Chaotic Dynamics in the $SU(2)$ Gauge Matrix Model

Classical Ergodicity Breaking and Quantum Phases

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Joint Work with Chaitanya Bhatt and Vijay Nenmeli

in progress

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Matrix Model for $SU(2)$ Yang-Mills

The $SU(2)$ Model in SVD Coordinates

A New Tetrahedral Symmetry

$SU(2)$ Matrix Model is the 3-d Henon-Heiles System

Stable Orbits and Their Classification

Perturbations About Periodic Orbits

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The $SU(2)$ matrix model is very easy to describe:

- Building blocks:
  - 2×2 hermitian matrices $M_i(t)$, $i = 1, 2, 3$.
  - Equivalently, a single 3×3 matrix $M \in \mathcal{M}_3(\mathbb{R})$.
- Physical rotations: $M \to RM$, $R \in SO(3)$,
  Gauge rotations: $M \to MS^T$, $S \in ad\, SU(2)$.
- Gauge group $ad\, SU(2)$ ($= SO(3)$) is finite-dimensional.
- The configuration space $\mathcal{C}_2 = \mathcal{M}_3(\mathbb{R})/SO(3)$.
- Nontrivial topology: $\mathcal{C}_2 \simeq \mathbb{R} \times (S^5 - \mathbb{R}P^2)$ (Narasimhan-Ramadas 1979).
- Curvature $F_{ij} = -\epsilon_{ijk} M_k - i[M_i, M_j]$.
- A natural reduction of $SU(2)$ YM on $S^3 \times \mathbb{R}$ to a matrix model.

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Matrix Model Dynamics

- For dynamics, we need a gauge-invariant Lagrangian.
- The electric $E_i \equiv D_t M_i$ and magnetic $B_i \equiv \frac{1}{2}\epsilon_{ijk} F_{jk}$:

$$E_i = \dot{M}_i - i[M_0, M_i], \qquad B_i = -M_i - \frac{i}{2}\epsilon_{ijk}[M_j, M_k].$$

- M_0: parallel transporter in the t direction. We set $M_0 = 0$.
- The matrix model Lagrangian is

$$L_{YM} = \frac{1}{2g^2}\mathrm{Tr}(E_i E_i - B_i B_i) = \frac{1}{2g^2}\mathrm{Tr}(D_t M_i D_t M_i) - V$$

(Balachandran-Queiroz-Vaidya 2015)

- $V(M)$ has upto quartic terms.
- The matrix model is just a multi-dimensional quartic oscillator.

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SVD Coordinates

- We write $M = RAS^T$, with $R \in O(3)$, $S \in SO(3)$ (Iwai 2010).
- A is diagonal with real entries.
- $R =$ physical rotations + parity; $S =$ gauge rotations.
- Define $\Omega = R^{-1}\dot{R}$ angular velocity, $\Lambda = S^{-1}\dot{S}$ gauge angular velocity to get

$$L_{YM} = \frac{1}{2g^2}\,\mathrm{Tr}\left(\dot{A}^2 - A^2(\Omega^2 + \Lambda^2) + 2\Omega A \Lambda A\right) - V(A)$$

- The Hamiltonian can be computed easily.
- Gauge invariant dynamics: set $\Lambda = 0$.
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$$V(M) = \frac{1}{2g^2}\left(\operatorname{Tr} M^T M - 6 \det M + \frac{1}{2}[(\operatorname{Tr} M^T M)^2 - \operatorname{Tr} M^T M M^T M]\right).$$

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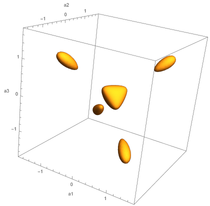
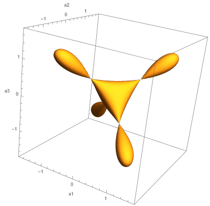
Tetrahedral Symmetry

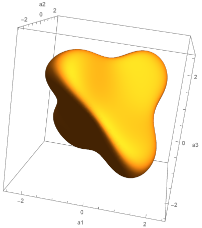
$g^2 V = 0.05$



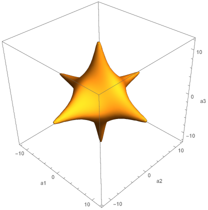
$g^2 V = 0.09375$



$g^2 V = 1.5$



$g^2 V = 75$



High energies: approximate octahedral symmetry.

Extrema of $V(M)$

- The extrema are at $a_1 = a_2 = a_3 = a$, for $a = 0, \frac{1}{2}, 1$.
- Compute the Hessian $\left[\frac{\partial^2 V}{\partial a_i \partial a_j}\right]$.
- It is positive definite at $M = 0, \mathbf{1}$ (minima).
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A 3-dimensional Henon-Heiles System

- In the color-0 spin-0 sector, the Hamiltonian is

$$H = \frac{g^2}{2}\left(p_{a_1}^2 + p_{a_2}^2 + p_{a_3}^2\right) + \frac{1}{2g^2}\left(a_1^2 + a_2^2 + a_3^2 - 6a_1 a_2 a_3 + (a_1^2 a_2^2 + a_1^2 a_3^2 + a_2^2 a_3^2)\right)$$

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Nonlinearity

- The constant energy surface is disconnected (5 pieces) for $g^2 E < \frac{3}{32}$ and connected for $g^2 E > \frac{3}{32}$.
- The critical energy $g^2 E_c = \frac{3}{32}$ is the saddle point $a_1 = a_2 = a_3 = \frac{1}{2}$.
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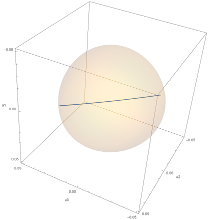
- Nonlinear normal modes are characterized by the stabilizer $G \subset \mathcal{T}_d \times \mathcal{T}$ (Montaldi-Roberts-Stewart 1990).

Conjugacy class of Stabilizer	No. of orbits
$D_{2d} \times \mathcal{T} \quad (\equiv A_4)$	3
$C_{3v} \times \mathcal{T} \quad (\equiv A_3)$	4
$C_{2v} \times \mathcal{T} \quad (\equiv A_2)$	6
$S_4 \wedge \mathcal{T}_2 \quad (\equiv B_4)$	6
$C_3 \wedge \mathcal{T}_s \quad (\equiv B_3)$	8

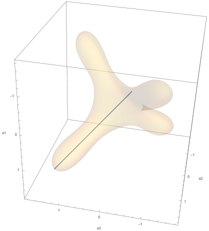
Table: Non-linear normal modes - Classification

Orbit Pictures

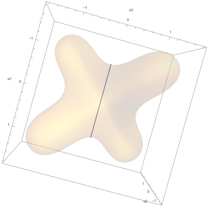
*A*2



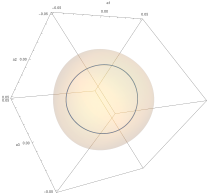
*A*3



*A*4



*B*3



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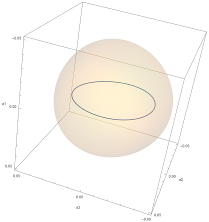
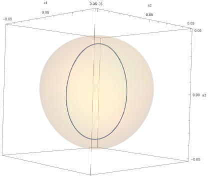


A non-critical closed orbit



- We will study perturbations about each type of orbit.
- Generically, the orbits destabilize for large enough perturbations.
- Typically, the orbits destabilize by becoming weakly ergodic (Lyapunov $= 0$) and then (strongly) ergodic (Lyapunov $\neq 0$).
- Orbits of $A4$ and $A3$ destabilize in a most interesting manner.
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Monodromy Matrix I

- Let $X(t)$ be a periodic solution of period T, and $x(t)$ a perturbation around X.
- Then $\frac{dx}{dt} = F[X(t)]x(t)$.
- Equivalently, there is a time-evolution matrix $U(t) : x(0) \to x(t)$.
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Stability of $A4$ Orbits

- Initial condition: $(a_1, a_2, a_3, p_{a_1}, p_{a_2}, p_{a_3}) = (A, 0, 0, 0, 0, 0)$.
- Fluctuation: $(x_1, x_2, x_3, p_1, p_2, p_3)$.
- Fluctuation equations:

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\dot{x}_1 &= p_1, \quad \dot{p}_1 = -x_1 \\
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- Using $b_\pm = \frac{(x_2 \pm x_3)}{\sqrt{2}}$ decouples the equations completely.

Stability of $A4$ Orbits

- The b_- obeys the Whittaker-Hill equation

$$\ddot{b}_-(s) + (\eta + 2\alpha\cos 2s + 2\beta\cos 4s)b_-(s) = 0.$$

(Here $s = \frac{t}{2}, \alpha = -6A, \beta = A^2, \eta = 4 + 2A^2$).

- This is the Schrödinger equation for a particle in the periodic potential $V(s) = -(\alpha\cos 2s + \beta\cos 4s)$ and energy $\eta/2$.
- The allowed energies form *bands*, and between them are *band gaps*.
- Allowed energy $\Leftrightarrow$ Bloch wave function $\Leftrightarrow$ stable perturbation.
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Stability of *A*4 Orbits

- Eigenvalues of U are $\{1, 1, \lambda, \lambda, \mu, \mu\}$.
- Only two possibilities:
  - Real eigenvalues $\Rightarrow$ exactly one out of $\{\lambda, \mu\ (= 1/\lambda)\}$ will be outside the unit circle $\Rightarrow$ unstable orbit.
  - Imaginary eigenvalues $\Rightarrow$ eigenvalues on the unit circle $\Rightarrow$ stable orbit.
- Enough to study $\gamma \equiv \lambda + \mu = \frac{(\mathrm{Tr}\, M - 2)}{2}$ (or equivalently $\mathrm{Tr}\, M$).
- Numerically, we see that we have
  - Instability for all $g^2 E < 3/2$.
  - Quasiperiodic transitions between stability and instability $g^2 E > 3/2$.

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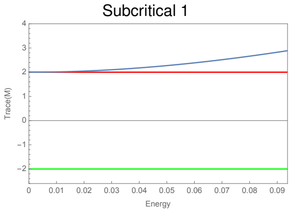

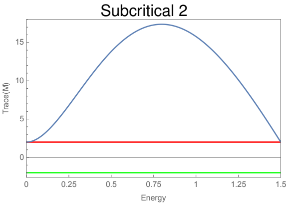

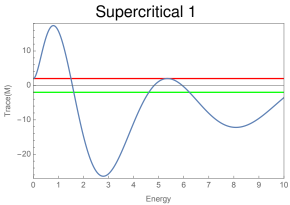

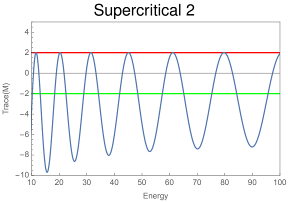

Stability of $A3$ Orbits

- Initial condition: $(a_1, a_2, a_3, p_{a_1}, p_{a_2}, p_{a_3}) = (A, A, A, 0, 0, 0)$.
- The orbit is now an elliptic function of t.
- The time period can be computed exactly in terms of (in)complete elliptic functions of the first kind.
- The perturbations again obey the Schrödinger equation for a (more complicated) periodic potential.
- Again, there is the structure of bands and band gaps.

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Stability $A3$ Orbits

Specifically, we find numerically that

- For $E < E_c$, $\operatorname{Tr} M$ oscillates between ± 2 with ever increasing frequency.
- Peak spacings vary geometrically as we approach E_c from below.
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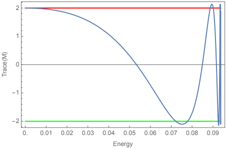
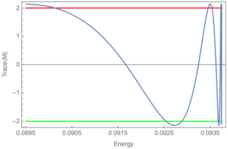
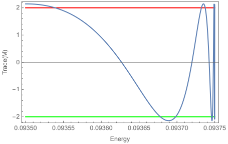


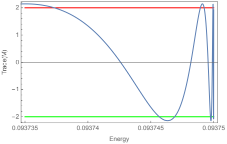
Subcritical 2



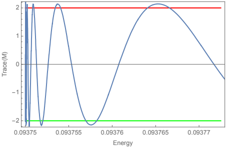
Subcritical 3



Subcritical 4



Supercritical 1



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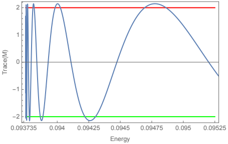

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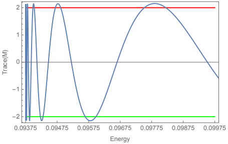

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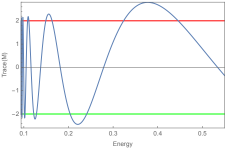

Stability of $A3$ Orbits

- The bands/gaps have a *self-similar* structure as a function of energy.
- We can compute the Feigenbaum constant on either side of E_c.
- Define $\nu = |1 - E_c/E|$, look at ratios of bifurcation energies ν_n/ν_{n+2}.
  - $\nu_n/\nu_{n+2} \to \delta_1 = e^{-\pi\sqrt{\frac{2}{5}}}$ for $E < E_c$.
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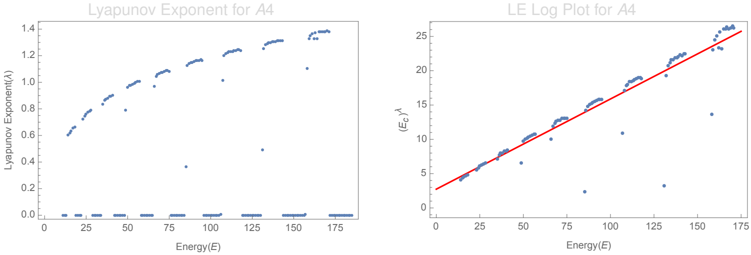
Lyapunov Exponents for *A*4

- Non-zero Lyapunov exponent means chaotic dynamics.
- The Lyapunov exponents reflect the band structure.



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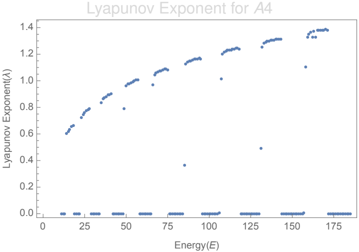

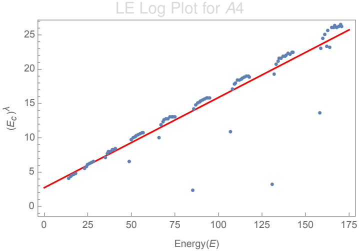

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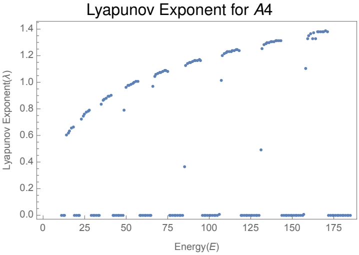

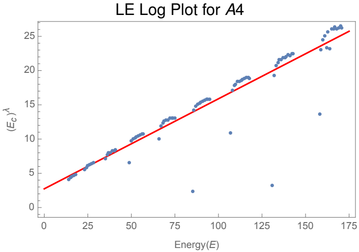

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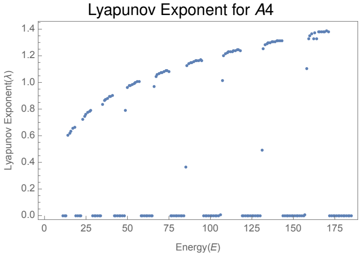

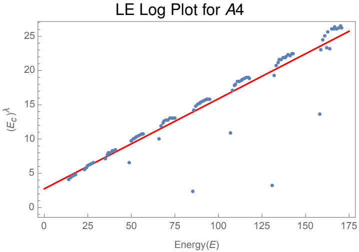

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Lyapunov Exponents for *A*3

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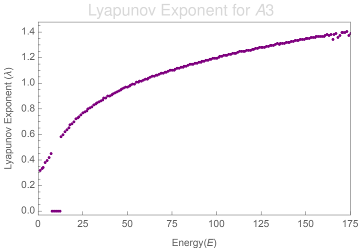

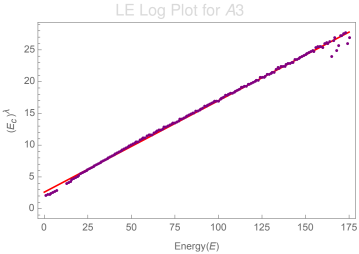

- Bands are wider, spaced apart by geometric progression.

Lyapunov Exponents for $A3$

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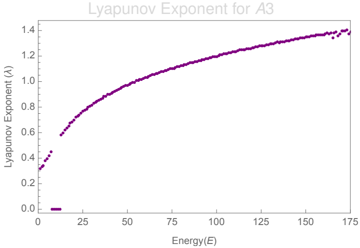

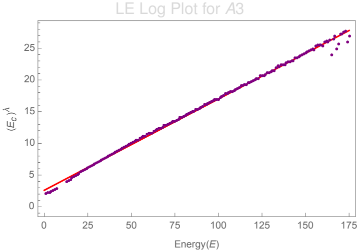

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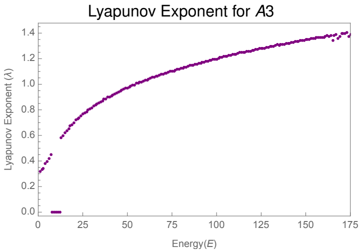

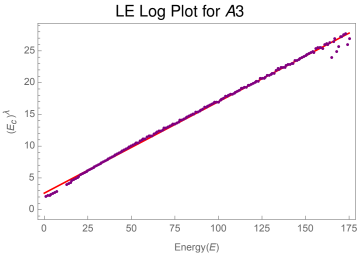

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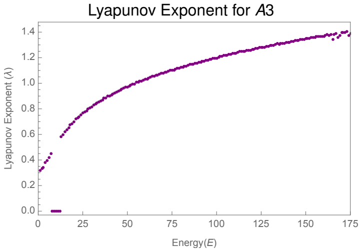

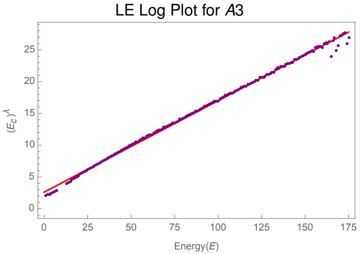

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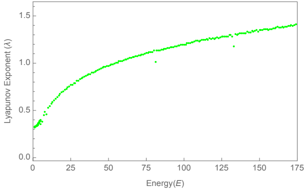
Lyapunov Exponents: Generic Perturbations

Lyapunov Exponents for arbitrary trajectories as a function of energy.



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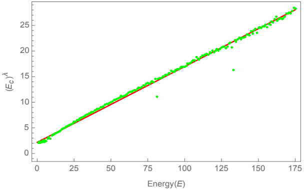

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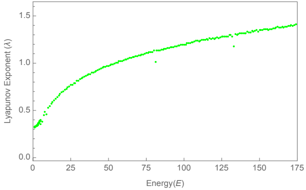

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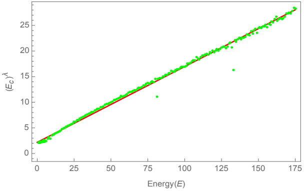

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Ergodicity and Thermodynamics

- For arbitrary initial conditions, the trajectories are strongly chaotic (LE $\neq 0$).
- Equipartition holds: $\langle p_1^2 \rangle = \langle p_2^2 \rangle = \langle p_3^2 \rangle, \quad \langle \cdot \rangle =$ time average. We have numerical verification.
- The LEs have an intimate relation to Kolmogorov-Sinai entropy: $\sum \text{LE} \lesssim S_{KS}$ (Pesin 1977).
- One can define a useful quantity called "temperature" T_B. (Berdichevskii 1988).
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Ergodicity Breaking

- The $A3$ and $A4$ trajectories are special.
- Under a generic perturbation, even in the chaotic domain, the trajectories do not explore all available phase space.
- Rather, they are confined to particular corners.
- These trajectories *break ergodicity!*
- Numerical evidence: equipartition does not hold.
- Ergodicity breaking: different phases or glasses.
- There are at least two *corners* ($A3$ and $A4$) of phase space from which the dynamics cannot break out.

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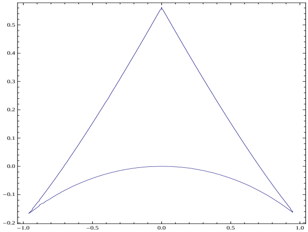
Quantum Image of Ergodicity Breaking

- Remarkably, the quantum $SU(2)$ model + quarks has quantum phases because of superselection sectors.
  (Pandey-Vaidya 2017)
- Quarks get trapped in special corners of the gauge field space.



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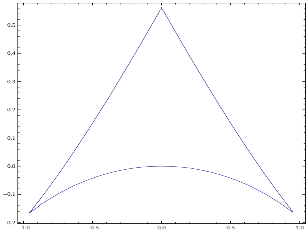

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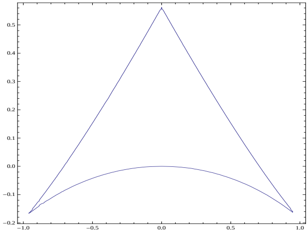

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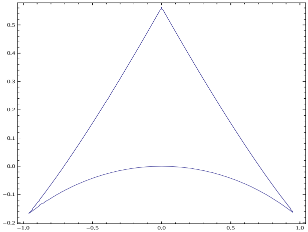

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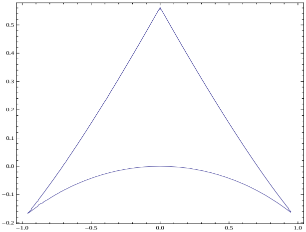

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- The $SU(2)$ model has an unexpected tetrahedral symmetry.
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- The model is chaotic. One of the routes is via Feigenbaum-like "period-doubling".
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