standard model.

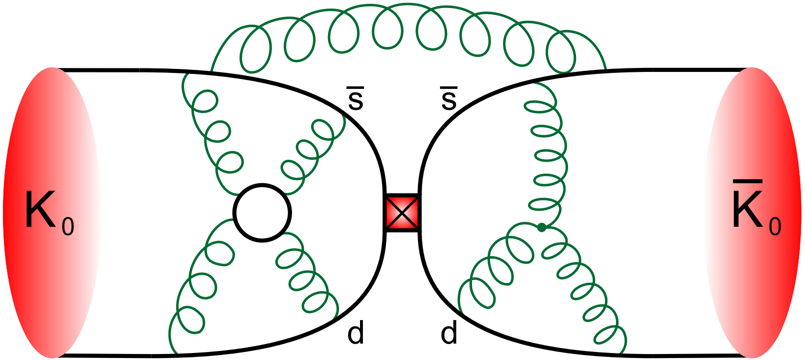
$$B_K = \frac{\langle \bar{K}_0|[\bar{s}\gamma_\mu(1-\gamma_5)d][\bar{s}\gamma_\mu(1-\gamma_5)d]|K_0\rangle}{\frac{8}{3}\langle \bar{K}_0|\bar{s}\gamma_\mu\gamma_5 d|0\rangle\langle 0|\bar{s}\gamma_\mu\gamma_5 d|K_0\rangle}$$

$$\hat{B}_K = C(\mu)B_K(\mu), \qquad C(\mu) = \alpha_s(\mu)^{-\frac{\gamma_0}{2b_0}}[1+\alpha_s(\mu)J_3]$$

- Experiment:

$$\varepsilon_K = (2.228 \pm 0.011) \times 10^{-3} \times e^{i\phi_\varepsilon}$$
$$\phi_\varepsilon = 43.52(5)^\circ$$

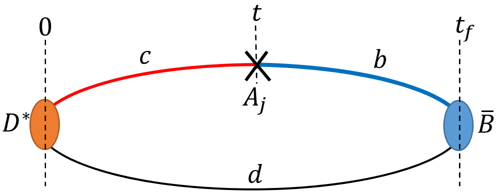
$\hat{B}_K$ on the lattice



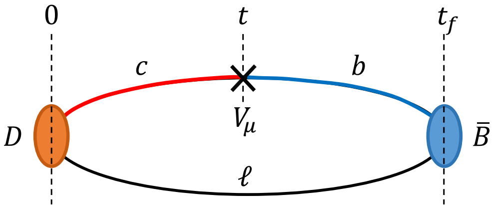
- This is one of the ∞ number of the Feynman diagrams that we need to calculate using lattice QCD tools.

$|V_{cb}|$ on the lattice

- $\bar{B} \to D^* \ell \bar{\nu}$ decay form factors:



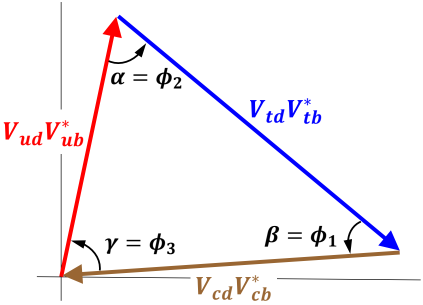
- $\bar{B} \to D \ell \bar{\nu}$ decay form factors:



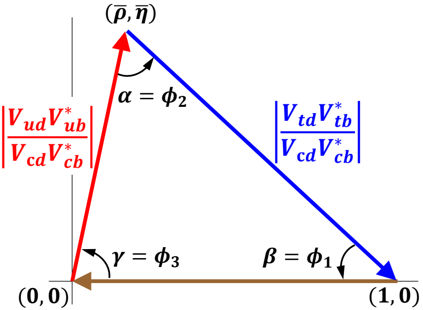
ε_K with lattice QCD inputs

Input Parameters $\bar{\rho}$ and $\bar{\eta}$: Wolfenstein Parameters

Unitarity Triangle $\rightarrow$ $(\bar{\rho},\ \bar{\eta})$



(a)



(b)

Global UT Fit and Angle-Only-Fit (AOF)

Global UT Fit

- Input: $|V_{ub}|/|V_{cb}|$, Δm_d, $\Delta m_s/\Delta m_d$, ε_K, and $\sin(2\beta)$.
- Determine the UT apex $(\bar{\rho}, \bar{\eta})$.
- Take λ from

$$|V_{us}| = \lambda + \mathcal{O}(\lambda^7)\,,$$

  which comes from K_{l3} and $K_{\mu 2}$.
- Disadvantage: unwanted correlation between $(\bar{\rho}, \bar{\eta})$ and ε_K.

AOF

- Input: $\sin(2\beta)$, $\cos(2\beta)$, $\sin(\gamma)$, $\cos(\gamma)$, $\sin(2\beta+\gamma)$, $\cos(2\beta+\gamma)$, and $\sin(2\alpha)$.
- Determine the UT apex $(\bar{\rho}, \bar{\eta})$.
- Take λ from $|V_{us}| = \lambda + \mathcal{O}(\lambda^7)\,,$ which comes from K_{l3} and $K_{\mu 2}$.
- Use $|V_{cb}|$ to determine A.

$$|V_{cb}| = A\lambda^2 + \mathcal{O}(\lambda^7)$$

- Advantage: NO correlation between $(\bar{\rho}, \bar{\eta})$ and ε_K.

Inputs of Angle-Only-Fit (AOF)

- $A_{\text{CP}}(J/\psi\ K_s) \to S_{\psi K_s} = \sin(2\beta)$ with assumption of $S_{\psi K_s} \ggg C_{\psi K_s}$.
- $(B \to DK) + (B \to [K\pi]_D K) +$ (Dalitz method) give $\sin(\gamma)$ and $\cos(\gamma)$.
- $S(D^-\pi^+)$ and $S(D^+\pi^-)$ give $\sin(2\beta+\gamma)$ and $\cos(2\beta+\gamma)$.
- $(B^0 \to \pi^+\pi^-) + (B^0 \to \rho^+\rho^-) + (B^0 \to (\rho\pi)^0)$ give $\sin(2\alpha)$.
- Combining all of these gives β, γ, and α, which leads to the UT apex $(\bar\rho,\ \bar\eta)$.

Wolfenstein Parameters

Input Parameters for Angle-Only-Fit (AOF)

- ε_K, $\hat{B}_K$, and $|V_{cb}|$ are used as inputs to determine the UT angles in the global fit of `UTfit` and `CKMfitter`.
- Instead, we can use angle-only-fit result for the UT apex $(\bar{\rho}, \bar{\eta})$.
- Then, we can take λ independently from

$$|V_{us}| = \lambda + \mathcal{O}(\lambda^7)\,,$$

  which comes from K_{l3} and $K_{\mu 2}$.
- Use $|V_{cb}|$ instead of A.

$$|V_{cb}| = A\lambda^2 + \mathcal{O}(\lambda^7)$$

λ	0.22500(24)	[1] CKMfitter 2021
	0.22500(100)	[2] UTfit 2018
	0.2249(5)	$\lvert V_{us}\rvert$ FLAG 2021
$\bar{\rho}$	0.1566(85)	[1] CKMfitter 2021
	0.1604(90)	[2] UTfit 2022
	0.146(22)	[3] UTfit-17 (AOF)
$\bar{\eta}$	0.3475(118)	[1] CKMfitter 2021
	0.3448(94)	[2] UTfit 2022
	0.333(16)	[3] UTfit-17 (AOF)

Input Parameter: $\hat{B}_K$

Input Parameter: $\hat{B}_K$ (FLAG 2021)

$\hat{B}_K$ in lattice QCD with $N_f = 2 + 1$.

Collaboration	Ref.	$\hat{B}_K$
SWME 2015	[4]	0.735(5)(36)
RBC/UKQCD 2014	[5]	0.7499(24)(150)
Laiho 2011	[6]	0.7628(38)(205)
BMW 2011	[7]	0.7727(81)(84)
FLAG 2021	[8]	0.7625(97)

Input Parameter $|V_{cb}|$

Input Parameter: Exclusive $|V_{cb}|$ in units of 1.0×10^{-3}

(a) Exclusive $|V_{cb}|$ (1.0×10^{-3})

channel	value	method	collaboration	Ref.
$\bar{B} \to D^* \ell \bar{\nu}$	39.0(2)(6)(6)	CLN	BELLE 2021	[9]
$\bar{B} \to D^* \ell \bar{\nu}$	38.9(3)(7)(6)	BGL	BELLE 2021	[9]
$\bar{B} \to D^* \ell \bar{\nu}$	38.40(84)	CLN	BABAR 2019	[10]
$\bar{B} \to D^* \ell \bar{\nu}$	38.36(90)	BGL	BABAR 2019	[10]
$\bar{B} \to D^* \ell \bar{\nu}$	38.57(78)	BGL	FNAL/MILC 2021[1]	[11]
ex-comb	39.48(68)	comb	FLAG 2021	[8]
ex-comb	39.25(56)	comb	HFLAV 2021	[12]
$\bar{B}_s \to D_s^* \ell \bar{\nu}$	41.4(6)(9)(12)	CLN	LHCb 2020	[13]
$\bar{B}_s \to D_s^* \ell \bar{\nu}$	42.3(8)(9)(12)	BGL	LHCb 2020	[13]

- Note that there is no difference between the CLN and BGL analyses within statistical and systematic uncertainty.

[1] They combined both BELLE and BABAR data in their analysis.

Input Parameter: Inclusive $|V_{cb}|$ in units of 1.0×10^{-3}

$|V_{cb}|$ in units of 1.0×10^{-3}.

(a) Exclusive $|V_{cb}|$

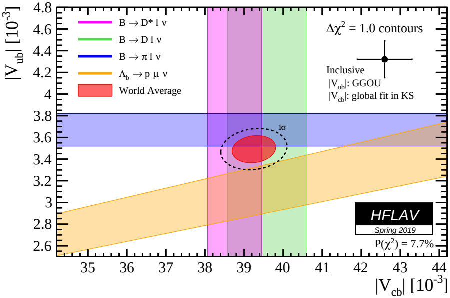
method	value	collaboration
CLN	39.0(2)(6)(6)	BELLE 2021
BGL	38.57(78)	FNAL/MILC 2021

(b) Inclusive $|V_{cb}|$

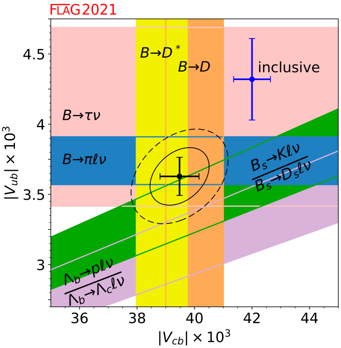
method	value	Ref.
kinetic scheme	42.16(51)	[14]
1S scheme	41.98(45)	[15]

- There is $3\sigma \sim 4\sigma$ difference in $|V_{cb}|$ between the exclusive and inclusive decay channels.
- This issue remains unresolved yet.

Current Status of $|V_{cb}|$ in 2022



(a) HFLAV 2021



(b) FLAG 2021

Input Parameter ξ_0

Input Parameter: ξ_0

Indirect Method

$$\xi_0 = \frac{\mathrm{Im} A_0}{\mathrm{Re}\, A_0}, \quad \xi_2 = \frac{\mathrm{Im} A_2}{\mathrm{Re}\, A_2}.$$

ξ_0	$-17.38(77) \times 10^{-5}$	RBC-UK-2021 [16]
ξ_2	$-5.64(70) \times 10^{-5}$	RBC-UK-2021 [16]

where $\mathcal{A}(K_0 \to \pi\pi(I)) \equiv A_I e^{i\delta_I} = |A_I| e^{i\xi_I} e^{i\delta_I}$

- RBC-UKQCD calculated $\mathrm{Im} A_2$: $\mathrm{Im} A_2 \to \xi_2 \to \varepsilon'_K/\varepsilon_K \to \xi_0$

$$\xi_0 = \xi_2 - \frac{\sqrt{2}|\varepsilon_K|}{\omega} \left(\frac{\varepsilon'_K}{\varepsilon_K} \right) \tag{1}$$

  Other inputs ω, ε_K and $\varepsilon'_K/\varepsilon_K$ are taken from the experimental values.
- Here, we choose an approximation of $\cos(\phi_{\epsilon'} - \phi_\epsilon) \approx 1$.
- $\phi_\epsilon = 43.52(5)$, $\phi_{\epsilon'} = 42.3(1.5)$
- Isopspin breaking effect: (at most 15% of ξ_0) $\to$ (1% in ε_K) $\to$ neglected!

Input Parameter: ξ_0

Direct Method

- RBC-UKQCD calculated $\mathrm{Im} A_0$. $\mathrm{Im} A_0 \to \xi_0$.

$$\xi_0 = \frac{\mathrm{Im} A_0}{\mathrm{Re}\, A_0} = -21.02(472) \times 10^{-5}$$

Other input $\mathrm{Re}\, A_0$ is taken from the experimental value.

- Here, we use the indirect method to determine ξ_0.

Input Parameter: ξ_0

Summary

Input Parameters: ξ_0

Method	Value	Ref.
Indirect	$-17.38(77) \times 10^{-5}$	RBC-UK-2021 [16]
Direct	$-21.02(472) \times 10^{-5}$	RBC-UK-2021 [16]

- Here, we use the results for ξ_0 obtained using the indirect method.

Input Parameter ξ_{LD}

Input Parameter: ξ_{LD}

$$\xi_{\text{LD}} = \frac{m'_{\text{LD}}}{\sqrt{2}\,\Delta M_K}$$

$$m'_{\text{LD}} = -\mathrm{Im}\left[\mathcal{P}\sum_{C} \frac{\langle \overline{K}^0 | H_{\text{w}} | C \rangle \langle C | H_{\text{w}} | K^0 \rangle}{m_{K^0} - E_C}\right]$$

- RBC-UKQCD rough estimate [PRD 88, 014508] gives

$$\xi_{\text{LD}} = (0 \pm 1.6)\% \quad \text{of } |\varepsilon_K|$$

- BGI estimate [PLB 68, 309, 2010] gives

$$\xi_{\text{LD}} = -0.4(3) \times \frac{\xi_0}{\sqrt{2}}$$

- Precision measurement of lattice QCD is not available yet.

Input Parameters m_c and m_t, charm and top quark masses

Input Parameter: charm quark mass $m_c(m_c)$

$m_c(m_c)$ in lattice QCD.

Collaboration	N_f	$m_c(m_c)$	Ref.
FLAG 2021	$2+1$	1.275(5)	[8]
FLAG 2021	$2+1+1$	1.278(13)	[8]

- The results for $m_c(m_c)$ with $N_f = 2+1+1$ are inconsistent with each other.
- Hence, we use the results for $m_c(m_c)$ with $N_f = 2+1$.

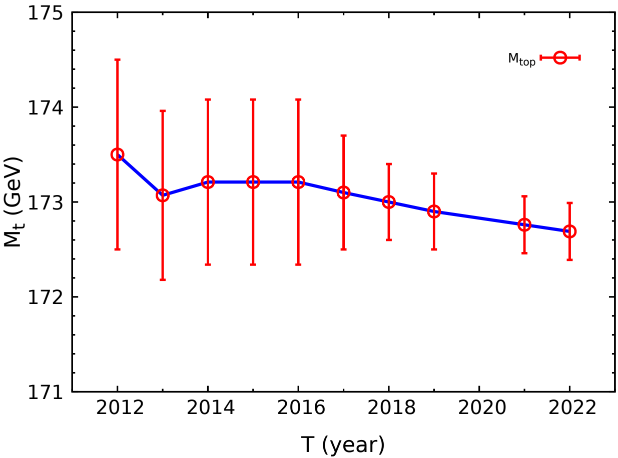
Input Parameter: top quark mass $m_t(m_t)$

$m_t(m_t)$ in the $\overline{\text{MS}}$ scheme in units of GeV.

Collaboration	M_t	$m_t(m_t)$	Ref.
PDG 2016	173.5 ± 1.1	$163.65 \pm 1.05 \pm 0.17$	[17]
PDG 2018	173.0 ± 0.4	$163.17 \pm 0.38 \pm 0.17$	[18]
PDG 2019	172.9 ± 0.4	$163.08 \pm 0.38 \pm 0.17$	[18]
PDG 2021	172.76 ± 0.30	$162.96 \pm 0.28 \pm 0.17$	[19]
PDG 2022	172.69 ± 0.30	$162.90 \pm 0.28 \pm 0.17$	[20]

- M_t is the pole mass of top quarks.
- CMS and ATLAS have done a great job in reducing the error.
- Here, we use the results for $m_t(m_t)$ obtained from PDG 2022.

Input Parameter: top quark pole mass M_t



- CMS and ATLAS have done a great job in reducing the error !!!

Input Parameter m_W, W-boson mass

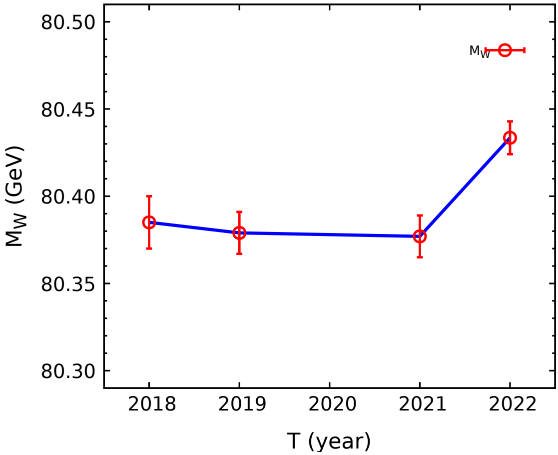
Input Parameter: W-boson mass m_W

W boson mass m_W in units of GeV.

Collaboration	m_W	Ref.
PDG 2022	80.377 ± 0.012	[20]
CDF 2022	80.4335 ± 0.0094	[21]

- We find some tension ($\sim 3.7\sigma$) between PDG and CDF.
- Here, we use results of CDF 2022.

Input Parameter: m_W history



- We find that ε_K decreases as m_W gets heavier.
- This means that $\Delta\varepsilon_K$ increases with heavier m_W.

Other Input Parameters

Other Input Parameters

Other input parameters.

Input Par.	Value	Ref.
G_F	$1.1663787(6) \times 10^{-5}\,\mathrm{GeV}^{-2}$	PDG 22 [20]
θ	$43.52(5)^\circ$	PDG 22 [20]
m_{K^0}	$497.611(13)\,\mathrm{MeV}$	PDG 22 [20]
ΔM_K	$3.484(6) \times 10^{-12}\,\mathrm{MeV}$	PDG 22 [20]
F_K	$155.7(3)\,\mathrm{MeV}$	FLAG 21 [8]

Higher order QCD corrections: η_{ij}.

Input	Value	Ref.
η_{cc}	1.72(27)	[22]
η_{tt}	0.5765(65)	[23]
η_{ct}	0.496(47)	[24]

Comment on η_{cc}

- Poor convergence in η_{cc}:

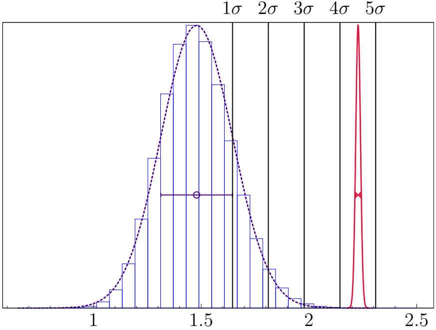
$$\begin{aligned}\eta_{cc} &= 1_{(\mathrm{LO})} + 0.37_{(\mathrm{NLO})} + 0.36_{(\mathrm{NNLO})} + (\mathrm{NNNLO}) \\ &= 1.72 \pm 0.27\end{aligned}$$

- We do not know the size of NNNLO but know that the sign is very likely to be positive.
- Then, it will decrease $|\varepsilon_K|^{\mathrm{SM}}_{\mathrm{Latt}}$ further, and increase the gap $\Delta\varepsilon_K$ more.
- Ultimately, it would be nice to calculate η_{cc} using tools in lattice QCD.
- Other strategy to get around the problem is proposed in Ref. [25].

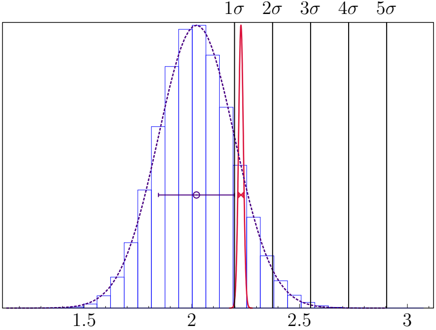
Results for ε_K

ε_K from Exclusive $|V_{cb}|$ (FNAL-MILC 2021)

RBC-UKQCD estimate for ξ_{LD}



Exclusive $|V_{cb}|$ (FNAL-21)



Inclusive $|V_{cb}|$ (1S)

- With exclusive $|V_{cb}|$ (FNAL-MILC 2021), it has 4.5σ tension.

$$|\varepsilon_K|^{\text{Exp}} = (2.228 \pm 0.011) \times 10^{-3}$$
$$|\varepsilon_K|^{\text{SM}}_{\text{excl}} = (1.479 \pm 0.166) \times 10^{-3}$$

Current Status of ε_K

- FLAG 2021 + PDG 2022: (in units of 1.0×10^{-3}, AOF)

$$|\varepsilon_K|^{\rm SM}_{\rm excl} = 1.479 \pm 0.166 \quad \text{for Exclusive } |V_{cb}| \text{ (Lattice QCD + BGL)}$$
$$|\varepsilon_K|^{\rm SM}_{\rm incl} = 2.022 \pm 0.176 \quad \text{for Inclusive } |V_{cb}| \text{ (Heavy Quark Expansion)}$$

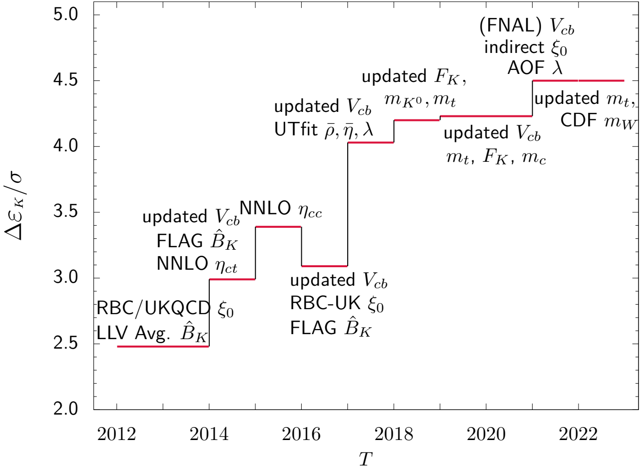
- Experiments:

$$|\varepsilon_K|^{\rm Exp} = 2.228 \pm 0.011$$

- Hence, we observe $4.5\sigma \sim 3.7\sigma$ difference between the SM theory (Lattice QCD) and experiments.

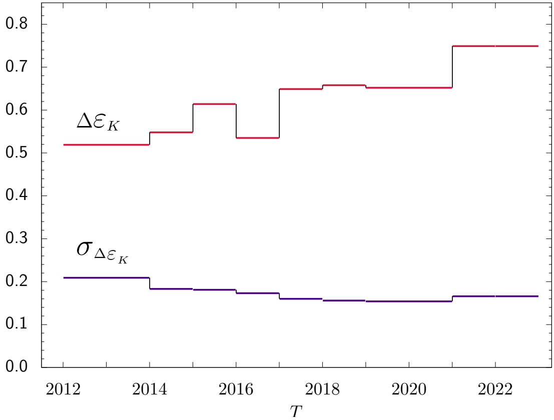
- What does this mean? $\longrightarrow$ Breakdown of SM ?

Time Evolution of $\Delta\varepsilon_K$ on the Lattice



- $\Delta\varepsilon_K \equiv |\varepsilon_K|^{\text{Exp}} - |\varepsilon_K|^{\text{SM}}_{\text{excl}}$
- We use exclusive $|V_{cb}|$ (BGL) and ξ_{LD} (RBC-UK).

Time Evolution of Average and Error for $\Delta\varepsilon_K$



- The average $\Delta\varepsilon_K$ has increased by 44% with some fluctuations.
- The error $\sigma_{\Delta\varepsilon_K}$ has decreased by 20% with some fluctuations.

Error Budget of $\Delta\varepsilon_K$: excl. $|V_{cb}|$ (BGL), ξ_{LD} (RBC-UK)

source	error (%)	memo
$\|V_{cb}\|$	44.7	Exclusive
$\bar{\eta}$	21.4	AOF
η_{ct}	18.1	$c-t$ Box
η_{cc}	7.7	$c-c$ Box
$\bar{\rho}$	3.2	AOF
ξ_{LD}	1.9	Long-distance
$\hat{B}_K$	1.5	FLAG
η_{tt}	0.61	$t-t$ Box
ξ_0	0.54	Indirect
λ	0.18	$\|V_{us}\|$ (PDG)
$\vdots$	$\vdots$	$\vdots$

- The error from $|V_{cb}|$ is dominant.

To Do List

- It would be nice to reduce overall errors on $|V_{cb}|$ down to $1/2$ of the current values.
  [OK action project: LANL-SWME report in Lattice 2022]
- It would be nice to reduce overall errors on $\bar{\eta}$. [BELLE2]
- It would be nice to reduce overall errors on ξ_0 and ξ_2 in lattice QCD. [RBC-UKQCD]
- It would be nice to reduce overall errors on $|V_{us}|$, $m_c(m_c)$, f_K in lattice QCD.

Summary and Conclusion

Summary

1. We find that

$$\Delta\varepsilon_K^{\text{excl}} = 4.5(3)\sigma \qquad \text{(Lattice QCD, CLN/BGL)} \qquad (2)$$

$$\Delta\varepsilon_K^{\text{incl}} = 1.2\sigma \qquad \text{(HQE, QCD Sum Rules)} \qquad (3)$$

2. We find that the results of CLN are consistent with those of BGL for the $\bar{B} \to D^*\ell\bar{\nu}$ decays at present. (Good news !!!)

3. Meanwhile, it would be very helpful to reduce the errors for $|V_{cb}|$, $|V_{us}|$, $\bar{\eta}$, ξ_0, ξ_2, $m_c(m_c)$, f_K, $\hat{B}_K$, and ξ_{LD} in lattice QCD.
$\bar{\eta} \quad \longleftarrow \quad \xi,\ f_{B_d},\ f_{B_s},\ B_{B_d},\ B_{B_s}, \cdots$

4. Please stay tuned for the update.

My personal opinion

- The Scripture says in [Isaiah 55:8-9] that
  8. "For my thoughts are not your thoughts, neither are your ways my ways," declares the LORD.
  9. "As the heavens are higher than the earth, so are my ways higher than your ways and my thoughts than your thoughts."
- I suspect that some of the fundamental postulates of the standard model (SM) might be wrong even in quark sector,
- even though we do not know yet which one may deviate from the regular orbit.

Thank God for your help !!!

Backup Slides

CKM matrix elements

Charged Current Lagrangian in Quark Sector of the SM

$$\mathcal{L}_W = \frac{g_w}{\sqrt{2}} \sum_{i=1,2,3} \sum_{k=1,2,3} \left[V_{jk} \bar{u}_{jL} \gamma^\mu d_{kL} W_\mu^+ + V_{jk}^* \bar{d}_{kL} \gamma^\mu u_{jL} W_\mu^- \right]$$

where

$$u_j = \begin{pmatrix} u \\ c \\ t \end{pmatrix}, \qquad d_k = \begin{pmatrix} d \\ s \\ b \end{pmatrix}$$

and

$$V_{jk} = \begin{pmatrix} V_{ud} & V_{us} & V_{ub} \\ V_{cd} & V_{cs} & V_{cb} \\ V_{td} & V_{ts} & V_{tb} \end{pmatrix}$$

CKM matrix elements

Standard Parametrization:

$$V = \begin{pmatrix} c_{12}c_{13} & s_{12}c_{13} & s_{13}e^{-i\delta} \\ -s_{12}c_{23} - c_{12}s_{23}s_{13}e^{i\delta} & c_{12}c_{23} - s_{12}s_{23}s_{13}e^{i\delta} & s_{23}c_{13} \\ s_{12}s_{23} - c_{12}c_{23}s_{13}e^{i\delta} & -s_{23}c_{12} - s_{12}c_{23}s_{13}e^{i\delta} & c_{23}c_{13} \end{pmatrix}$$

Wolfenstein Parametrization:

$$s_{12} = \lambda, \qquad s_{23} = A\lambda^2, \qquad s_{13} = A\lambda^3\sqrt{\rho^2 + \eta^2},$$

$$V = \begin{pmatrix} 1 - \frac{\lambda^2}{2} & \lambda & A\lambda^3(\rho - i\eta) \\ -\lambda & 1 - \frac{\lambda^2}{2} & A\lambda^2 \\ A\lambda^3(1 - \rho - i\eta) & -A\lambda^2 & 1 \end{pmatrix} + \mathcal{O}(\lambda^4)$$

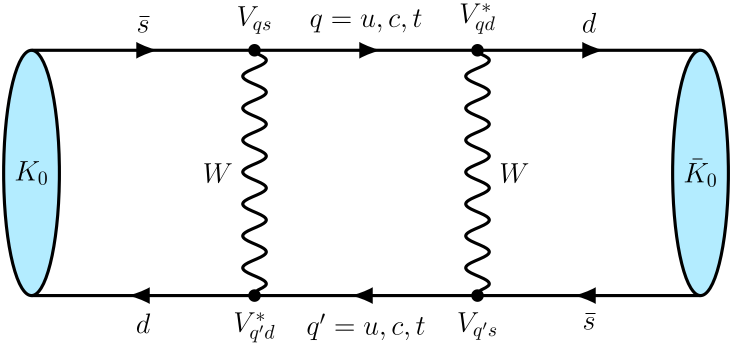
where $\lambda = |V_{us}| \cong 0.22, \quad A \cong 0.83, \quad \rho \cong 0.16, \quad \eta \cong 0.35$

[Meson]–[Anti-Meson] Mixing

$M - \overline{M}$ Mixing

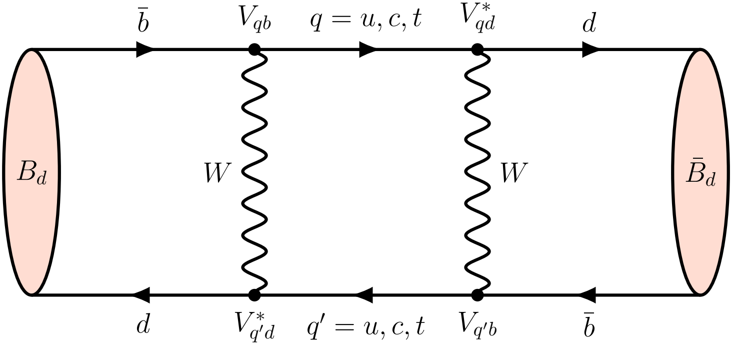
- It is possible only for the 4 neutral mesons.
- $K_0 \cong \bar{s}d \longleftrightarrow \overline{K}_0 \cong s\bar{d}$ $\qquad 497.611(13)\,\mathrm{MeV} \cong 0.5\,\mathrm{GeV}$
- $D_0 \cong c\bar{u} \longleftrightarrow \overline{D}_0 \cong \bar{c}u$ $\qquad 1864.83(5)\,\mathrm{MeV} \cong 1.9\,\mathrm{GeV}$
- $B_d \cong \bar{b}d \longleftrightarrow \overline{B}_d \cong b\bar{d}$ $\qquad 5279.64(13)\,\mathrm{MeV} \cong 5.3\,\mathrm{GeV}$
- $B_s \cong \bar{b}s \longleftrightarrow \overline{B}_s \cong b\bar{s}$ $\qquad 5366.88(17)\,\mathrm{MeV} \cong 5.4\,\mathrm{GeV}$

$K_0 - \overline{K}_0$ Mixing



- This is the main topic of the talk.
- Hence, we will discuss it later.

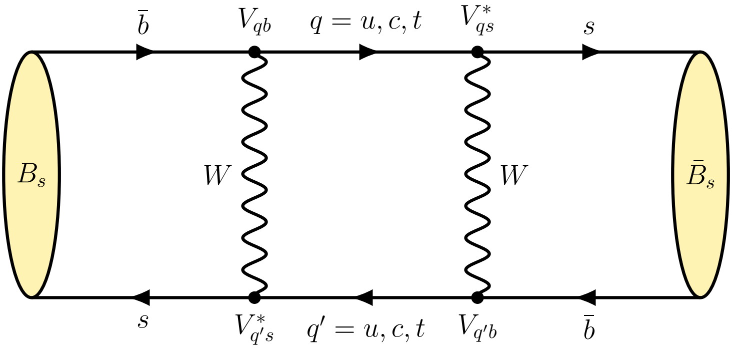
$B_d - \overline{B}_d$ Mixing



- $t - t$ box $\rightarrow x_t(V_{tb}V^*_{td})^2 \cong x_t A^2\lambda^6(1-\rho+i\eta)^2$ with $x_t = (m_t/m_W)^2$
- $c - c$ box $\rightarrow x_c(V_{cb}V^*_{cd})^2 \cong x_c A^2\lambda^6 \cong \frac{1}{16000} \times$ [$t - t$ box]
- $c - t$ box $\rightarrow \sqrt{x_c x_t}(V_{cb}V^*_{cd} \cdot V_{tb}V^*_{td}) \cong -\sqrt{x_c x_t}A^2\lambda^6(1-\rho+i\eta)$

$$\Delta m_d = \frac{G_F^2}{6\pi^2} M_{B_d} f_{B_d}^2 \hat{B}_{B_d} M_W^2 S(x_t)(V_{tb}V^*_{td})^2$$

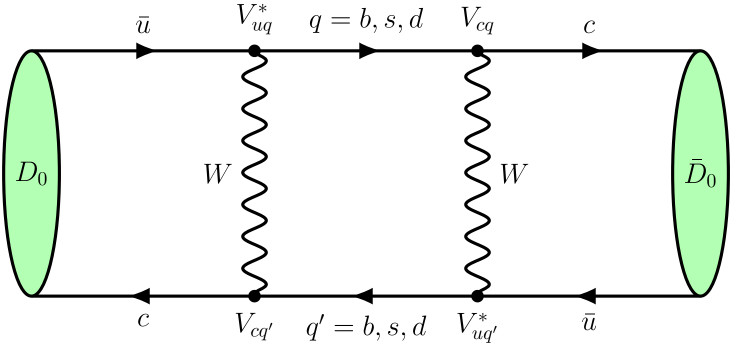
$B_s - \overline{B}_s$ Mixing



- $t - t$ box $\rightarrow x_t(V_{tb}V^*_{ts})^2 \cong x_t A^2\lambda^4$ with $x_t = (m_t/m_W)^2$
- $c - c$ box $\rightarrow x_c(V_{cb}V^*_{cs})^2 \cong x_c A^2\lambda^4 \cong \frac{1}{16000} \times [t - t \text{ box}]$
- $c - t$ box $\rightarrow \sqrt{x_c x_t}(V_{cb}V^*_{cs} \cdot V_{tb}V^*_{ts}) \cong -\sqrt{x_c x_t}A^2\lambda^4$

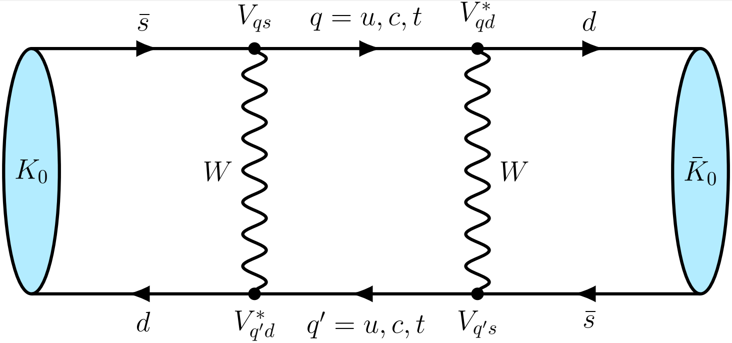
$$\Delta m_s = \frac{G_F^2}{6\pi^2} M_{B_s} f_{B_s}^2 \hat{B}_{B_s} M_W^2 S(x_t)(V_{tb}V^*_{ts})^2$$

$D_0 - \overline{D}_0$ Mixing



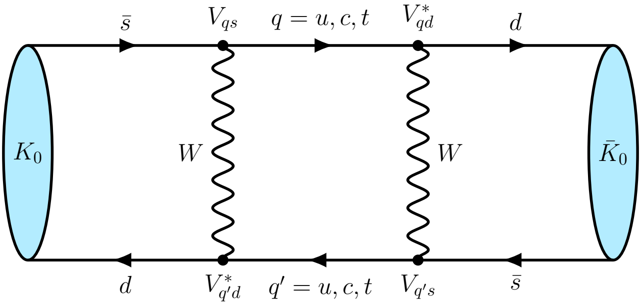
- $b - b$ box $\rightarrow x_b(V_{cb}V_{ub}^*)^2 \cong x_b A^4 \lambda^{10}(\rho + i\eta)^2$ with $x_b = (m_b/m_W)^2$
- $s - s$ box $\rightarrow x_s(V_{cs}V_{us}^*)^2 \cong x_s\lambda^2 \cong 200 \times [b - b \text{ box}]$
- $d - d$ box $\rightarrow x_d(V_{cd}V_{ud}^* \cdot V_{cd}V_{ud}^*) \cong x_d\lambda^2 \cong [b - b \text{ box}]$
- Hence, the long distance effect from the $s - s$ box becomes dominant and important. $\rightarrow$ Very tough in lattice QCD.

ΔM_K : Real Part of $K_0 - \overline{K}_0$ Mixing



- $t - t$ box $\rightarrow x_t(V_{ts}V^*_{td})^2 \cong x_t A^4 \lambda^{10}(1 - \rho + i\eta)^2$ with $x_t = (m_t/m_W)^2$
- $c - c$ box $\rightarrow x_c(V_{cs}V^*_{cd})^2 \cong x_c \lambda^2 \cong 25\times\ \mathrm{Re}[t - t$ box$]$
- $u - u$ box $\rightarrow x_u(V_{us}V^*_{ud})^2 \cong x_u \lambda^2 \cong \frac{1}{2800}\times\ \mathrm{Re}[t - t$ box$]$
- Hence, the $c - c$ box becomes dominant. Hence, the long distance effect ($\approx 30\%$) becomes important.

ε_K : Imaginary Part of $K_0 - \overline{K}_0$ Mixing



- $t - t \to x_t \mathrm{Im}(V_{ts} V_{td}^*)^2 \cong 2 x_t A^4 \lambda^{10} (1 - \rho)\eta$ with $x_t = (m_t/m_W)^2$
- $c - c \to x_c \mathrm{Im}(V_{cs} V_{cd}^*)^2 \cong -2 x_c A^2 \lambda^6 \eta \cong -\frac{1}{25} \times \mathrm{Re}[t - t \text{ box}]$
- $c - t \to 2\sqrt{x_c x_t}\, \mathrm{Re}(V_{cs} V_{cd}^*) \mathrm{Im}(V_{ts} V_{td}^*) \cong 2\sqrt{x_c x_t} A^2 \lambda^6 \eta \cong +\frac{1}{5} \times \mathrm{Re}[t - t \text{ box}]$
- Hence, the $t - t$ box is dominant (86%), the $c - t$ box is sub-dominant (17%), and the $c - c$ box is small and negative (−3.4%).

Input Parameter: Exclusive $|V_{cb}|$ in units of 1.0×10^{-3}

(a) Exclusive $|V_{cb}|$ (1.0×10^{-3})

channel	value	method	collaboration	Ref.
$\bar{B} \to D^* \ell \bar{\nu}$	39.0(2)(6)(6)	CLN	BELLE 2021	[9]
$\bar{B} \to D^* \ell \bar{\nu}$	38.9(3)(7)(6)	BGL	BELLE 2021	[9]
$\bar{B} \to D^* \ell \bar{\nu}$	38.40(84)	CLN	BABAR 2019	[10]
$\bar{B} \to D^* \ell \bar{\nu}$	38.36(90)	BGL	BABAR 2019	[10]
$\bar{B} \to D^* \ell \bar{\nu}$	38.57(78)	BGL	FNAL/MILC 2021[2]	[11]
ex-comb	39.48(68)	comb	FLAG 2021	[8]
ex-comb	39.25(56)	comb	HFLAV 2021	[12]
$\bar{B}_s \to D_s^* \ell \bar{\nu}$	41.4(6)(9)(12)	CLN	LHCb 2020	[13]
$\bar{B}_s \to D_s^* \ell \bar{\nu}$	42.3(8)(9)(12)	BGL	LHCb 2020	[13]

- There is no difference between the CLN and BGL analyses.
- Refer to BABAR 2019 [26] and BELLE 2019 [27].
- Hence, the CLN method turns out to be consistent with BGL within our limited knowledge.

[2] They combined both BELLE and BABAR data in their analysis.

$\bar{B} \to D^*\ell\bar{\nu}$ Form Factor Parametrization: CLN vs. BGL I

- Consider the $\bar{B} \to D^*\ell\bar{\nu}$ decays:

$$\frac{d\Gamma(\bar{B} \to D^*\ell\bar{\nu})}{dw} = \frac{G_F^2 m_{D^*}^3}{48\pi^3}(m_B - m_{D^*})^2\ \chi(w)\ \eta_{\mathrm{EW}}^2\ \mathcal{F}^2(w)\ |V_{cb}|^2$$

- In order that the experiments determine $|\mathcal{F}(w)| \cdot |V_{cb}|$, they must know a specific functional form of $\mathcal{F}(w)$.
- The theory provides the functional form and parametrization for $\mathcal{F}(w)$.
- Popular parametrizations are CLN and BGL.
- CLN depends on the HQET, but BGL does NOT.
- HQET is the heavy quark effective theory, as if the chiral perturbation theory is the low energy effective theory of QCD.

$\bar{B} \to D^*\ell\bar{\nu}$ Form Factor Parametrization: CLN vs. BGL II

- CLN: Caprini, Lellouch, and Neubert [28]

$$\mathcal{F}(w) = h_{A_1}(w) \times \frac{1}{Y(w)} \times X(w)$$

$$h_{A_1}(w) = h_{A_1}(1)\left[1 - 8\rho^2 z + (53\rho^2 - 15)z^2 - (231\rho^2 - 91)z^3\right]$$

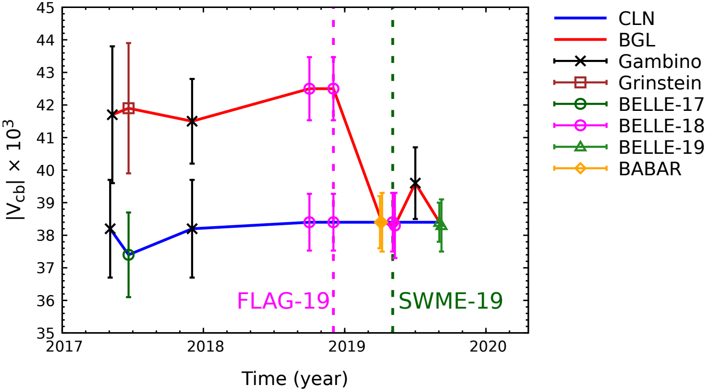
$$z = \frac{\sqrt{w+1} - \sqrt{2}}{\sqrt{w+1} + \sqrt{2}}, \qquad w \equiv v_B \cdot v_{D^*} = \frac{E_{D^*}}{m_{D^*}}$$

where z is a conformal mapping variable. $\to$ z expansion.

- BGL: Boyd, Grinstein, and Lebed [29]

$$\mathcal{F}(w) = \frac{1}{\phi(z)P(z)} \sum_{n=0}^{\infty} a_n z^n(z)$$

CLN vs. BGL in $\bar{B} \to D^* \ell \bar{\nu}$ decays



- At present, we find that BGL is consistent with CLN.
  $\Longrightarrow$ Resolved ???

CLN vs. BGL: Martin Jung's claim in 19/08 INT workshop

- BELLE 2019 used $\mathrm{BGL}_{(102)}$ fit (6 parameters).
- Martin used $\mathrm{BGL}_{(222)}$ fit (9 parameters) [PLB795 (2019) 386].
- χ^2/d.o.f. $= 32.5/35$ ($\mathrm{BGL}_{(102)}$) and $31.2/32$ ($\mathrm{BGL}_{(222)}$).
  $\rightarrow$ No distinction !!!
- Martin claims that the correct error for $|V_{cb}|$ is 50% larger than that of BELLE 2019.
- Martin also suggested that the slope and curvature of $R_1(w)$ and $R_2(w)$ at zero recoil should be calculated in lattice QCD.

Input Parameter: Inclusive $|V_{cb}|$ in units of 1.0×10^{-3}

$|V_{cb}|$ in units of 1.0×10^{-3}.

(a) Exclusive $|V_{cb}|$

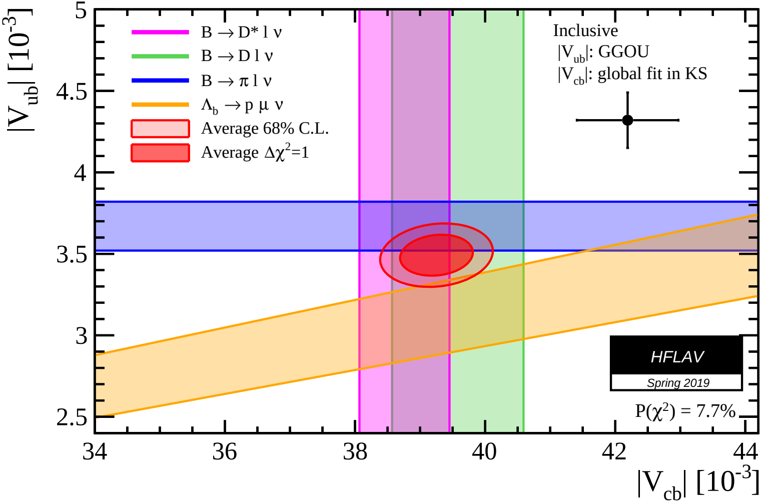
method	value	collaboration
CLN	39.0(2)(6)(6)	BELLE 2021
BGL	38.57(78)	FNAL/MILC 2021

(b) Inclusive $|V_{cb}|$

method	value	Ref.
kinetic scheme	42.16(51)	[14]
1S scheme	41.98(45)	[15]

- There is $3\sigma \sim 4\sigma$ difference in $|V_{cb}|$ between the exclusive and inclusive decay channels.
- This issue remains unresolved yet.

Current Status of $|V_{cb}|$ in 2019



$R(D)$ and $R(D^*)$

$R(D)$ and $R(D^*)$

- Definition:

$$R(D) \equiv \frac{\mathcal{B}(B \to D\tau\nu_\tau)}{\mathcal{B}(B \to D\ell\nu_\ell)}, \qquad R(D^*) \equiv \frac{\mathcal{B}(B \to D^*\tau\nu_\tau)}{\mathcal{B}(B \to D^*\ell\nu_\ell)}$$

- Results of HFLAV 2017

channel	SM (Lattice QCD)	Experiment	Difference
$R(D)$	0.300(8)	0.403(40)(24)	2.2σ
$R(D^*)$	0.252(3)	0.310(15)(8)	3.4σ

- Results of HFLAV 2019 (Preliminary)

channel	SM (Lattice QCD)	Experiment	Difference
$R(D)$	0.299(3)	0.340(27)(13)	1.4σ
$R(D^*)$	0.258(5)	0.295(11)(8)	2.5σ

Calculation of $R(D^*)$ and $R(D)$ I

- We can calculate the semi-leptonic form factors of the $\bar{B} \to D^{(*)}\ell\bar{\nu}$ decays using tools in lattice QCD.

- Then, we can obtain the form factors $\mathcal{F}(w)$ and $\mathcal{G}(w)$ for the full range of w (the recoil parameter) using CLN, BGL, and BCL.

- The recoil parameter w is

$$w = v_B \cdot v_D = \frac{m_B^2 + m_D^2 - q^2}{2m_B m_D} = \sqrt{1 + v_D^2}$$

- $q^2 \in [m_\ell^2, (m_B - m_D)^2] \to w \in [1, x_\ell]$, where $x_\ell = \dfrac{m_B^2 + m_D^2 - m_\ell^2}{2m_B m_D}$

Calculation of $R(D^*)$ and $R(D)$ II

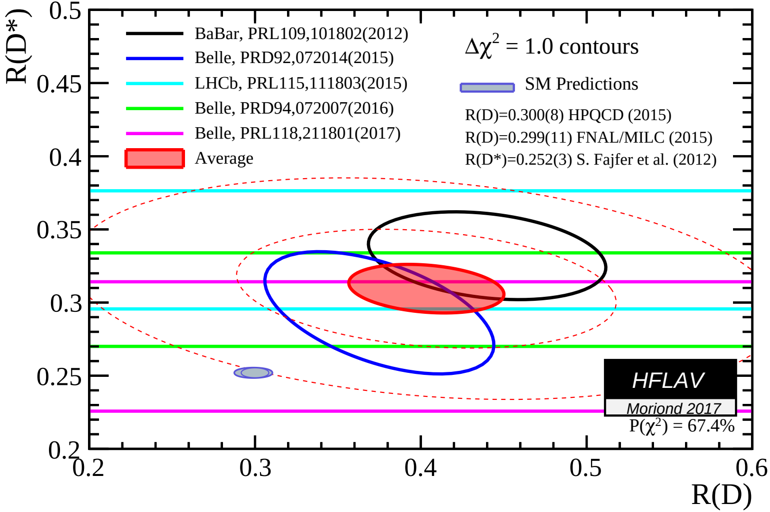
- How to calculate $R(D^*)$:

$$R(D^*) = \frac{\mathcal{B}(B \to D^* \tau \nu_\tau)}{\mathcal{B}(B \to D^* \ell \nu_\ell)} = \frac{\displaystyle\int_1^{x_\tau} dw \left[\frac{d\Gamma}{dw}(w, m_\tau)\right]}{\displaystyle\int_1^{x_\ell} dw \left[\frac{d\Gamma}{dw}(w, m_\ell)\right]},$$

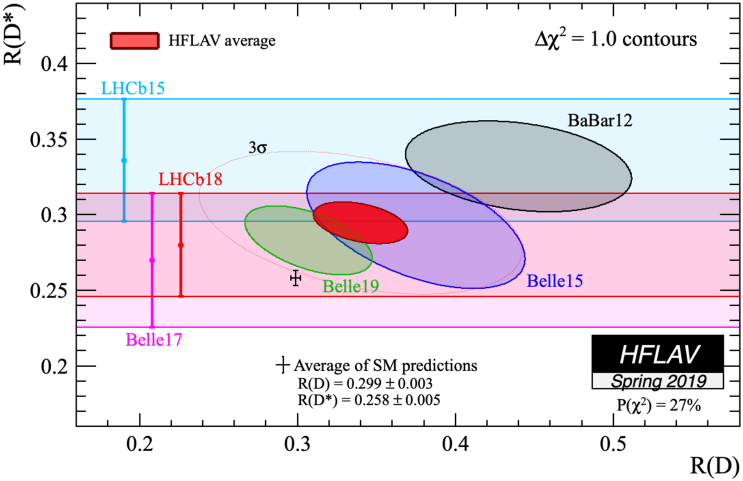
where $\ell = \{e,\ \mu\}$, $x_\tau = 1.355$, $x_\mu = 1.503$, and

$$\frac{d\Gamma}{dw} = \frac{G_F^2 M_{D^*}^3}{4\pi^3}(M_B - M_{D^*})^2 \sqrt{w^2 - 1}\ |\eta_{\mathrm{EM}}|^2\, |V_{cb}|^2\, \chi(w)\, |\mathcal{F}(w)|^2$$

$R(D)$ and $R(D^*)$ (2017)



$R(D)$ and $R(D^*)$ (2019, preliminary)



$|V_{cb}|$ on the lattice

Why the OK action?

Calculation of $|V_{cb}|$ on the lattice

1. Exclusive $\bar{B} \to D^* \ell \bar{\nu}$ at zero recoil [Fermilab-MILC (2014), HPQCD (2018)]
   - Gold-plated: most precise in experimental and lattice errors.
   - Form factor calculation using the 3-point function $\langle D^* | A^\mu | B \rangle$ on the lattice.
2. Exclusive $\bar{B} \to D \ell \bar{\nu}$ at non-zero recoil [Fermilab-MILC (2015), HPQCD (2015)]
   - Near the zero recoil, the experimental precision is poor due to to the phase space suppression.
   - Form factor calculation using the 3-point function $\langle D | V^\mu | B \rangle$ on the lattice.
3. Inclusive $B \to X_c \ell \bar{\nu}$ [S. Hashimoto (2017)]
   - Preliminary, Calculate the 4-point function on the lattice,

$$\langle B | T\{J_\mu^\dagger(q) J_\nu(0)\} | B \rangle, \qquad \text{where } J_\mu = \bar{c}\gamma_\mu(1-\gamma_5)b.$$

Decay mode	Method	$\lvert V_{cb}\rvert \times 10^3$ [HFLAV (2017)]
$\bar{B} \to D^* \ell \bar{\nu}$	Lattice	39.05(47)(58)
$\bar{B} \to D \ell \bar{\nu}$	Lattice	39.18(94)(36)
$B \to X_c \ell \bar{\nu}$	QCD sum rule	42.03(39)

Limitation of Fermilab action calculation

- On the lattice, we have **discretization error** by construction.
- For the $\bar{B} \to D^{(*)}\ell\bar{\nu}$ study, the heavy quark discretization error (HQDE) for charm quark is dominant. ($\lambda \sim \Lambda/2m_Q$)
- The Fermilab action calculation of h_{A_1} ($\bar{B} \to D^*\ell\bar{\nu}$ semileptonic form factor) has $\mathcal{O}(\alpha_s\lambda^2)$ and $\mathcal{O}(\lambda^3) \sim 1\%$ discretization error.
- To achieve a sub-percent ($< 1\%$) precision, we have to use new action: **Oktay-Kronfeld action**, $\mathcal{O}(\lambda^3)$ improved action where its discretization error appears at $\mathcal{O}(\lambda^4) \sim 0.2\%$.

Limitation of Fermilab action calculation

- We expect the improvement in charm quark discretization error from the Fermilab/MILC results [PRD89, 114504 (2014) and PRD92, 034506 (2015)] for the $\bar{B} \to D^{(*)}\ell\bar{\nu}$ semileptonic form factors.

form factor	h_{A_1}	f_+
decay channel	$\bar{B} \to D^*\ell\bar{\nu}$	$\bar{B} \to D\ell\bar{\nu}$
statistics	0.4	0.7
matching	0.4	0.7
χPT	0.5	0.6
$g_{D^*D\pi}$	0.3	-
c discretization	$1.0 \to (0.2)_{\mathrm{OK}}$	$0.4 \to (0.1)_{\mathrm{OK}}$
others	0.1	0.2
total	$1.4 \to (0.8)_{\mathrm{OK}}$	$1.2 \to (1.1)_{\mathrm{OK}}$

- BELLE2 has been running since April, 2019, and the target statistics is 50 times larger than BELLE.

The OK action

OK Action (mass form)

$$S_{\rm OK} = S_{\rm Fermilab} + S_{\rm new}\,, \qquad S_{\rm Fermilab} = S_0 + S_B + S_E$$

$$\begin{aligned} S_0 &= m_0 \sum_x \bar\psi(x)\psi(x) + \sum_x \bar\psi(x)\gamma_4 D_4 \psi(x) - \tfrac{1}{2} a \sum_x \bar\psi(x)\triangle_4\psi(x) \\ &\quad + \zeta \sum_x \bar\psi(x)\vec\gamma\cdot\vec D\psi(x) - \tfrac{1}{2} r_s \zeta a \sum_x \bar\psi(x)\triangle^{(3)}\psi(x) \\ &= \mathcal{O}(1) + \mathcal{O}(\lambda) \qquad [\lambda \sim a\Lambda,\, \Lambda/m_Q] \\ S_B &= -\tfrac{1}{2} c_B \zeta a \sum_x \bar\psi(x) i\vec\Sigma\cdot\vec B\psi(x) \to \mathcal{O}(\lambda) \\ S_E &= -\tfrac{1}{2} c_E \zeta a \sum_x \bar\psi(x)\vec\alpha\cdot\vec E\psi(x) \to \mathcal{O}(\lambda^2) \qquad (c_E \neq c_B :\ \text{OK action}) \\ m_0 &= \frac{1}{2\kappa_t} - (1 + 3 r_s\zeta + 18 c_4) \end{aligned}$$

[M. B. Oktay and A. S. Kronfeld, PRD **78**, 014504 (2008)]

[A. El-Khadra, A. S. Kronfeld and P. B. Mackenzie, PRD **55**, 3933 (1997)]

OK Action (mass form)

$$
\begin{aligned}
S_{\text{new}} = \mathcal{O}(\lambda^3) = {} & c_1 a^2 \sum_x \bar{\psi}(x) \sum_i \gamma_i D_i \triangle_i \psi(x) \\
& + c_2 a^2 \sum_x \bar{\psi}(x) \{ \vec{\gamma} \cdot \vec{D}, \triangle^{(3)} \} \psi(x) \\
& + c_3 a^2 \sum_x \bar{\psi}(x) \{ \vec{\gamma} \cdot \vec{D}, i \vec{\Sigma} \cdot \vec{B} \} \psi(x) \\
& + c_{EE} a^2 \sum_x \bar{\psi}(x) \{ \gamma_4 D_4, \vec{\alpha} \cdot \vec{E} \} \psi(x) \\
& + c_4 a^3 \sum_x \bar{\psi}(x) \sum_i {\triangle_i}^2 \psi(x) \\
& + c_5 a^3 \sum_x \bar{\psi}(x) \sum_i \sum_{j \neq i} \{ \mathrm{i} \Sigma_i B_i, \triangle_j \} \psi(x)
\end{aligned}
$$

Inconsistency Parameter

Improvement Test: Inconsistency Parameter

$$I \equiv \frac{2\delta M_{\overline{Q}q} - (\delta M_{\overline{Q}Q} + \delta M_{\overline{q}q})}{2M_{2\overline{Q}q}} = \frac{2\delta B_{\overline{Q}q} - (\delta B_{\overline{Q}Q} + \delta B_{\overline{q}q})}{2M_{2\overline{Q}q}}$$

$$M_{1\overline{Q}q} = m_{1\overline{Q}} + m_{1q} + B_{1\overline{Q}q} \qquad \delta M_{\overline{Q}q} = M_{2\overline{Q}q} - M_{1\overline{Q}q}$$

$$M_{2\overline{Q}q} = m_{2\overline{Q}} + m_{2q} + B_{2\overline{Q}q} \qquad \delta B_{\overline{Q}q} = B_{2\overline{Q}q} - B_{1\overline{Q}q}$$

[S. Collins *et al.*, NPB **47**, 455 (1996) , A. S. Kronfeld, NPB **53**, 401 (1997)]

- Inconsistency parameter I can be used to examine the improvements by $\mathcal{O}(\boldsymbol{p}^4)$ terms in the action. The OK action is designed to improve these terms and matched at tree-level.
- Binding energies B_1 and B_2 are of order $\mathcal{O}(\boldsymbol{p}^2)$. Because the kinetic meson mass M_2 appears with a factor $\boldsymbol{p}^2$, the leading contribution of binding energy B_2 is generated by $\mathcal{O}(\boldsymbol{p}^4)$ terms in the action.

$$E = M_1 + \frac{\boldsymbol{p}^2}{2M_2} + \cdots = M_1 + \frac{\boldsymbol{p}^2}{2(m_{2\overline{Q}} + m_{2q})}\left[1 - \frac{B_{2\overline{Q}q}}{(m_{2\overline{Q}} + m_{2q})} + \cdots\right] + \cdots$$

Improvement Test: Inconsistency Parameter

$$I \cong \frac{2\delta M_{\overline{Q}q} - \delta M_{\overline{Q}Q}}{2M_{2\overline{Q}q}} \cong \frac{2\delta B_{\overline{Q}q} - \delta B_{\overline{Q}Q}}{2M_{2\overline{Q}q}}$$

- Considering non-relativistic limit of quark and anti-quark system, for S-wave case ($\mu_2^{-1} = m_{2\overline{Q}}^{-1} + m_{2q}^{-1}$),

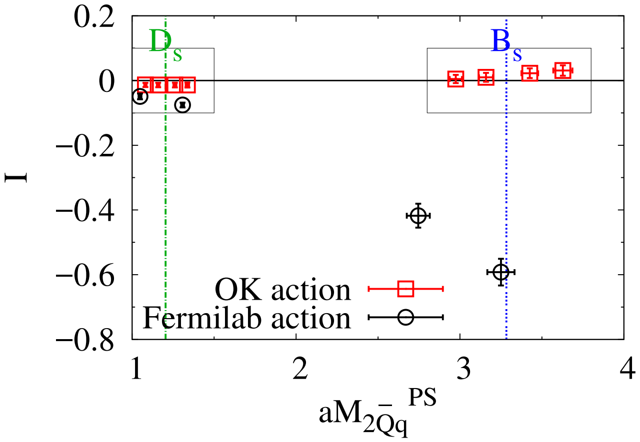
$$\begin{aligned} \delta B_{\overline{Q}q} \quad &= \frac{5}{3}\frac{\langle \boldsymbol{p}^2 \rangle}{2\mu_2}\left[\mu_2\left(\frac{m_{2\overline{Q}}^2}{m_{4\overline{Q}}^3} + \frac{m_{2q}^2}{m_{4q}^3}\right) - 1\right] \quad (m_4 : c_1, c_3) \\ &+ \frac{4}{3}a^3\frac{\langle \boldsymbol{p}^2 \rangle}{2\mu_2}\mu_2(w_{4\overline{Q}}m_{2\overline{Q}}^2 + w_{4q}m_{2q}^2) \quad (w_4 : c_2, c_4) \\ &+ \mathcal{O}(p^4) \end{aligned}$$

[A. S. Kronfeld, NPB **53**, 401 (1997) , C. Bernard *et al.*, PRD **83**, 034503 (2011)]

- Leading contribution of $\mathcal{O}(\boldsymbol{p}^2)$ in δB vanishes when $w_4 = 0$, $m_2 = m_4$, not only for S-wave states but also for higher harmonics.
- This condition is satisfied exactly at tree-level, and we expect I is close to 0.

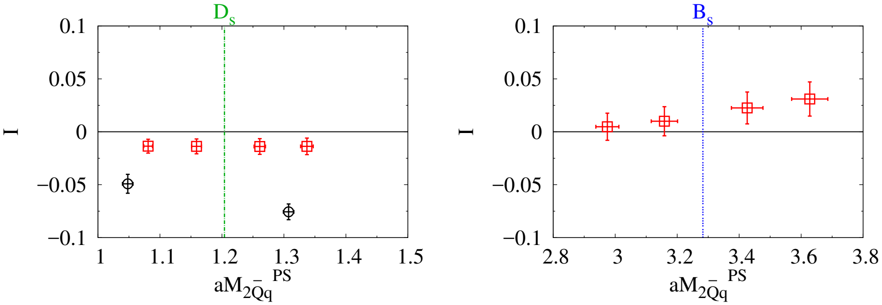
Improvement by the OK action: Inconsistency

$$I \equiv \frac{2\delta M_{\bar{Q}q} - (\delta M_{\bar{Q}Q} + \delta M_{\bar{q}q})}{2M_{2\bar{Q}q}} = \frac{2\delta B_{\bar{Q}q} - (\delta B_{\bar{Q}Q} + \delta B_{\bar{q}q})}{2M_{2\bar{Q}q}}$$



Inconsistency

a12m310, $\kappa_{\text{crit}} = 0.051211$ (nonperturbative)



[Yong-Chull Jang et al., EPJC 77:768]

$\bar{B} \to D^* \ell \bar{\nu}$ Form Factors

$|V_{cb}|$ from the exclusive $\bar{B} \to D^* \ell \bar{\nu}$ I

1. $\bar{B} \to D^* \ell \bar{\nu}$ HQET form factors: $h_{A_1}(w)$, $h_{A_2}(w)$, $h_{A_3}(w)$, and $h_V(w)$

$$\frac{\langle D^*(p_{D^*}, \epsilon) | A^\mu | \bar{B}(p_B) \rangle}{\sqrt{m_B m_{D^*}}} = i h_{A_1}(w)(w+1)\epsilon^{*\mu} - i h_{A_2}(w)(\epsilon^* \cdot v_B) v_B^\mu - i h_{A_3}(w)(\epsilon^* \cdot v_B) v_{D^*}^\mu$$

$$\frac{\langle D^*(p_{D^*}, \epsilon) | V^\mu | \bar{B}(p_B) \rangle}{\sqrt{m_B m_{D^*}}} = \varepsilon^{\mu\nu}{}_{\rho\sigma} \epsilon^*_\nu v_B^\rho v_{D^*}^\sigma h_V(w)$$

2. $R_i(w)$ form factor ratios:

$$R_1(w) \equiv \frac{h_V(w)}{h_{A_1}(w)}$$

$$R_2(w) \equiv \frac{h_{A_3}(w) + r\, h_{A_2}(w)}{h_{A_1}(w)} \qquad \text{with} \qquad r = M_{D^*}/M_B$$

$|V_{cb}|$ from the exclusive $\bar{B} \to D^*\ell\bar{\nu}$ II

3. **Experiment**: determine $\dfrac{d\Gamma}{dw}$ as a function of w (= recoil parameter).

$$\frac{d\Gamma}{dw} = \frac{G_F^2 M_{D^*}^3}{4\pi^3}(M_B - M_{D^*})^2\sqrt{w^2-1}\;|\eta_{\mathrm{EM}}|^2\,|V_{cb}|^2\,\chi(w)\,|\mathcal{F}(w)|^2$$

where $w = v_B \cdot v_{D^*}$, $r = M_{D^*}/M_B$, and

$$\begin{aligned}
\chi(w) &= (1+w)^2\lambda(w)\\
\lambda(w) &= \frac{1}{12}\left(1+\frac{4w}{w+1}t^2(w)\right)\\
t^2(w) &= \frac{1-2wr+r^2}{(1-r)^2}\\
\mathcal{F}(w) &= h_{A_1}(w)\sqrt{\frac{H_0^2(w)+H_+^2(w)+H_-^2(w)}{\lambda(w)}}
\end{aligned}$$

$|V_{cb}|$ from the exclusive $\bar{B} \to D^* \ell \bar{\nu}$ III

$$H_0(w) = \frac{w - r - X_3(w) - rX_2(w)}{1 - r} = \frac{w - r - R_2(w)}{1 - r}$$

$$H_\pm(w) = t(w)(1 \mp X_V(w))$$

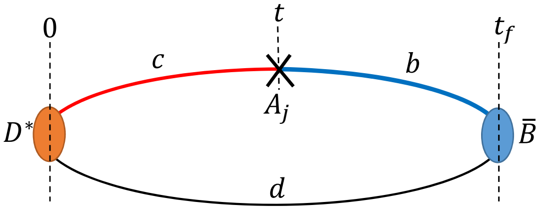
$$X_2(w) = (w - 1)\frac{h_{A_2}(w)}{h_{A_1}(w)}$$

$$X_3(w) = (w - 1)\frac{h_{A_3}(w)}{h_{A_1}(w)}$$

$$X_V(w) = \sqrt{\frac{w - 1}{w + 1}}\frac{h_V(w)}{h_{A_1}(w)} = \sqrt{\frac{w - 1}{w + 1}} \cdot R_1(w)$$

$|V_{cb}|$ from the $\bar{B} \to D^* \ell \bar{\nu}$ decays at zero recoil ($w = 1$)

1. **Lattice QCD**: Calculate $\mathcal{F}(1) = h_{A_1}(1)$ from the following matrix element



2. **Determine $|\mathbf{V_{cb}}|$ by combining experiment with lattice QCD results for $\mathcal{F}(1)$**

$\bar{B} \to D^* \ell \bar{\nu}$ Form Factor: $h_{A_1}(w=1)$

$\bar{B} \to D^* \ell \bar{\nu}$ at zero recoil: $h_{A_1}(1)$ on the lattice

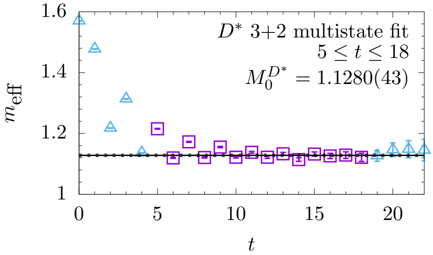
$$|h_{A_1}(1)|^2 = \frac{\langle D^*|A^j_{cb}|\bar{B}\rangle\langle \bar{B}|A^j_{bc}|D^*\rangle}{\langle D^*|V^4_{cc}|D^*\rangle\langle \bar{B}|V^4_{bb}|\bar{B}\rangle} \times \rho^2_{A_j}\,, \quad \text{with} \quad \rho^2_{A_j} = \frac{Z^{cb}_{A_j}\, Z^{bc}_{A_j}}{Z^{cc}_{V_4}\, Z^{bb}_{V_4}}$$

First, we calculate 2-point correlation functions on the lattice:

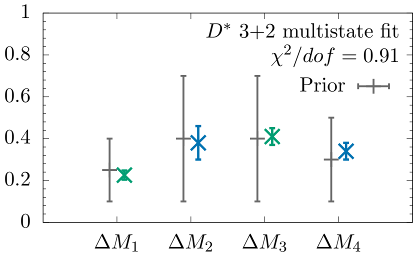
$$\begin{aligned} C^{\rm 2pt}_X(t) &= \langle O^\dagger_X(t) O_X(0)\rangle \\ &= |\mathcal{A}_0|^2 e^{-M_0 t}\Bigg(1 + \left|\frac{\mathcal{A}_2}{\mathcal{A}_0}\right|^2 e^{-\Delta M_2 t} + \left|\frac{\mathcal{A}_4}{\mathcal{A}_0}\right|^2 e^{-(\Delta M_2 + \Delta M_4)t} + \cdots \\ &\quad - (-1)^t \left|\frac{\mathcal{A}_1}{\mathcal{A}_0}\right|^2 e^{-\Delta M_1 t} - (-1)^t \left|\frac{\mathcal{A}_3}{\mathcal{A}_0}\right|^2 e^{-(\Delta M_1 + \Delta M_3)t} + \cdots \Bigg) + (t \leftrightarrow T - t) \end{aligned}$$

where $X = B, D^*$, $\mathcal{A}_n \equiv \langle n|O_X|\Omega\rangle$, and
$\Delta M_n \equiv M_n - M_{n-2}$ with $\Delta M_1 = M_1 - M_0$ and $n \geq 2$.

Multi-state fitting (3+2 fit)



(a) Effective mass



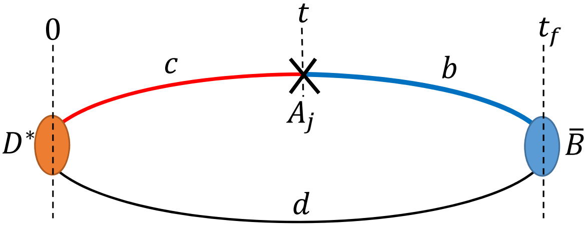
(b) Multi-state fit

$$m_{\rm eff}(t) = \frac{1}{2} \ln \left(\frac{C^{\rm 2pt}(t)}{C^{\rm 2pt}(t+2)} \right)$$

$$\Delta M_n = M_n - M_{n-2} \qquad \text{for } n \geq 2$$

$$\Delta M_1 = M_1 - M_0$$

3-point correlation function



$$C_{A_j}^{B\to D^*}(t, t_f) = \sum_{\vec{x},\vec{y}} \langle O_{D^*}^\dagger(0) A_j^{cb}(\vec{y}, t) O_B(\vec{x}, t_f)\rangle \qquad (0 < t < t_f)$$

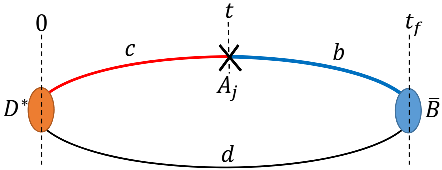
Interpolating operators for mesons

$$O_B = \bar{\psi}_b \gamma_5 \psi_l, \qquad O_{D^*} = \bar{\psi}_c \gamma_j \psi_l$$

Improved axial current operator

$$A_j^{cb} = \bar{\Psi}_c \gamma_j \gamma_5 \Psi_b,$$

3-point correlation function: current improvement



$$A_j^{cb} = \bar{\Psi}_c \gamma_j \gamma_5 \Psi_b,$$

The $\mathcal{O}(\lambda^3)$ improved field with 11 parameters (d_i): [Jaehoon Leem, Lattice 2017]

$$\begin{aligned}
\Psi(x) = e^{M_1/2}\Big[1 &+ d_1 \boldsymbol{\gamma}\cdot\boldsymbol{D} && \to \mathcal{O}(\lambda^1)\\
&+ d_2\Delta^{(3)} + d_B i\boldsymbol{\Sigma}\cdot\boldsymbol{B} + d_E \boldsymbol{\alpha}\cdot\boldsymbol{E} && \to \mathcal{O}(\lambda^2)\\
&+ d_{rE}\{\boldsymbol{\gamma}\cdot\boldsymbol{D}, \boldsymbol{\alpha}\cdot\boldsymbol{E}\} + d_3\sum_i \gamma_i D_i \Delta_i + d_4\{\boldsymbol{\gamma}\cdot\boldsymbol{D}, \Delta^{(3)}\} && \\
&+ d_5\{\boldsymbol{\gamma}\cdot\boldsymbol{D}, i\boldsymbol{\Sigma}\cdot\boldsymbol{B}\} + d_{EE}\{\gamma_4 D_4, \boldsymbol{\alpha}\cdot\boldsymbol{E}\} && \to \mathcal{O}(\lambda^3)\\
&+ d_6[\gamma_4 D_4, \Delta^{(3)}] + d_7[\gamma_4 D_4, i\boldsymbol{\Sigma}\cdot\boldsymbol{B}]\Big]\psi(x). &&
\end{aligned}$$

Calculate $R = |h_{A_1}(1)/\rho_{A_j}|^2$ using two different analysis

1. **Direct analysis on $C_J^{X \to Y}$**

$$C_{A_j}^{B \to D^*}(t, t_f) = B^{B \to D^*} e^{-M_D^* t} e^{-M_B(t_f - t)} (1 + \hat{c}^{B \to D^*}(t, t_f))$$

where $B^{B \to D^*} = \mathcal{A}_0^{D^*} \langle D^* | A_j^{cb} | B \rangle \mathcal{A}_0^B$, and $\hat{c}^{B \to D^*}$ represents the contamination from the excited states of B and D^* mesons.

$$R = \frac{B^{B \to D^*} \cdot B^{D^* \to B}}{B^{B \to B} \cdot B^{D^* \to D^*}}$$

2. **Analysis on R**

$$R(t, t_f) \equiv \frac{C_{A_1}^{B \to D^*}(t, t_f) C_{A_1}^{D^* \to B}(t, t_f)}{C_{V_4}^{B \to B}(t, t_f) C_{V_4}^{D^* \to D^*}(t, t_f)}$$

$$= \frac{B^{B \to D^*} \cdot B^{D^* \to B}}{B^{B \to B} \cdot B^{D^* \to D^*}} [1 + \hat{c}^{B \to D^*}(t, t_f) + \hat{c}^{D^* \to B}(t, t_f) - \hat{c}^{B \to B}(t, t_f) - \hat{c}^{D^* \to D^*}(t, t_f) \cdots].$$

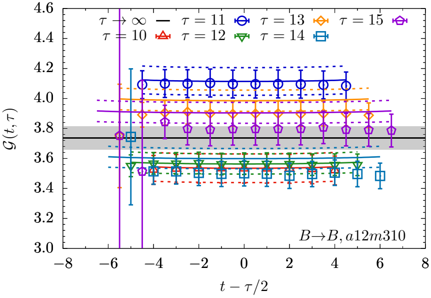
Direct analysis on $C_J^{X\to Y}$

$$
\begin{aligned}
C_{A_j}^{B\to D^*}(t,\tau) &= \langle O_{D^*}^\dagger(0) A_j^{cb}(t) O_B(\tau)\rangle \qquad (0<t<\tau)\\
&= \mathcal{A}_0^{D^*}\mathcal{A}_0^{B}\langle D_0^*|A_j^{cb}|B_0\rangle e^{-M_{B_0}(\tau-t)}e^{-M_{D_0^*}t}\\
&- \mathcal{A}_0^{D^*}\mathcal{A}_1^{B}\langle D_0^*|A_j^{cb}|B_1\rangle (-1)^{\tau-t} e^{-M_{B_1}(\tau-t)}e^{-M_{D_0^*}t}\\
&- \mathcal{A}_1^{D^*}\mathcal{A}_0^{B}\langle D_1^*|A_j^{cb}|B_0\rangle (-1)^{t} e^{-M_{B_0}(\tau-t)}e^{-M_{D_1^*}t}\\
&+ \mathcal{A}_1^{D^*}\mathcal{A}_1^{B}\langle D_1^*|A_j^{cb}|B_1\rangle (-1)^{\tau} e^{-M_{B_1}(\tau-t)}e^{-M_{D_1^*}t}\\
&+ \mathcal{A}_2^{D^*}\mathcal{A}_0^{B}\langle D_2^*|A_j^{cb}|B_0\rangle e^{-M_{B_0}(\tau-t)}e^{-M_{D_2^*}t}\\
&+ \mathcal{A}_0^{D^*}\mathcal{A}_2^{B}\langle D_0^*|A_j^{cb}|B_2\rangle e^{-M_{B_2}(\tau-t)}e^{-M_{D_0^*}t}\\
&- \mathcal{A}_2^{D^*}\mathcal{A}_1^{B}\langle D_2^*|A_j^{cb}|B_1\rangle (-1)^{\tau-t} e^{-M_{B_1}(\tau-t)}e^{-M_{D_2^*}t}\\
&- \mathcal{A}_1^{D^*}\mathcal{A}_2^{B}\langle D_1^*|A_j^{cb}|B_2\rangle (-1)^{t} e^{-M_{B_2}(\tau-t)}e^{-M_{D_1^*}t}\\
&+ \mathcal{A}_2^{D^*}\mathcal{A}_2^{B}\langle D_2^*|A_j^{cb}|B_2\rangle e^{-M_{B_2}(\tau-t)}e^{-M_{D_2^*}t} + \cdots .
\end{aligned}
$$

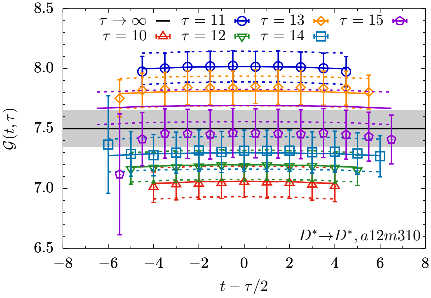
We take $\mathcal{A}_i^{D^*}$, $\mathcal{A}_j^{B}$, $M_{D_i^*}$ and M_{B_j} from the 2-point fitting.

Fitting results for the 3-point correlation functions (1)

$$\mathcal{G}(t,\tau) \equiv \frac{C_{A_j}^{X\to Y}(t,\tau)}{\mathcal{A}_0^Y \mathcal{A}_0^X e^{-M_{X_0}(\tau-t)} e^{-M_{Y_0} t}} = \langle Y_0 | A_j^{cb} | X_0 \rangle + \cdots ,$$



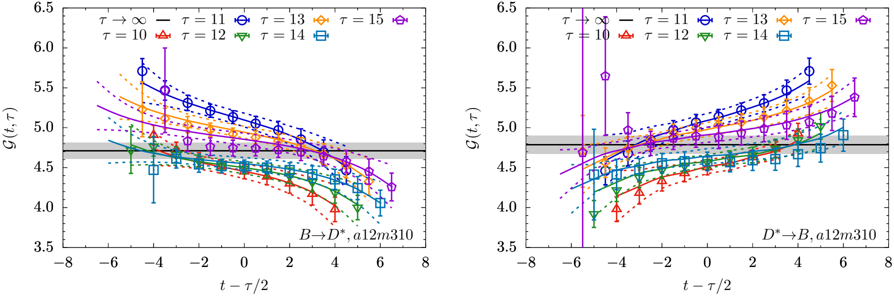
(c) $B \to B$



(d) $D^* \to D^*$

Fitting results for the 3-point correlation functions (2)

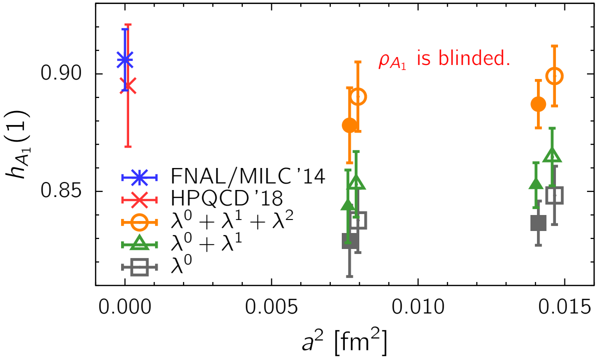
$$\mathcal{G}(t,\tau) \equiv \frac{C_{A_j}^{X\to Y}(t,\tau)}{\mathcal{A}_0^Y \mathcal{A}_0^X e^{-M_{X_0}(\tau-t)} e^{-M_{Y_0} t}} = \langle Y_0 | A_j^{cb} | X_0 \rangle + \cdots ,$$



(e) $B \to D^*$

(f) $D^* \to B$

$\bar{B} \to D^* \ell \bar{\nu}$ Form Factor at Zero Recoil : $h_{A_1}(w=1)$



- ρ_{A_j} is blinded: $\rho^2_{A_j} = \dfrac{Z^{bc}_{A_j} Z^{cb}_{A_j}}{Z^{bb}_{V_4} Z^{cc}_{V_4}} \to 1.$
- Non-perturbative calculation of ρ_{A_j} is underway.
- Preliminary results!!! [PoS (Lattice2019) 056]

Summary

- This is the first numerical study with the OK action using the currents improved up to $\mathcal{O}(\lambda^3)$.
- We have obtained preliminary results for $\dfrac{|h_{A_1}(1)|}{\rho_{A_j}}$ of $\bar{B} \to D^* \ell \bar{\nu}$ decays.

[To do list]

- Non-perturbative calculation of matching factor ρ_{A_j}.
- Extending measurement to superfine and ultrafine ensembles.
- Chiral-continuum extrapolation
- Accumulate more statistics

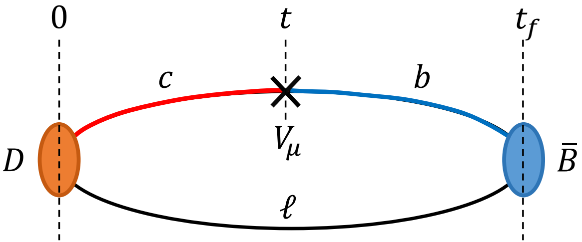
$\bar{B} \to D\ell\bar{\nu}$ Form Factors: $h_\pm(w)$

$\bar{B} \to D\ell\bar{\nu}$ Form Factors: $h_\pm(w)$ on the lattice

$$\frac{\langle D(M_D, \boldsymbol{p}')|V_\mu|B(M_B, \mathbf{0})\rangle}{\sqrt{2M_D}\sqrt{2M_B}} = \frac{1}{2}\left\{h_+(w)(v+v')_\mu + h_-(w)(v-v')_\mu\right\} ,$$

- B meson is at rest: $v = \dfrac{p}{M_B} = (1, \mathbf{0})$.
- D meson is moving with velocity: $v' = \dfrac{p'}{M_D} = (\dfrac{E_D}{M_D}, \dfrac{\boldsymbol{p}'}{M_D})$.
- Recoil parameter: $w = v \cdot v'$.

3-point correlation function



$$C_{V_\mu}^{B\to D}(t, t_f) = \sum_{\vec{x},\vec{y}} \langle O_D^\dagger(0) V_\mu^{cb}(\vec{y}, t) O_B(\vec{x}, t_f)\rangle \qquad (0 < t < t_f)$$

Interpolating operators for mesons

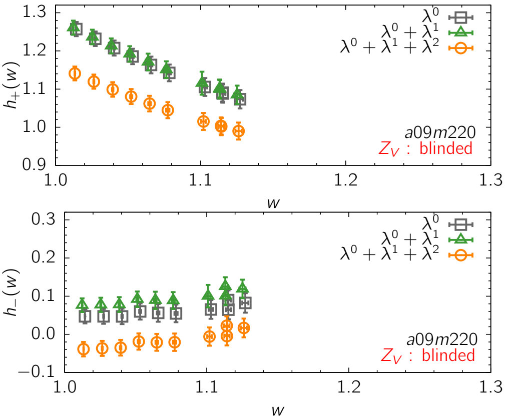
$$O_B = \bar{\psi}_b \gamma_5 \psi_\ell, \qquad O_D = \bar{\psi}_c \gamma_5 \psi_\ell$$

Improved vector current operator

$$V_\mu^{cb} = \bar{\Psi}_c \gamma_\mu \Psi_b,$$

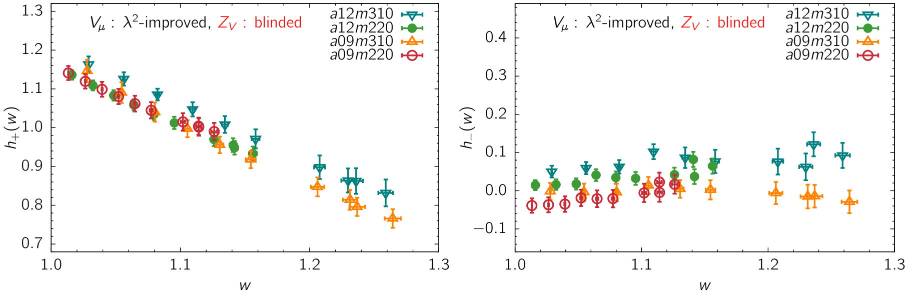
$\bar{B} \to D\ell\bar{\nu}$ Form Factors $h_\pm(w)$

$a \cong 0.09\,\mathrm{fm}$ & $m_\pi \cong 220\,\mathrm{MeV}$



- MILC HISQ lattice at $a \cong 0.09\,\mathrm{fm}$ and $m_\pi \cong 220\,\mathrm{MeV}$.
- Z_V is blinded. $\to$ Preliminary results!!! [PoS (LATTICE 2019) 056]

$\bar{B} \to D\ell\bar{\nu}$ Form Factors $h_\pm(w)$



- MILC HISQ lattices at $a \cong 0.12\,\mathrm{fm}$ and $a \cong 0.09\,\mathrm{fm}$
- Z_V is blinded. (NPR is underway.)
- The vector current is improved up to the λ^2 order.
- Preliminary results!!! [PoS (LATTICE 2019) 056]

Summary

- This is the first numerical study with the OK action using the currents improved up to $\mathcal{O}(\lambda^3)$.
- We produced 3-point correlation functions, and obtained preliminary results for $\dfrac{h_\pm(w)}{Z_V}$ of the $\bar{B} \to D\ell\bar{\nu}$ decays.

[To do list]

- Non-perturbative (NPR) calculation of matching factors: Z_V.
- Extending measurement to superfine, ultrafine, and anker-point ensembles.
- Chiral-continuum extrapolation
- Accumulate more statistics

Current status of measurements and data analysis

label	2–point	3–point	NPR	meson mass	analysis
a12m299	○	○	×	△	△
a12m216	○	○	×	×	△
a12m133	×	×	×	×	×
a09m301	○	○	×	×	△
a09m215	○	○	×	×	△
a09m130	×	×	×	×	×
a06m304	○	○	×	×	×
a06m224	△	×	×	×	×
a06m135	×	×	×	×	×
a042m294	×	×	×	×	×
a042m134	×	×	×	×	×
a03m294	×	×	×	×	×

label $=$ $N_f = 2+1+1$ MILC HISQ ensemble ID

example: a12m299 $\to$ $a = 0.12\,\mathrm{fm}$ and $m_\pi = 299\,\mathrm{MeV}$

MILC HISQ Ensembles

label	$a(\,\mathrm{fm})$	geometry	$m_\pi(\,\mathrm{MeV})$	am_ℓ	am_s	am_c
a12m299	0.12	$24^3 \times 64$	299	0.0102	0.0509	0.635
a12m216	0.12	$32^3 \times 64$	216	0.00507	0.0507	0.628
a12m133	0.12	$48^3 \times 64$	133	0.00184	0.0507	0.628
a09m301	0.09	$32^3 \times 96$	301	0.0074	0.037	0.440
a09m215	0.09	$48^3 \times 96$	215	0.00363	0.0363	0.430
a09m130	0.09	$64^3 \times 96$	130	0.0012	0.0363	0.432
a06m304	0.06	$48^3 \times 144$	304	0.0048	0.024	0.286
a06m224	0.06	$64^3 \times 144$	224	0.0024	0.024	0.286
a06m135	0.06	$96^3 \times 192$	135	0.0008	0.022	0.260
a042m294	0.042	$64^3 \times 192$	294	0.00316	0.0158	0.188
a042m134	0.042	$144^3 \times 288$	134	0.000569	0.01555	0.1827
a03m294	0.03	$96^3 \times 288$	294	0.00223	0.01115	0.1316

label = MILC HISQ ensemble ID

example: a12m299 $\rightarrow a = 0.12\,\mathrm{fm}$ and $m_\pi = 299\,\mathrm{MeV}$

CLN

CLN: Caprini, Lellouch, Neubert I

- Consider $\bar{B} \to D^* \ell \bar{\nu}$ decays.

$$\frac{d\Gamma(\bar{B} \to D^* \ell \bar{\nu})}{dw} = \frac{G_F^2 m_{D^*}^3}{48\pi^3} (m_B - m_{D^*})^2 \, \chi(w) \, \eta_{\mathrm{EW}}^2 \, \mathcal{F}^2(w) \, |V_{cb}|^2$$

- Here, G_F is Fermi constant, η_{EW} is a small electroweak correction, and $\mathcal{F}(w)$ is the form factor.
- The kinematic factor $\chi(w)$ is

$$\chi(w) = \sqrt{w^2 - 1}(w+1)^2 \times Y(w)$$
$$Y(w) = \left[1 + \frac{4w}{w+1} \frac{1 - 2wr + r^2}{(1-r)^2}\right]$$

CLN: Caprini, Lellouch, Neubert II

- The form factor can be rewritten as follows,

$$\mathcal{F}^2(w) = h_{A_1}^2(w) \times \frac{1}{Y(w)} \times \left\{ 2\frac{1-2wr+r^2}{(1-r)^2}\left[1+\frac{w-1}{w+1}R_1^2(w)\right] + \left[1+\frac{w-1}{1-r}\Big(1-R_2(w)\Big)\right]^2 \right\}$$

- So far the formalism is quite general.

CLN: Caprini, Lellouch, Neubert III

- CLN method [28]: ($\approx$ model-dependent approximation)

$$h_{A_1}(w) = h_{A_1}(1)\left[1 - 8\rho^2 z + (53\rho^2 - 15)z^2 - (231\rho^2 - 91)z^3\right] \quad (4)$$

$$R_1(w) = R_1(1) - 0.12(w-1) + 0.05(w-1)^2 \quad (5)$$

$$R_2(w) = R_2(1) + 0.11(w-1) - 0.06(w-1)^2 \quad (6)$$

where z is a conformal mapping variable:

$$z = \frac{\sqrt{w+1} - \sqrt{2}}{\sqrt{w+1} + \sqrt{2}} \quad (7)$$

CLN: Caprini, Lellouch, Neubert IV

- The trouble is that the slopes and curvatures of $R_1(w)$ and $R_2(w)$ are fixed by the HQET perturbation theory (zero-recoil expansion). The HQET results for the slopes and curvatures have about 10% uncertainty of order $\mathcal{O}(\Lambda^2/m_c^2)$ and $\mathcal{O}(\alpha_s \Lambda/m_c)$.
- Hence, CLN can NOT have precision better than 2% by construction.
- The trouble is that the experimental results have errors less than 2% and that the lattice QCD results for the form factors have such a high precision that the errors are below the 2% level.
- At any rate, the experimental group (HFLAV 2017) uses CLN to fit the experimental data to determine four parameters: $\eta_{\rm EW}\mathcal{F}(1)|V_{cb}|$, ρ^2, $R_1(1)$, $R_2(1)$.
- Lattice QCD determines $\mathcal{F}(1)$ very well.
- $\eta_{\rm EW}$ is very well known.
- Hence, we can determine exclusive $|V_{cb}|$ out of this.

BGL

BGL: Boyd, Grinstein, Lebed I

- BGL is model-independent.
- BGL is constructed on three building blocks:
  1. Dispersion relation
  2. Crossing symmetry
  3. Analytic continuation: analyticity
- Consider the 2-point function:

$$
\begin{aligned}
\Pi_J^{\mu\nu}(q) &= (q^\mu q^\nu - q^2 g^{\mu\nu})\Pi_J^T(q^2) + g^{\mu\nu}\Pi_J^L(q^2) \\
&\equiv i \int d^4 x e^{iq\cdot x} \langle 0 | T J^\mu(x) [J^\nu(0)]^\dagger | 0 \rangle
\end{aligned} \tag{8}
$$

- In general, $\Pi_J^{T,L}(q^2)$ is not finite.

BGL: Boyd, Grinstein, Lebed II

- Hence, we need to make one or two subtractions to obtain finite dispersion relations:

$$\chi_J^L(q^2) = \frac{\partial \Pi_J^L}{\partial q^2} = \frac{1}{\pi}\int_0^\infty dt \frac{\mathrm{Im}\Pi_J^L(t)}{(t-q^2)^2} \tag{9}$$

$$\chi_J^T(q^2) = \frac{\partial \Pi_J^T}{\partial q^2} = \frac{1}{\pi}\int_0^\infty dt \frac{\mathrm{Im}\Pi_J^T(t)}{(t-q^2)^2} \tag{10}$$

- Källen-Lehmann spectral decomposition:

$$\begin{aligned}(q^\mu q^\nu - q^2 g^{\mu\nu})&\mathrm{Im}\Pi_J^T(q^2) + g^{\mu\nu}\mathrm{Im}\Pi_J^L(q^2)\\ &= \frac{1}{2}\sum_X (2\pi)^4 \delta^4(q-p_X)\langle 0|J^\mu(0)|X\rangle\langle X|[J^\nu(0)]^\dagger|0\rangle\end{aligned} \tag{11}$$

BGL: Boyd, Grinstein, Lebed III

- Multiply $\xi_\mu \xi_\nu^*$ on both sides:

$$\left[(q^\mu q^\nu - q^2 g^{\mu\nu})\mathrm{Im}\Pi_J^T(q^2) + g^{\mu\nu}\mathrm{Im}\Pi_J^L(q^2)\right]\xi_\mu \xi_\nu^* \geq 0 \tag{12}$$

  for any complex 4-vector ξ_μ.

- From this we can prove the positivity:

$$\mathrm{Im}\Pi_J^T(q^2) \geq 0 \tag{13}$$

$$\mathrm{Im}\Pi_J^L(q^2) \geq 0 \tag{14}$$

BGL: Boyd, Grinstein, Lebed IV

- Consider the two body state of $X = H_b(p_1)H_c(p_2)$.

$$\begin{aligned}\mathrm{Im}\Pi_J^{ii}(q^2) =& \frac{1}{2}\int \frac{d^3p_1 d^3p_2}{(2\pi)^3 4E_1E_2}\delta^4(q-p_1-p_2) \\ & \times \sum_{\text{pol}} \langle 0|J^i|H_b(p_1)H_c(p_2)\rangle\langle H_b(p_1)H_c(p_2)|[J^i]^\dagger|0\rangle \\ & + \cdots \end{aligned} \tag{15}$$

- Here, the ellipsis $(\cdots)$ represents strictly positive contributions from the higher resonances and multi-particle states.
- We may assume that $H_b = B, B^*$ meson states, and $H_c = D, D^*$ meson states.

BGL: Boyd, Grinstein, Lebed V

- Let us consider a simple example of $H_b = B$ and $H_c = D^*$.

$$\mathrm{Im}\Pi_J^{ii}(t) \geq k(t)|\mathcal{F}(t)|^2 \tag{16}$$

where $t = q^2$, $k(t)$ is a calculable kinematic function arising from two-body phase space.

- Let us use the crossing symmetry and analytic continuation:

$$\langle 0|J^i|H_b(p_1)H_c(p_2)\rangle = \mathcal{F}(t) \qquad (t_+ \leq t < \infty) \tag{17}$$

$$\langle \bar{H}_b(-p_1)|J^i|H_c(p_2)\rangle = \mathcal{F}(t) \qquad (m_\ell^2 \leq t < t_-) \tag{18}$$

BGL: Boyd, Grinstein, Lebed VI

- Hadronic moments $\chi_J^{(n)}$:

$$\chi_J^{(n)} \equiv \frac{1}{\Gamma(n+3)} \left.\frac{\partial^{n+2}\Pi_J^{ii}}{\partial^{n+2}q^2}\right|_{q^2=0}$$
$$= \frac{1}{\pi}\int_0^\infty dt\, \left.\frac{\mathrm{Im}\Pi_J^{ii}(t)}{(t-q^2)^{n+3}}\right|_{q^2=0} \tag{19}$$

- Hence, the inequality is

$$\chi_J^{(n)} \geq \frac{1}{\pi}\int_{t_+}^\infty dt \frac{k(t)|\mathcal{F}(t)|^2}{t^{n+3}} \tag{20}$$
$$\longrightarrow \quad \frac{1}{\pi}\int_{t_+}^\infty dt |h^{(n)}(t)F(t)|^2 \leq 1 \tag{21}$$

BGL: Boyd, Grinstein, Lebed VII

where

$$[h^{(n)}(t)]^2 = \frac{k(t)}{t^{n+3}\chi_J^{(n)}} \geq 0\,. \tag{22}$$

- Let us introduce the conformal mapping function:

$$z(t, t_s) = \frac{\sqrt{t_+ - t} - \sqrt{t_+ - t_s}}{\sqrt{t_+ - t} + \sqrt{t_+ - t_s}} \tag{23}$$

- The inequality can be rewritten as follows,

$$\frac{1}{\pi}\int_{t_+}^{\infty} dt \left|\frac{dz(t, t_0)}{dt}\right| |\phi(t, t_0)P(t)F(t)|^2 \leq 1\,, \tag{24}$$

BGL: Boyd, Grinstein, Lebed VIII

- Here, the outer function ϕ is

$$\phi(t, t_0) = \tilde{P}(t)\frac{h^{(n)}(t)}{\sqrt{\left|\frac{dz(t, t_0)}{dt}\right|}} \tag{25}$$

- Here, the factor $\tilde{P}(t)$ removes the sub-threshold poles and branch cuts in $h^{(n)}(t)$.

$$\tilde{P}(t) = \prod_{i=1}^{\tilde{N}} z(t, t_{s_i}) \prod_{j=1}^{\tilde{M}} \sqrt{z(t, t_{s_j})} \tag{26}$$

BGL: Boyd, Grinstein, Lebed IX

- The Blaschke factor $P(t)$ removes all the sub-threshold poles in $\mathcal{F}(t)$.

$$P(t) \equiv \prod_{i=1}^{N} \frac{z - z_{P_i}}{1 - z z_{P_i}^*} = \prod_{i=1}^{N} \frac{z - z_{P_i}}{1 - z z_{P_i}} \tag{27}$$

$$z_{P_i} \equiv z(t_{P_i}, t_-) = \frac{\sqrt{t_+ - t_{P_i}} - \sqrt{t_+ - t_-}}{\sqrt{t_+ - t_{P_i}} + \sqrt{t_+ - t_-}} \tag{28}$$

  where $t_{P_i} = M_{P_i}^2$ represents the pole positions of $F(t)$ below the threshold $(t_{P_i} < t_+)$.
- $|\tilde{P}(t)| = 1$ and $|P(t)| = 1$ for $t_+ \leq t < \infty$.
- Hence, $\phi(t, t_0)P(t)\mathcal{F}(t)$ is analytic even in the sub-threshold region.

BGL: Boyd, Grinstein, Lebed X

- BGL method for the form factor parametrization:

$$F(t) = \frac{1}{\phi(t, t_0)P(t)} \sum_{n=0}^{\infty} a_n z^n(t, t_0) \tag{29}$$

- After the Fourier analysis, the inequality is

$$\sum_{n=0}^{\infty} |a_n|^2 \leq 1\,. \tag{30}$$

- This is called the unitarity conditions (the weak version).

B_s meson mass

Measurement

Gauge Ensemble, Heavy Quark κ, Meson Momentum

- MILC asqtad $N_f = 2 + 1$

a(fm)	$N_L^3 \times N_T$	β	am'_l	am'_s	u_0	a^{-1}(GeV)	N_{conf}	$N_{t_{\mathrm{src}}}$
0.12	$20^3 \times 64$	6.79	0.02	0.05	0.8688	1.683^{+43}_{-16}	500	6

- 11 momenta $|\boldsymbol{p}a| = 0, 0.099, \cdots, 1.26$

Measurement: Interpolating Operator

- Meson correlator

$$C(t,\boldsymbol{p}) = \sum_{\boldsymbol{x}} e^{\mathrm{i}\boldsymbol{p}\cdot\boldsymbol{x}} \langle \mathcal{O}^\dagger(t,\boldsymbol{x})\mathcal{O}(0,\boldsymbol{0})\rangle$$

- Heavy-light meson interpolating operator

$$\mathcal{O}_{\mathrm{t}}(x) = \bar{\psi}_\alpha(x)\Gamma_{\alpha\beta}\Omega_{\beta\mathrm{t}}(x)\chi(x)$$

$$\Gamma = \begin{cases} \gamma_5 & \text{(Pseudo-scalar)} \\ \gamma_\mu & \text{(Vector)} \end{cases} , \quad \Omega(x) \equiv \gamma_1^{x_1}\gamma_2^{x_2}\gamma_3^{x_3}\gamma_4^{x_4}$$

- Quarkonium interpolating operator

$$\mathcal{O}(x) = \bar{\psi}_\alpha(x)\Gamma_{\alpha\beta}\psi_\beta(x)$$

[Wingate *et al.*, PRD **67**, 054505 (2003) , C. Bernard *et al.*, PRD **83**, 034503 (2011)]

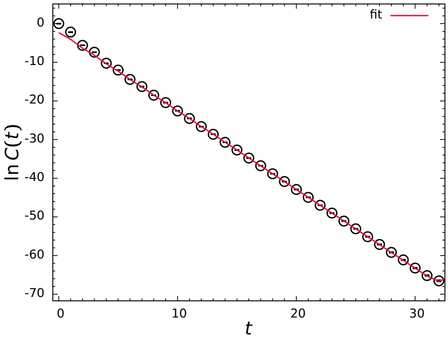
Correlator Fit

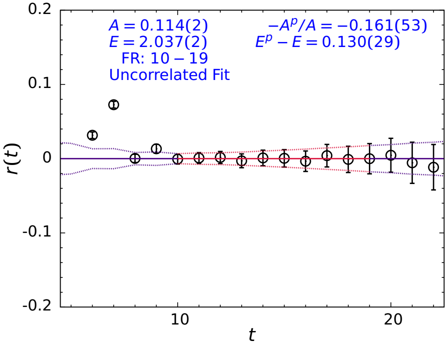
- fit function

$$f(t) = A\{e^{-Et} + e^{-E(T-t)}\} + (-1)^t A^p\{e^{-E^p t} + e^{-E^p(T-t)}\}$$

- fit residual

$$r(t) = \frac{C(t) - f(t)}{|C(t)|}, \text{where } C(t) \text{ is data.}$$





$[\overline{Q}q$, PS, $\kappa = 0.041$, $\boldsymbol{p} = \mathbf{0}]$

Correlator Fit: Effective Mass

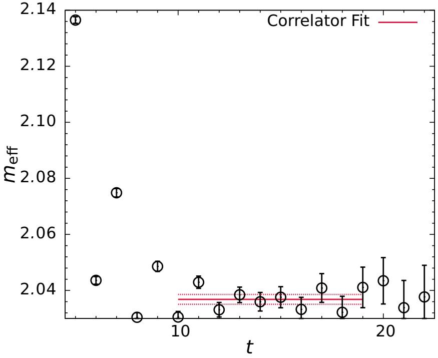
$$m_{\text{eff}}(t) = \frac{1}{2} \ln \left(\frac{C(t)}{C(t+2)} \right)$$

For small t,

$$\begin{aligned} C(t) \cong& A(e^{-Et} + \beta e^{-(E+\Delta E)t}) \\ =& A e^{-Et}(1 + \beta e^{-(\Delta E)t}) \ , \end{aligned}$$

$$\begin{cases} \beta > 0 & \text{(excited state)} \\ \beta \sim -(-1)^t & \text{(time parity state)} \end{cases}$$

$$m_{\text{eff}} \approx E + \beta(\Delta E) e^{-(\Delta E)t}$$

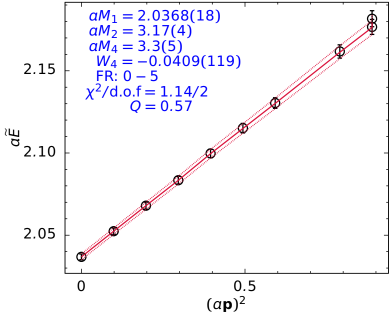


$[\overline{Q}q, \text{PS}, \kappa = 0.041, \boldsymbol{p} = \mathbf{0}]$

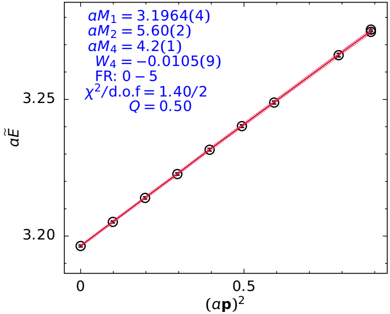
Dispersion Relation

$$E = M_1 + \frac{\boldsymbol{p}^2}{2M_2} - \frac{(\boldsymbol{p}^2)^2}{8M_4^3} - \frac{a^3 W_4}{6}\sum_i p_i^4$$

$$\widetilde{E} = E + \frac{a^3 W_4}{6}\sum_i p_i^4 \ , \qquad \boldsymbol{n} = (2,2,1)\ ,(3,0,0)$$



$[\overline{Q}q,\ \mathrm{PS},\ \kappa = 0.041]$



$[\overline{Q}Q,\ \mathrm{PS},\ \kappa = 0.041]$

Improvement Test: Inconsistency Parameter

$$I \equiv \frac{2\delta M_{\overline{Q}q} - (\delta M_{\overline{Q}Q} + \delta M_{\overline{q}q})}{2M_{2\overline{Q}q}} = \frac{2\delta B_{\overline{Q}q} - (\delta B_{\overline{Q}Q} + \delta B_{\overline{q}q})}{2M_{2\overline{Q}q}}$$

$$M_{1\overline{Q}q} = m_{1\overline{Q}} + m_{1q} + B_{1\overline{Q}q} \qquad \delta M_{\overline{Q}q} = M_{2\overline{Q}q} - M_{1\overline{Q}q}$$

$$M_{2\overline{Q}q} = m_{2\overline{Q}} + m_{2q} + B_{2\overline{Q}q} \qquad \delta B_{\overline{Q}q} = B_{2\overline{Q}q} - B_{1\overline{Q}q}$$

[S. Collins *et al.*, NPB **47**, 455 (1996) , A. S. Kronfeld, NPB **53**, 401 (1997)]

- Inconsistency parameter I can be used to examine the improvements by $\mathcal{O}(\boldsymbol{p}^4)$ terms in the action. The OK action is designed to improve these terms and matched at tree-level.
- Binding energies B_1 and B_2 are of order $\mathcal{O}(\boldsymbol{p}^2)$. Because the kinetic meson mass M_2 appears with a factor $\boldsymbol{p}^2$, the leading contribution of binding energy B_2 is generated by $\mathcal{O}(\boldsymbol{p}^4)$ terms in the action.

$$E = M_1 + \frac{\boldsymbol{p}^2}{2M_2} + \cdots = M_1 + \frac{\boldsymbol{p}^2}{2(m_{2\overline{Q}} + m_{2q})}\left[1 - \frac{B_{2\overline{Q}q}}{(m_{2\overline{Q}} + m_{2q})} + \cdots\right] + \cdots$$

Improvement Test: Inconsistency Parameter

$$I \cong \frac{2\delta M_{\overline{Q}q} - \delta M_{\overline{Q}Q}}{2M_{2\overline{Q}q}} \cong \frac{2\delta B_{\overline{Q}q} - \delta B_{\overline{Q}Q}}{2M_{2\overline{Q}q}}$$

- Considering non-relativistic limit of quark and anti-quark system, for S-wave case ($\mu_2^{-1} = m_{2\overline{Q}}^{-1} + m_{2q}^{-1}$),

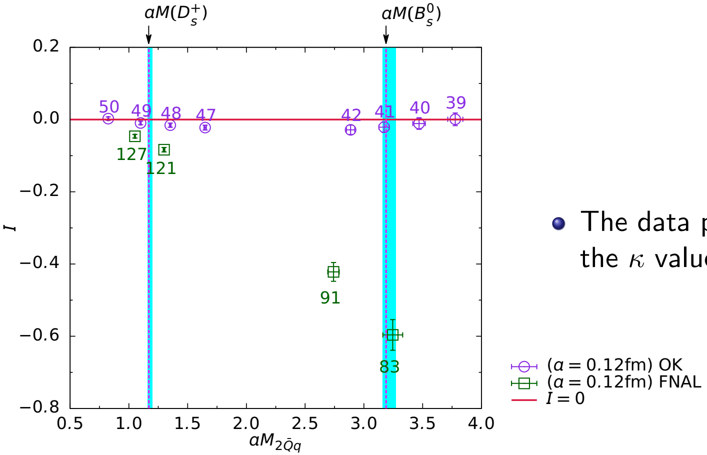
$$\begin{aligned} \delta B_{\overline{Q}q} \quad &= \frac{5}{3}\frac{\langle \boldsymbol{p}^2 \rangle}{2\mu_2}\left[\mu_2\left(\frac{m_{2\overline{Q}}^2}{m_{4\overline{Q}}^3} + \frac{m_{2q}^2}{m_{4q}^3}\right) - 1\right] \qquad (m_4 : c_1, c_3) \\ &+ \frac{4}{3}a^3\frac{\langle \boldsymbol{p}^2 \rangle}{2\mu_2}\mu_2(w_{4\overline{Q}}m_{2\overline{Q}}^2 + w_{4q}m_{2q}^2) \quad (w_4 : c_2, c_4) \\ &+ \mathcal{O}(p^4) \end{aligned}$$

[A. S. Kronfeld, NPB **53**, 401 (1997) , C. Bernard *et al.*, PRD **83**, 034503 (2011)]

- Leading contribution of $\mathcal{O}(\boldsymbol{p}^2)$ in δB vanishes when $w_4 = 0$, $m_2 = m_4$, not only for S-wave states but also for higher harmonics.
- This condition is satisfied exactly at tree-level, and we expect I is close to 0.

Improvement Test: Inconsistency Parameter

- The coarse ($a = 0.12$fm) ensemble data covers the B_s^0 mass and shows significant improvement compared to the Fermilab action.



- The data point labels denote the κ values.

Improvement Test: Hyperfine Splitting Δ

$$\Delta_1 = M_1^* - M_1\,,\ \Delta_2 = M_2^* - M_2$$

Recall,

$$\begin{aligned} M^{(*)}_{1\overline{Q}q} &= m_{1\overline{Q}} + m_{1q} + B^{(*)}_{1\overline{Q}q} \\ M^{(*)}_{2\overline{Q}q} &= m_{2\overline{Q}} + m_{2q} + B^{(*)}_{2\overline{Q}q} \\ \delta B^{(*)} &= B_2^{(*)} - B_1^{(*)} \end{aligned}$$

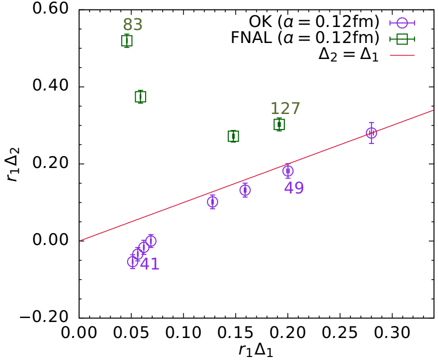
Then,

$$\Delta_2 = \Delta_1 + \delta B^* - \delta B$$

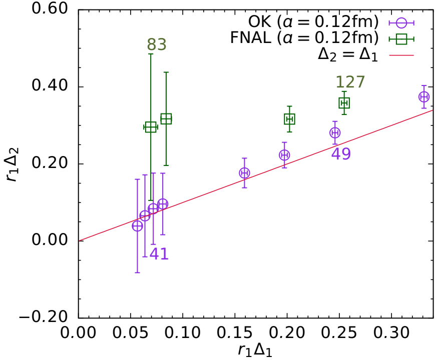
- The difference in hyperfine splittings $\Delta_2 - \Delta_1$ also can be used to examine the improvement from $\mathcal{O}(\boldsymbol{p}^4)$ terms in the action.

Improvement Test: Hyperfine Splitting Δ

$$\Delta_2 = \Delta_1 + \delta B^* - \delta B$$



Quarkonium



Heavy-light

κ Tuning

$N_f = 2 + 1 + 1$ MILC HISQ lattice

a (fm)	Volume	$\hat{m}'/m'_s$	$M_\pi L$	M_π (MeV)	N_{conf}
0.12	$24^3 \times 64$	1/5	4.54	305.3(4)	1040
	$24^3 \times 64$	1/10	3.22	218.1(4)	1020
	$32^3 \times 64$	1/10	4.29	216.9(2)	1000
	$40^3 \times 64$	1/10	5.36	217.0(2)	1028
	$48^3 \times 64$	1/27	3.88	131.7(1)	1000
0.09	$32^3 \times 96$	1/5	4.50	312.7(6)	1011
	$48^3 \times 96$	1/10	4.71	220.3(2)	1000
	$64^3 \times 96$	1/27	3.66	128.2(1)	1047
0.06	$48^3 \times 144$	1/5	4.51	319.3(5)	1016
	$64^3 \times 144$	1/10	4.30	229.2(4)	1246
	$96^3 \times 192$	1/27	3.69	135.5(2)	858
0.042	$64^3 \times 192$	1/5	4.35	309.3(9)	1133
	$144^3 \times 288$	1/27	4.17	134.2(2)	381
0.03	$96^3 \times 288$	1/5	4.84	308.7(1.2)	609

$|V_{cb}|$ from the exclusive decay $\bar{B} \to D^* \ell \bar{\nu}$

$$\frac{d\Gamma}{dw}(\bar{B} \to D^* l \bar{\nu}) = \frac{G_F^2 |V_{cb}|^2 M_B^5}{4\pi^3} r^{*3} (1 - r^*)^2 (w^2 - 1)^{\frac{1}{2}} \eta_C |\eta_{EW}|^2 \chi(w) |\mathcal{F}(w)|^2$$

- $w = v_B \cdot v_{D^*}$, $r^* = \frac{M_{D^*}}{M_B}$
- η_C: Coulomb attraction, $\eta_{EW} = 1.0066$: the one-loop electroweak correction
- $\chi(w)$: Phase-space factor
- $\mathcal{F}(w)$: Form factor ($\leftarrow$LATTICE)

Heavy quarks on the lattice: Fermilab method

The most updated version of $|V_{cb}|$ calculation is done using the Fermilab action to control the c, b heavy quark discretization errors. It is generalized version of the Wilson clover action [El-Khadra, Kronfeld, and Mackenzie, PRD55, 3933 (1997)]

$$S_{\text{Fermilab}} = S_0 + S_E + S_B$$

$$S_0 = a^4 \sum_x \bar{\psi}(x)\Big[m_0 + \gamma_4 D_4 - \frac{a}{2}\Delta_4 + \zeta\Big(\gamma\cdot\boldsymbol{D} - \frac{r_s a}{2}\Delta^{(3)}\Big)\Big]\psi(x)$$

$$S_E = -\frac{1}{2}c_E\zeta a^5 \sum_x \bar{\psi}(x)\boldsymbol{\alpha}\cdot\boldsymbol{E}\psi(x), \quad S_B = -\frac{1}{2}c_B\zeta a^5 \sum_x \bar{\psi}(x) i\boldsymbol{\Sigma}\cdot\boldsymbol{B}\psi(x).$$

- The Wilson term breaks the chiral symmetry explicitly, and the mass gets additive renormalization. $\rightarrow$ We tune κ and κ_{crit} to the physical quark.

$$am_0 = \frac{1}{2u_0}\left(\frac{1}{\kappa} - \frac{1}{\kappa_{\text{crit}}}\right)$$

Oktay-Kronfeld action

The OK action is an improved version of the Fermilab action such that the bilinear operators are tree-level matched to QCD through $\mathcal{O}(\lambda^3)$ in HQET power counting where $\lambda \sim a\Lambda \sim \Lambda/(2m_Q)$ [Oktay and Kronfeld, PRD78, 014504 (2008)]

$$S_{\text{OK}} = S_{\text{Fermilab}} + S_{\text{new}}$$

$$S_{\text{new}} = a^6 \sum_x \bar{\psi}(x) \Big[c_1 \sum_i \gamma_i D_i \Delta_i + c_2 \{\boldsymbol{\gamma}\cdot\boldsymbol{D}, \Delta^{(3)}\} + c_3 \{\boldsymbol{\gamma}\cdot\boldsymbol{D}, i\boldsymbol{\Sigma}\cdot\boldsymbol{B}\}$$

$$+ c_{EE}\{\gamma_4 D_4, \boldsymbol{\alpha}\cdot\boldsymbol{E}\} + c_4 \sum_i \Delta_i^2 + c_5 \sum_{i\neq j} \{i\Sigma_i B_i, \Delta_j\} \Big] \psi(x)$$

- The matching determines c_B, c_E, $c_{1,\cdots,5}$ and c_{EE} as a function of m_0. We have a tree-level value for the κ_{crit}

$$\kappa_{\text{crit}}^{\text{tree}} = [2u_0(1+3\zeta r_s + 18c_4)]^{-1} = 0.053850 \qquad (\zeta = r_s = 1)$$

where $u_0 = 0.86372$ for MILC HISQ lattice (a12m310, $24^3 \times 64$)

Fermilab method

We write non-relativistic dispersion relation,

$$E(\mathbf{p}) = M_1 + \frac{\mathbf{p}^2}{2M_2} - \frac{(\mathbf{p}^2)^2}{8M_4^3} - \frac{a^3 W_4}{6} \sum_i p_i^4 + \cdots .$$

- M_1: rest mass
- M_2: kinetic mass → Tuning to the physical mass
- M_4: quartic mass
- W_4: Lorentz symmetry breaking term

(Example) Tree-level relation between the bare quark mass m_0 and the kinetic quark mass m_2

$$\frac{1}{am_2} = \frac{2\zeta^2}{am_0(2 + am_0)} + \frac{r_s \zeta}{1 + am_0}$$

Nonperturbative determination of κ_{crit}

- $M_2(\kappa, \kappa_{\text{crit}})$: Light kinetic meson mass (600∼950 MeV)
- $m_2(\kappa, \kappa_{\text{crit}})$: kinetic quark mass

Let us suppose the meson mass relation

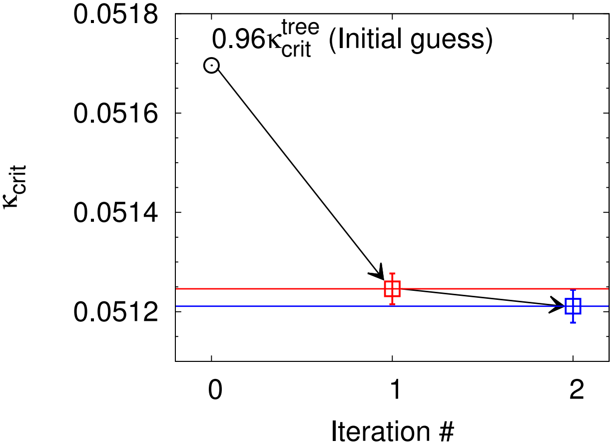
$$M_2^2 = A + Bm_2 + Cm_2^2.$$

The fitting using the true value of κ_{crit} will give $A = 0$. Note that the action depends on both κ and κ_{crit}. We determine κ_{crit} iteratively, as follows.

1. Start with an initial guess $\kappa'_{\text{crit}} = 0.96\kappa^{\text{tree}}_{\text{crit}}$
2. Determine the OK action coefficients using κ'_{crit}
3. Produce 2-pt pion correlators, and determine kinetic meson mass $M_2(\kappa, \kappa'_{\text{crit}})$ using various κ in the range (600∼950 MeV)
4. Find κ_{crit} such that fitting in terms of $m_2(\kappa, \kappa_{\text{crit}})$ gives $A = 0$.
5. Update $\kappa'_{\text{crit}} = \kappa_{\text{crit}}$ and go to the step 2.

Nonperturbative determination of $\kappa_{\rm crit}$: result

$N_f = 2 + 1 + 1$ MILC HISQ ensemble (a12m310),
$N_{\rm conf} = 130$, point source



$\kappa_{\rm crit} = 0.051211(33)(4)$

κ tuning using D_s and B_s masses

- $M_2(\kappa, \kappa_{\text{crit}})$: Heavy-light meson mass
- $m_2(\kappa, \kappa_{\text{crit}})$: kinetic quark mass

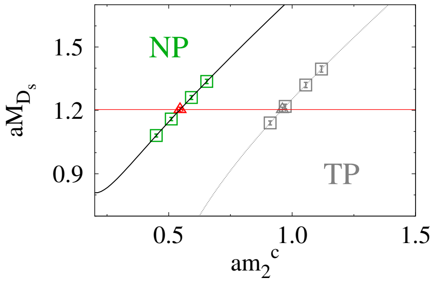
We use the HQET expansion of heavy-light meson masses M_2 as a fitting function:

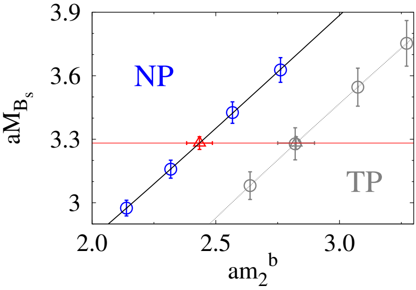
$$aM_2 = am_2 + d_0 + \frac{d_1}{am_2} + \frac{d_2}{(am_2)^2}.$$

1. Determine the OK action coefficients using charm and bottom type κ values with nonperturbative κ_{crit}.
2. Produce 2-pt B_s, D_s correlators, and determine $M_2(\kappa, \kappa_{\text{crit}})$
3. Determine the coefficients d_0, d_1 and d_2 using least-χ^2 fitting
4. Find m_2^{tuned} that gives the physical meson mass $M^{\text{Phys}} = M_2(m_2^{\text{tuned}})$.
5. obtain κ^{tuned} such that $m_2^{\text{tuned}} = m_2(m_0^{\text{tuned}})$ and $m_0^{\text{tuned}} = m_0(\kappa^{\text{tuned}}, \kappa_{\text{crit}})$.

κ tuning using D_s and B_s masses: results

- $N_f = 2+1+1$ MILC HISQ ensemble (a12m310)
- HISQ propagators ($am_s = 0.0509$) with point source
- OK propagators ($\kappa_{\text{crit}} = 0.051211$ and $\kappa_{\text{crit}}^{\text{tree}}$) with covariant Gaussian smearing.

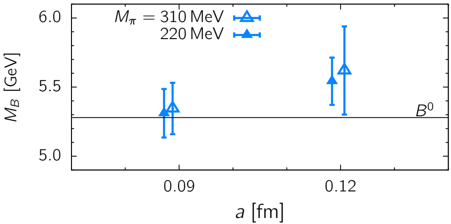


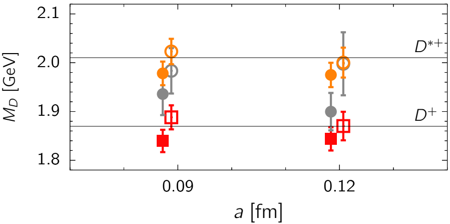


$$\kappa_c = 0.048524(33)(43), \qquad \kappa_b = 0.04102(14)(9)$$

Meson Spectrum of B and $D^{(*)}$

$$E = M_1 + \frac{\boldsymbol{p}^2}{2M_2} - \frac{(\boldsymbol{p}^2)^2}{8M_4^3} - \frac{a^3 w_4}{6} \sum_i p_i^4 + \cdots .$$





- Meson masses (M_B, M_D) can be obtained from the kinetic mass M_2.
- M_{D^*} (gray) : kinetic mass (M_2).
- M_{D^*} (orange) : $M(D^*) = M_2(D) + M_1(D^*) - M_1(D) \rightarrow$ smaller errors.

References I

[1] J. Charles et al.
CP violation and the CKM matrix: Assessing the impact of the asymmetric B factories.
Eur.Phys.J., C41:1–131, 2005.
updated results and plots available at: http://ckmfitter.in2p3.fr.

[2] M. Bona et al.
The Unitarity Triangle Fit in the Standard Model and Hadronic Parameters from Lattice QCD: A Reappraisal after the Measurements of Delta m(s) and BR(B —¿ tau nu(tau)).
JHEP, 10:081, 2006.
Standard Model fit results: Summer 2016 (ICHEP 2016): http://www.utfit.org.

[3] Guido Martinelli et al.
Private communication with UTfit.
http://www.utfit.org/UTfit/, 2017.

[4] Benjamin J. Choi et al.
Kaon BSM B-parameters using improved staggered fermions from $N_f = 2 + 1$ unquenched QCD.
Phys. Rev., D93(1):014511, 2016.

References II

[5] T. Blum et al.
Domain wall QCD with physical quark masses.
Phys. Rev., D93(7):074505, 2016.

[6] Jack Laiho and Ruth S. Van de Water.
Pseudoscalar decay constants, light-quark masses, and B_K from mixed-action lattice QCD.
PoS, LATTICE2011:293, 2011.

[7] S. Durr et al.
Precision computation of the kaon bag parameter.
Phys. Lett., B705:477–481, 2011.

[8] Y. Aoki et al.
FLAG Review 2021.
11 2021.

[9] E. Waheed et al.
Measurement of the CKM matrix element $|V_{cb}|$ from $B^0 \to D^{*-}\ell^+\nu_\ell$ at Belle.
Phys. Rev. D, 100(5):052007, 2019.
[Erratum: Phys.Rev.D 103, 079901 (2021)].

References III

[10] J. P. Lees et al.
Extraction of form Factors from a Four-Dimensional Angular Analysis of $\overline{B} \to D^{*}\ell^{-}\overline{\nu}_{\ell}$.
Phys. Rev. Lett., 123(9):091801, 2019.

[11] A. Bazavov et al.
Semileptonic form factors for $B \to D^{*}\ell\nu$ at nonzero recoil from $2+1$-flavor lattice QCD.
5 2021.

[12] Yasmine Sara Amhis et al.
Averages of b-hadron, c-hadron, and τ-lepton properties as of 2018.
Eur. Phys. J. C, 81(3):226, 2021.

[13] Roel Aaij et al.
Measurement of $|V_{cb}|$ with $B_s^0 \to D_s^{(*)-}\mu^{+}\nu_{\mu}$ decays.
Phys. Rev. D, 101(7):072004, 2020.

[14] Marzia Bordone, Bernat Capdevila, and Paolo Gambino.
Three loop calculations and inclusive Vcb.
Phys. Lett. B, 822:136679, 2021.

[15] Y. Amhis et al.
Averages of b-hadron, c-hadron, and τ-lepton properties as of summer 2016.
Eur. Phys. J., C77(12):895, 2017.

References IV

[16] R. Abbott et al.
Direct CP violation and the $\Delta I = 1/2$ rule in $K \to \pi\pi$ decay from the Standard Model.
4 2020.

[17] C. Patrignani et al.
Review of Particle Physics.
Chin. Phys., C40(10):100001, 2016.
https://pdg.lbl.gov/ .

[18] M. Tanabashi et al.
Review of Particle Physics.
Phys. Rev., D98(3):030001, 2018.

[19] P.A. Zyla et al.
Review of Particle Physics.
PTEP, 2020(8):083C01, 2020.

[20] R.L. Workman et al.
The Review of Particle Physics (2022).
PTEP, 2022:083C01, 2022.

References V

[21] T. Aaltonen et al.
High-precision measurement of the W boson mass with the CDF II detector.
Science, 376(6589):170–176, 2022.

[22] Jon A. Bailey, Yong-Chull Jang, Weonjong Lee, and Sungwoo Park.
Standard Model evaluation of ε_K using lattice QCD inputs for $\hat{B}_K$ and V_{cb}.
Phys. Rev., D92(3):034510, 2015.

[23] Andrzej J. Buras and Diego Guadagnoli.
Phys.Rev., D78:033005, 2008.

[24] Joachim Brod and Martin Gorbahn.
ϵ_K at Next-to-Next-to-Leading Order: The Charm-Top-Quark Contribution.
Phys.Rev., D82:094026, 2010.

[25] Joachim Brod, Martin Gorbahn, and Emmanuel Stamou.
Standard-Model Prediction of ϵ_K with Manifest Quark-Mixing Unitarity.
Phys. Rev. Lett., 125(17):171803, 2020.

[26] J. P. Lees et al.
A test of heavy quark effective theory using a four-dimensional angular analysis of $\overline{B} \to D^* \ell^- \overline{\nu}_\ell$.
2019.

References VI

[27] A. Abdesselam et al.
Measurement of CKM Matrix Element $|V_{cb}|$ from $\bar{B} \to D^{*+}\ell^-\bar{\nu}_\ell$.
2018.

[28] Irinel Caprini, Laurent Lellouch, and Matthias Neubert.
Dispersive bounds on the shape of anti-B —¿ D(*) lepton anti-neutrino form-factors.
Nucl. Phys., B530:153–181, 1998.

[29] C. Glenn Boyd, Benjamin Grinstein, and Richard F. Lebed.
Precision corrections to dispersive bounds on form-factors.
Phys. Rev., D56:6895–6911, 1997.