Correspondence of topological classification between quantum graph extra dimension and topological matter

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Based on arXiv:2204.03834 [hep-th]

"Corfu 2022: Workshop on Standard Model and Beyond," Corfu, Greece, Sept. 5, 2022

Talk Plan

- Introduction
  - Quantum graph
  - Classification of topological matter
- Setup
- Results
- Summary and Discussion

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- **Introduction**
  - **Quantum graph**
  - **Classification of topological matter**
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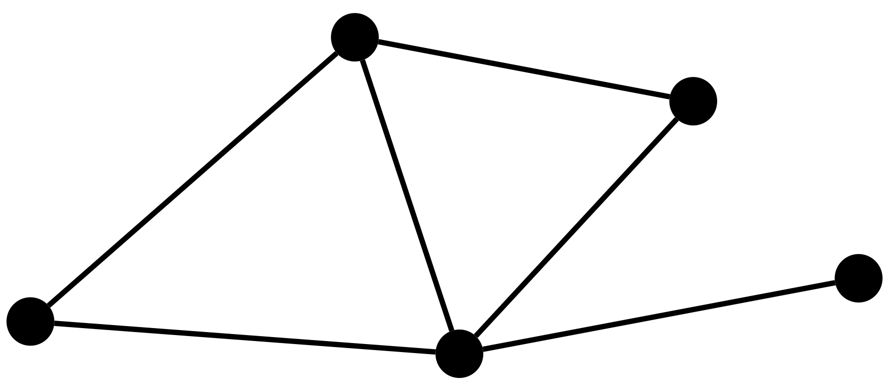
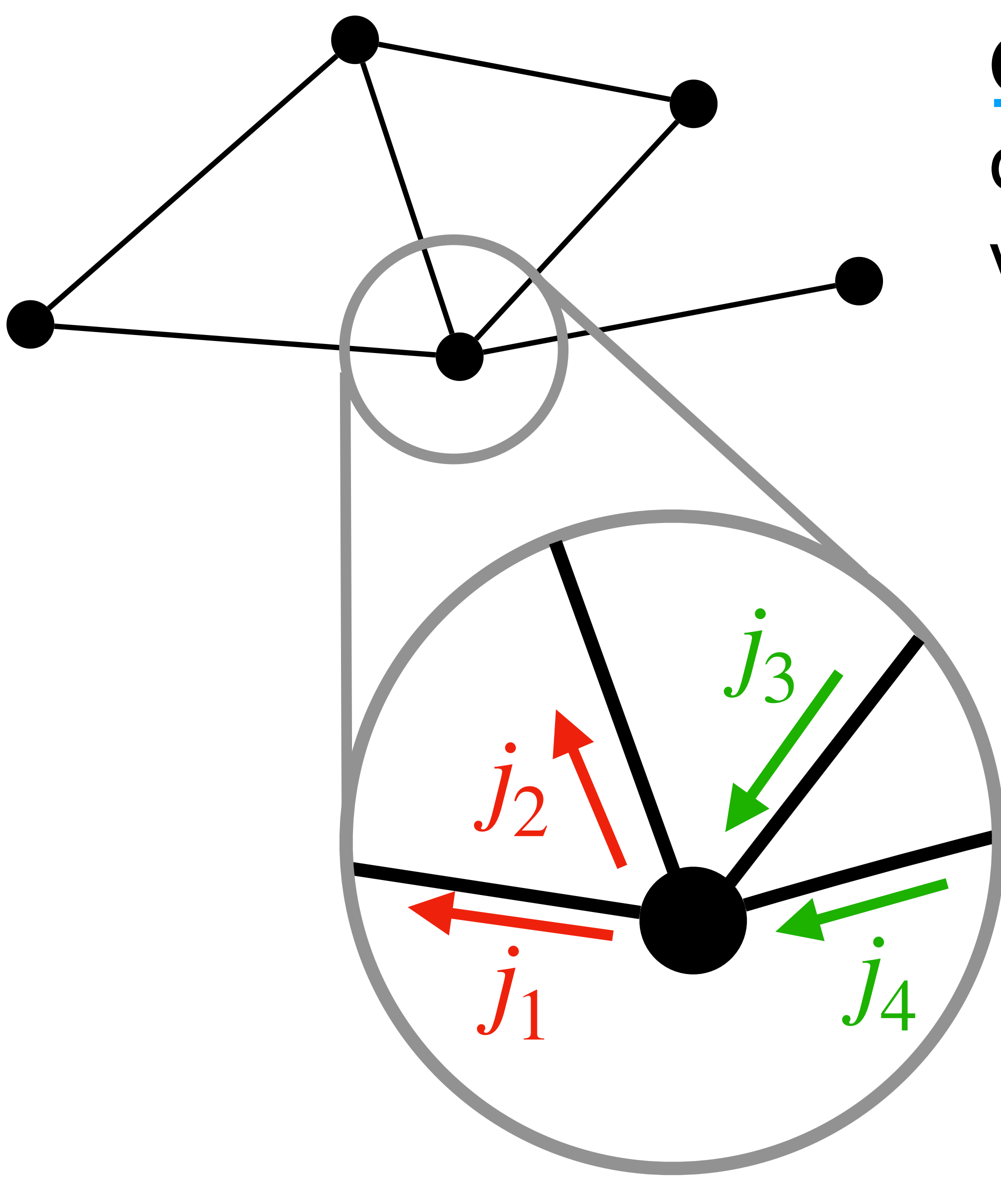
Quantum graph and B.C.



Quantum mechanical system on one-dim. circuits consisting of lines and vertices

Quantum graph and B.C.



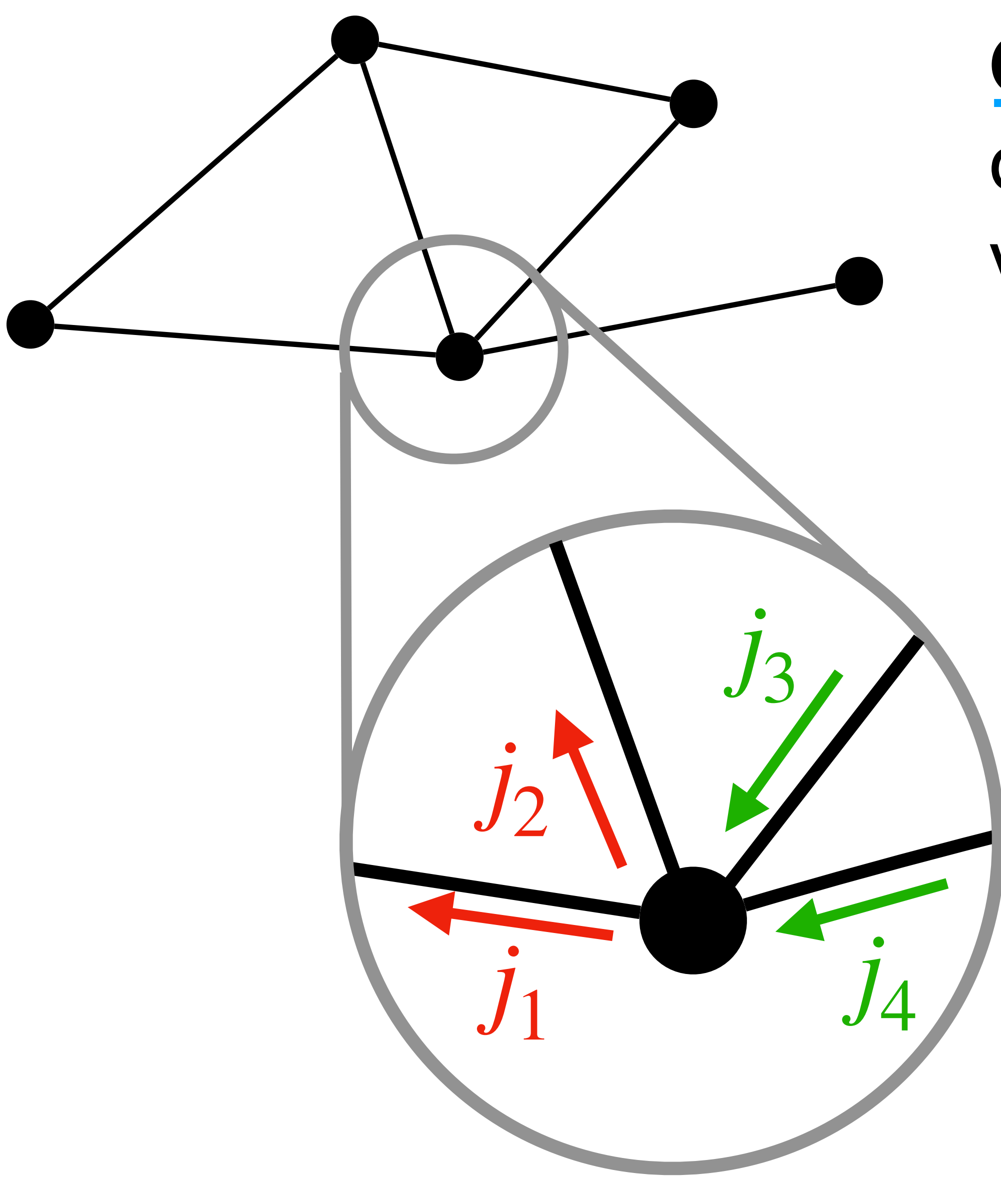
Quantum mechanical system on one-dim. circuits consisting of lines and vertices

↕

Current conservation in this system

$$j_1 + j_2 = j_3 + j_4$$

Quantum graph and B.C.



Quantum mechanical system on one-dim. circuits consisting of lines and vertices

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Current conservation in this system

$$j_1 + j_2 = j_3 + j_4$$

↕

Appropriate boundary conditions at the vertices

Topological matter

- **Altland-Zirnbauer classification**

  Classification of Hamiltonian in free fermion systems according to 10 symmetries

  [Altland and Zirnbauer, Phys. Rev. B55 (1997) 1142.]

Topological matter

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Classification of Hamiltonian in free fermion systems according to 10 symmetries

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Hamiltonian: $H^2 = 1, \quad H^\dagger = H$

Time reversal sym.	Particle-hole sym.	Chiral sym.
$THT^{-1} = H$	$CHC^{-1} = -H$	$\Gamma H \Gamma^{-1} = -H$
$(T^2 = \pm 1)$	$(C^2 = \pm 1)$	$(\Gamma = T \times C, \Gamma^2 = 1)$

Can be classified into 10 classes by combining 3 symmetries

→

Topological matter

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Can be classified into 10 classes by combining 3 symmetries

$\longrightarrow$

Each classes are characterized by a zero-th homotopy of the classification space of *H.*

$$\mathbb{Z}, \quad \mathbb{Z}_2, \quad 2\mathbb{Z}, \quad 0$$

Topological matter

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$\boxed{\mathbb{Z}, \quad \mathbb{Z}_2, \quad 2\mathbb{Z},} \quad 0$

Goals of this talk

AZ classification

- **Setup**

  Free fermion system

  $$H^2 = 1, \quad H^\dagger = H$$

- **Quantity that characterizes the class**

  Topological number π_0

Our model

- **Setup**

  5d Dirac fermion on the spacetime shown Fig. 1.

  ?

- **Quantity that characterizes the class**

  ?

4d Minkowski spacetime (x^μ)
+ Quantum graph (y) = 5d spacetime

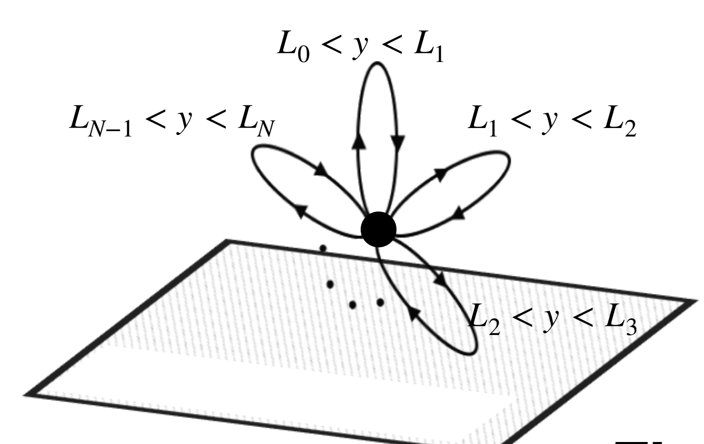

Fig. 1

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  - Quantum graph
  - Classification of topological matter
- **Setup**
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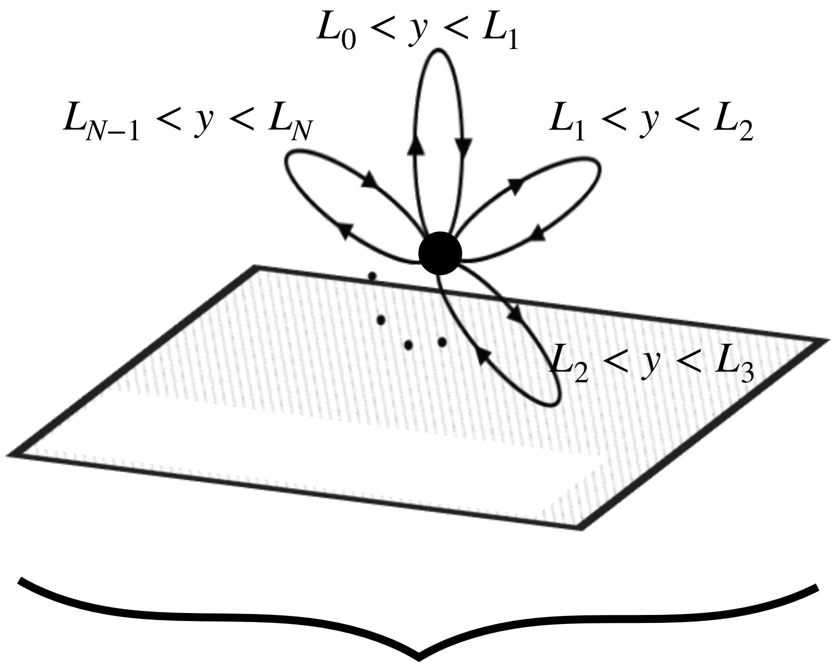
Setup



4-dim Minkowski spacetime (x^μ)
+ Quantum graph (y)

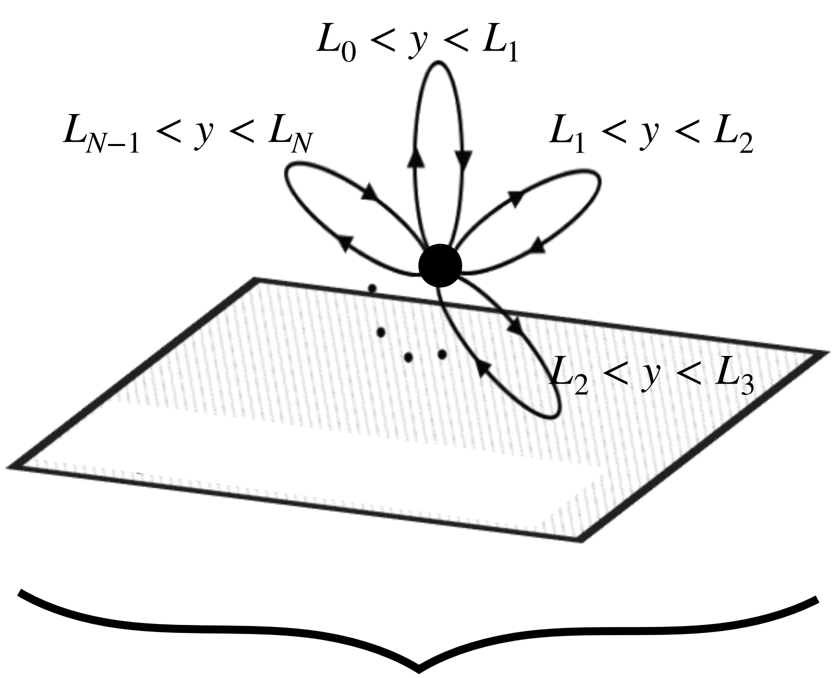
= 5-dim spacetime

# of segments (N)
= even

Action of 5d Dirac free fermion

$$S = \int d^4x \sum_{a=1}^{N} \int_{L_{a-1}+\varepsilon}^{L_{a-\varepsilon}} dy\ \overline{\Psi}(x, y)[i\gamma^\mu \partial_\mu + i\gamma^y \partial_y + M]\Psi(x, y)$$

Setup



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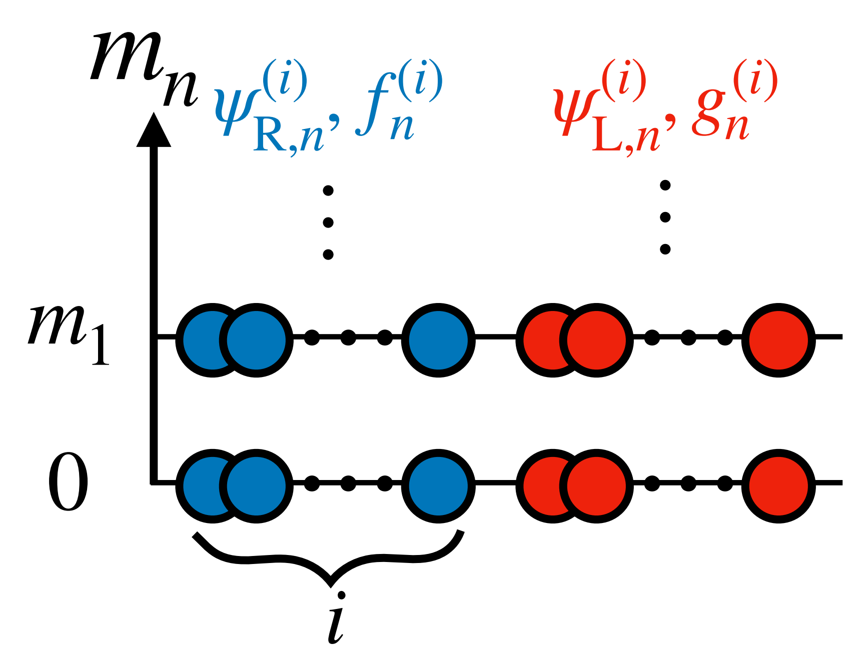
Kaluza-Klein expansion

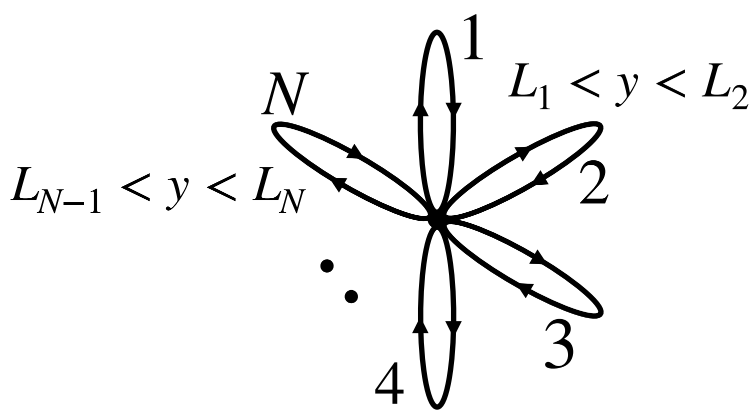
$$\Psi(x,y) = \sum_{n=0}^{\infty} \sum_{i} [\psi_{\mathrm{R},n}^{(i)}(x) f_n^{(i)}(y) + \psi_{\mathrm{L},n}^{(i)}(x) g_n^{(i)}(y)]$$

$$\gamma^5 \psi_{\mathrm{R/L},n}^{(i)} = \psi_{\mathrm{R/L},n}^{(i)}$$

Diagrammatic meaning of KK expansion $\longrightarrow$



B.C. in this setup

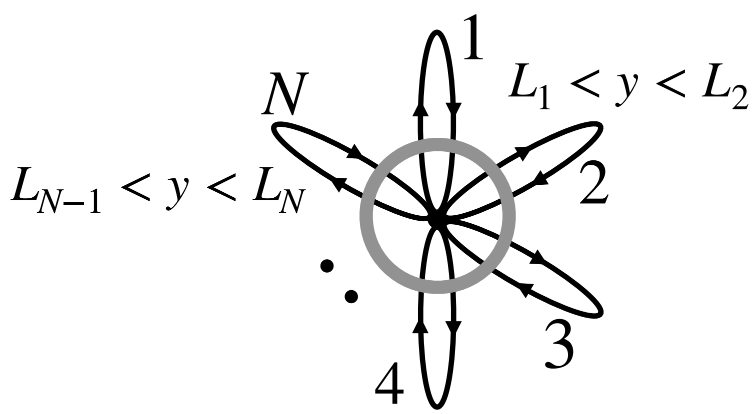


Quantum system

Dirac fermion on 4+1 dim.

(4: 4d Minkowski spacetime, 1: quantum graph)

B.C. in this setup



Quantum system

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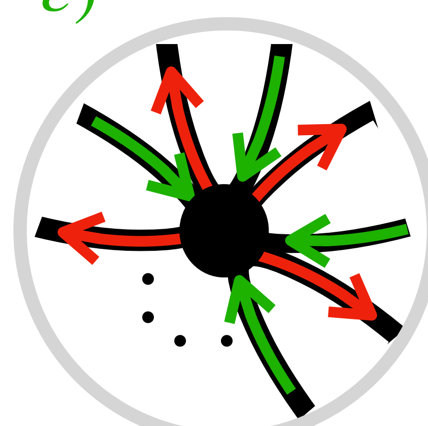
Current conservation

$$\sum_{a=1}^{N} j(L_{a-1}+\varepsilon) = \sum_{a=1}^{N} j(L_a - \varepsilon)$$

$$j(x,y) = \overline{\psi}(x,y)\gamma^5\psi(x,y)$$

y: quantum graph



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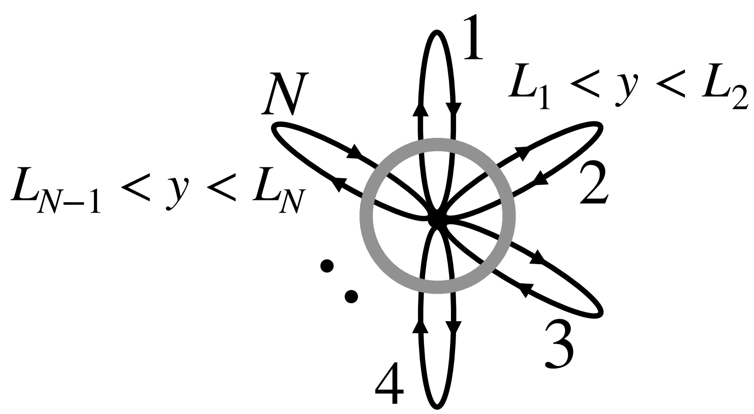

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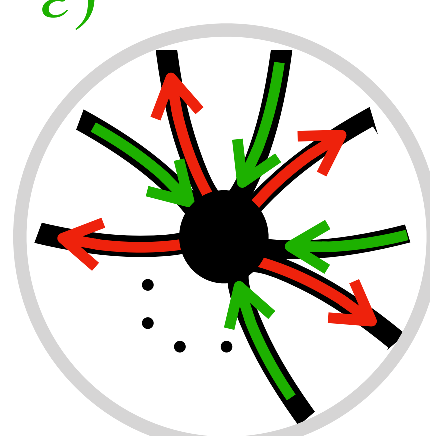

$\longleftrightarrow$

Boundary conditions

$$(1_{2N} - U_{\mathrm{B}})\vec{F} = \vec{0}$$

$$(1_{2N} + U_{\mathrm{B}})\vec{G} = \vec{0}$$

$$\vec{F} \equiv \begin{pmatrix} f(L_0+\varepsilon) \\ f(L_1-\varepsilon) \\ \vdots \\ f(L_N-\varepsilon) \end{pmatrix}, \quad \vec{G} \equiv \begin{pmatrix} g(L_0+\varepsilon) \\ -g(L_1-\varepsilon) \\ \vdots \\ -g(L_N-\varepsilon) \end{pmatrix}$$

$$U_{\mathrm{B}} \in U(2N), \quad U_{\mathrm{B}}^2 = 1_{2N}$$

The boundary conditions are characterized by the parameters of U(2N) with $U^2 = I_{2N}$.

B.C. and Witten index

- **Boundary condition (B.C.)**

  We can classify into 2N+1 classes that do not transfer for continuous deformation. ($\because$ $U_{\mathrm{B}} \in U(2N)$ with $U_{\mathrm{B}}^2 = 1_{2N}$)

B.C. and Witten index

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$$U_{\mathrm{B}} = V \begin{pmatrix} 1_{2N-k} & 0 \\ 0 & -1_k \end{pmatrix} V^\dagger, \quad V \in U(2N) \quad k = 0,1,\cdots,2N$$

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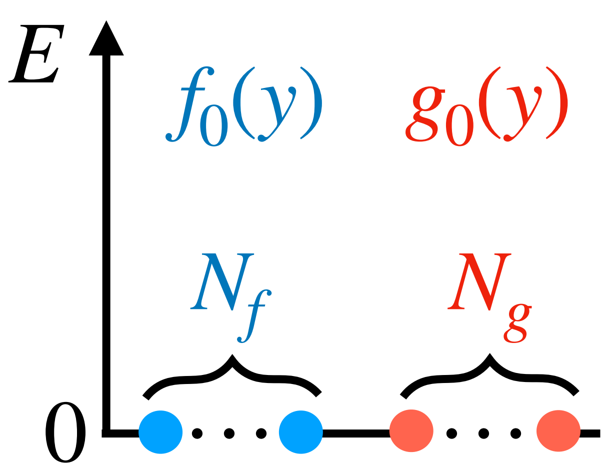

$$N_f - N_g = N - k$$

f_0, g_0 : zero-energy state

[J. Phy. A 52 (2019) 455401]

B.C. and Witten index

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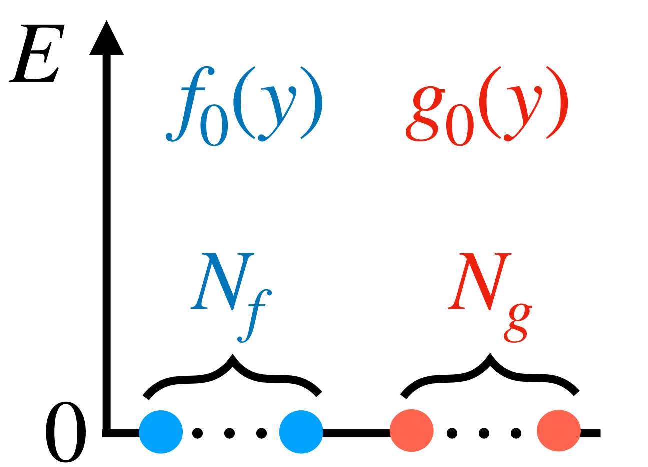

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Interesting features

- **Multiple degeneracy in zero energy state**
- **N-k is topological invariants in the space of boundary conditions.**

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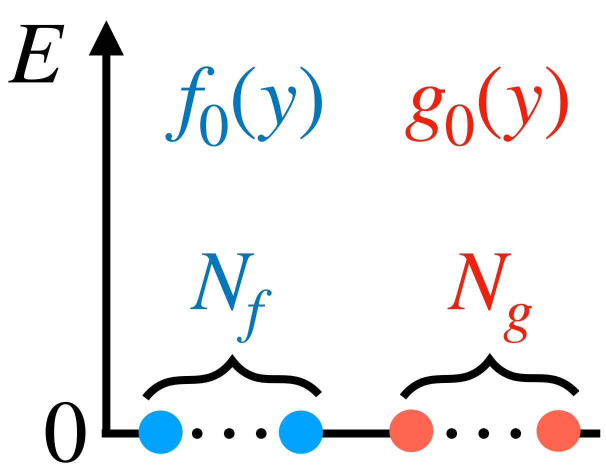

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Interesting features

- **Multiple degeneracy in zero energy state**
- **N-k is topological invariants in the space of boundary conditions.**

⟶ It is called “**Witten index**”.

Symmetries in this model

Time-reversal (T²=±1)

$$\Psi(x,y) \xrightarrow{T} \Psi^T(x,y) = U_T \Psi^*(x,y)$$

$$f_n^{(i)}(y) \xrightarrow{T} f_n^{(i)*}(y), \quad g_n^{(i)}(y) \xrightarrow{T} g_n^{(i)*}(y)$$

$$U_T \gamma^A U_T^{-1} = \begin{cases} (\gamma^0)^* & (A=0) \\ -(\gamma^i)^* & (i \neq 0) \end{cases}$$

Charge conjugation (C²=±1)

$$\Psi(x,y) \xrightarrow{C} \Psi^C(x,y) = C\mathcal{P}_y \overline{\Psi}^\top(x,y)$$

$$f_n^{(i)}(y) \xrightarrow{C} g_n^{(i)*}(\tilde{y}), \quad g_n^{(i)}(y) \xrightarrow{C} f_n^{(i)*}(\tilde{y})$$

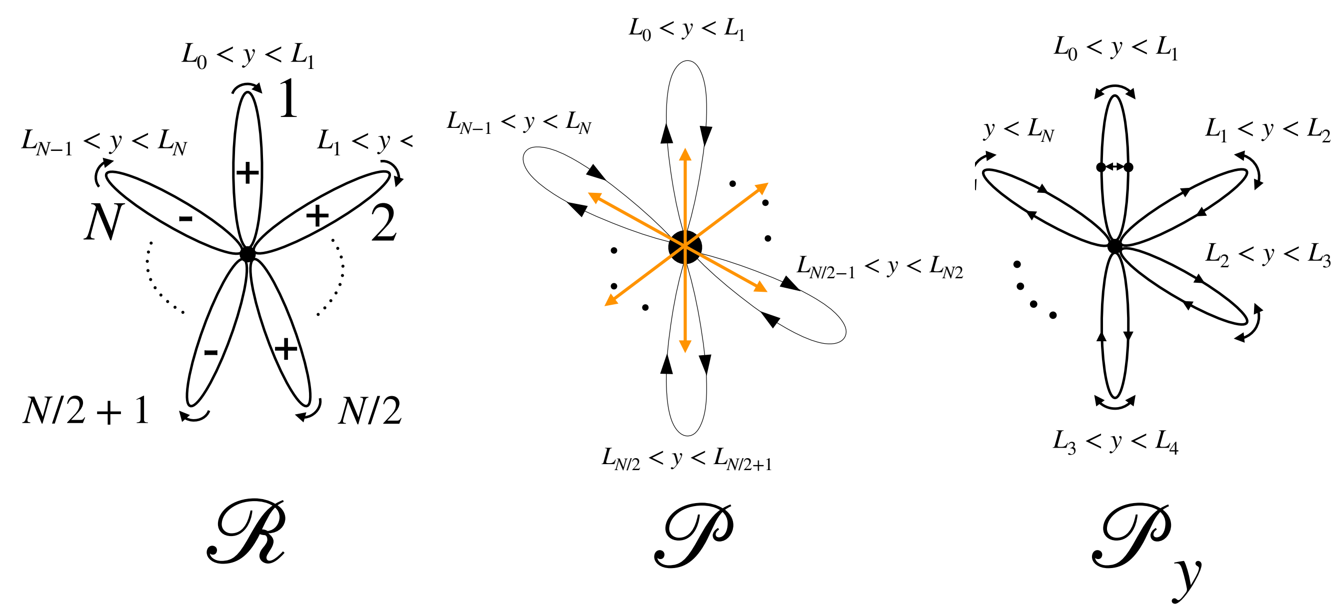
$$C(\gamma^A)^\top C^{-1} = \begin{cases} -\gamma^\mu & (\mu = 0,\ldots,3) \\ \gamma^y & (A=y) \end{cases}$$

Parity transformation (P²=1)

$$\Psi(x,y) \xrightarrow{P} \Psi^P(x,y) = \gamma^0 \mathcal{P}_y \Psi(x,y)$$

$$f_n^{(i)}(y) \xrightarrow{P} g_n^{(i)}(\tilde{y}), \quad g_n^{(i)}(y) \xrightarrow{P} f_n^{(i)}(\tilde{y})$$

Transformations in y-direction



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Result (1)

■ Correspondence to AZ classes

AZ classes

- **Setup**

  gapped free-fermion system

  $$H^2 = 1, \quad H^\dagger = H$$

- **Classification of *H***

  $$THT^{-1} = H \quad (T^2 = \pm 1)$$

  $$CHC^{-1} = -H \quad (C^2 = \pm 1)$$

  $$\Gamma H \Gamma^{-1} = -H \quad (\Gamma^2 = 1)$$

  classified into 10 classes

- **Quantity that characterizes the class**

  Topological number π_0

⟷

Our model

- **Setup**

  Dirac fermion on quantum graph extra dimension model

  $$U_{\mathrm{B}}^2 = 1, \quad U_{\mathrm{B}}^\dagger = U_{\mathrm{B}}$$

- **Classification of U_{B}**

  $$\hat{T} U_{\mathrm{B}} \hat{T}^{-1} = U_{\mathrm{B}} \quad (\hat{T}^2 = \pm 1)$$

  $$\hat{C} U_{\mathrm{B}} \hat{C}^{-1} = -U_{\mathrm{B}} \quad (\hat{C}^2 = \pm 1)$$

  $$\hat{\Gamma} U_{\mathrm{B}} \hat{\Gamma}^{-1} = -U_{\mathrm{B}} \quad (\hat{\Gamma}^2 = 1)$$

  (+ symmetry in y-direction)

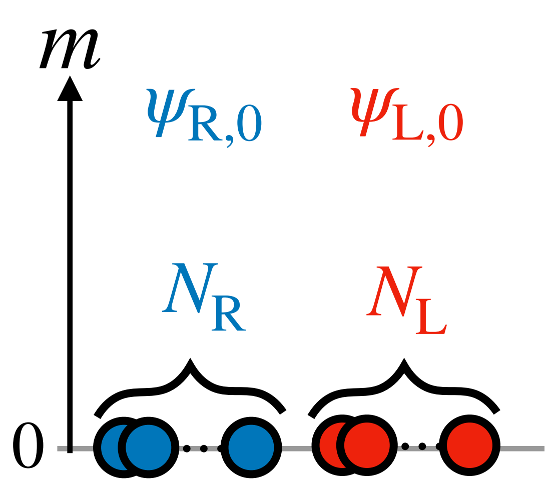
- **Quantity that characterizes the class**

  Witten index or parity of the chiral zero-mode

Result (2)

■ Correspondence of topological

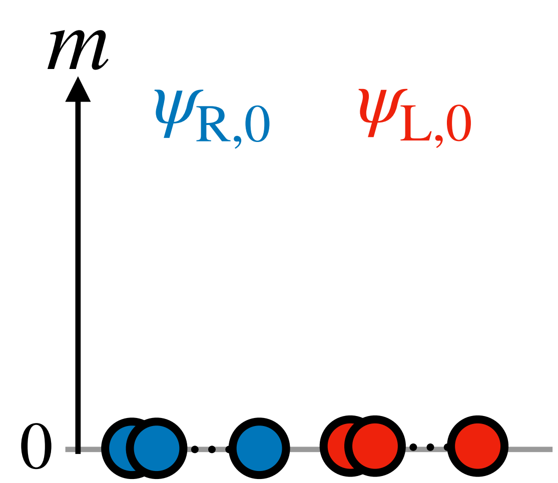
$$U_B = V \begin{pmatrix} 1_{2N-k} & 0 \\ 0 & -1_k \end{pmatrix} V^\dagger$$



$$N_R - N_L = N - k$$

class	T sym.	C sym.	Γ sym.
A	0	0	0
AI	+1	0	0

$\mathbb{Z}$ ⟷ $N - k$

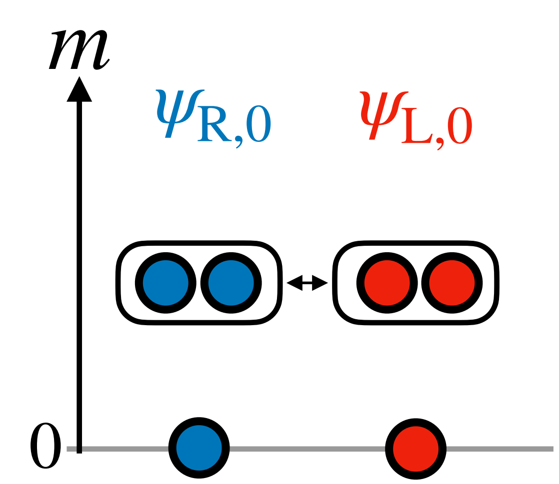


$$N_R - N_L = N - 2b$$

$(b = 0, 1, \cdots, N,\ N \in 2Z)$

class	T sym.	C sym.	Γ sym.
AII	-1	0	0

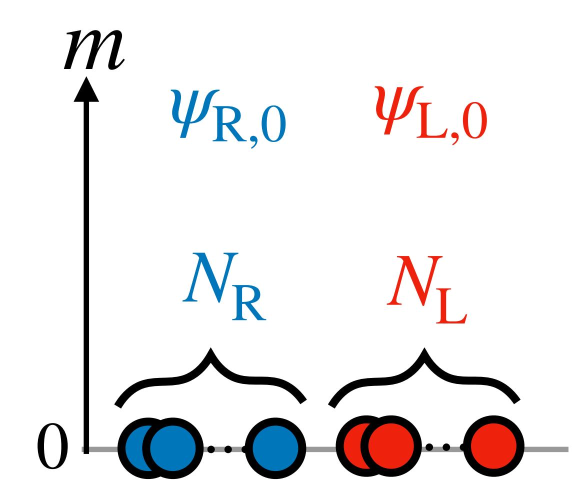
$2\mathbb{Z}$ ⟷ $N - 2b$



$$N_R - N_L = 0$$

class	T sym.	C sym.	Γ sym.
BDI	+1	+1	+1
D	0	+1	0

$\mathbb{Z}_2$ ⟷ parity of N_R



$$N_R - N_L = 0$$

class	T sym.	C sym.	Γ sym.
AIII	0	0	1
DIII	-1	+1	1
CII	-1	-1	1
C	0	-1	0
CI	+1	-1	1

0 ⟷ 0

(0 : No symmetry, ±1 : Positive or Negative of square)

Summary and Discussion

■ Summary

- We show the correspondence between quantum graph extra dimension model and topological classification of matter (AZ classification).

Hamiltonian		Boundary condition
$H^2 = 1, \quad H^\dagger = H$	$\longleftrightarrow$	$U_\mathrm{B}^2 = 1, \quad U_\mathrm{B}^\dagger = U_\mathrm{B}$

■ Discussion

- We investigate the correspondence in real condensed matter system (superfluid ^{3}He-A phase which could reproduce 2-dim Minkowski + quantum graph).

Buck up slides

Application to various areas

■ Applicable to various research areas

This because graphs have fascinating structures arising from boundary conditions.

Energy scale

- Extra dimensional model

SM scale

- Quantum chaos
- SUSY quantum mechanics
- Scattering theory on graph
- Nanotechnology

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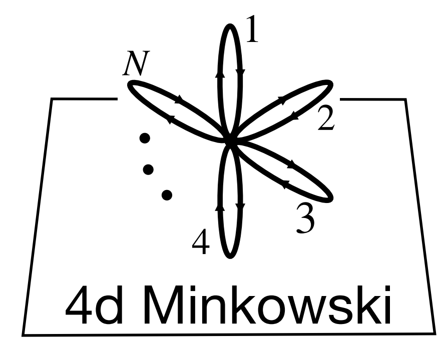
Quantum graph



5d space-time

Problem in the SM

- Generation problem
- Mass hierarchy

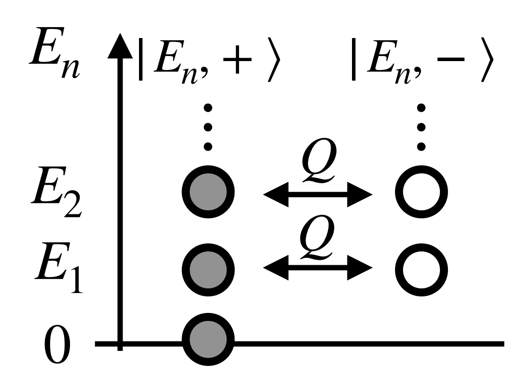
Be solvable

SUSY algebra

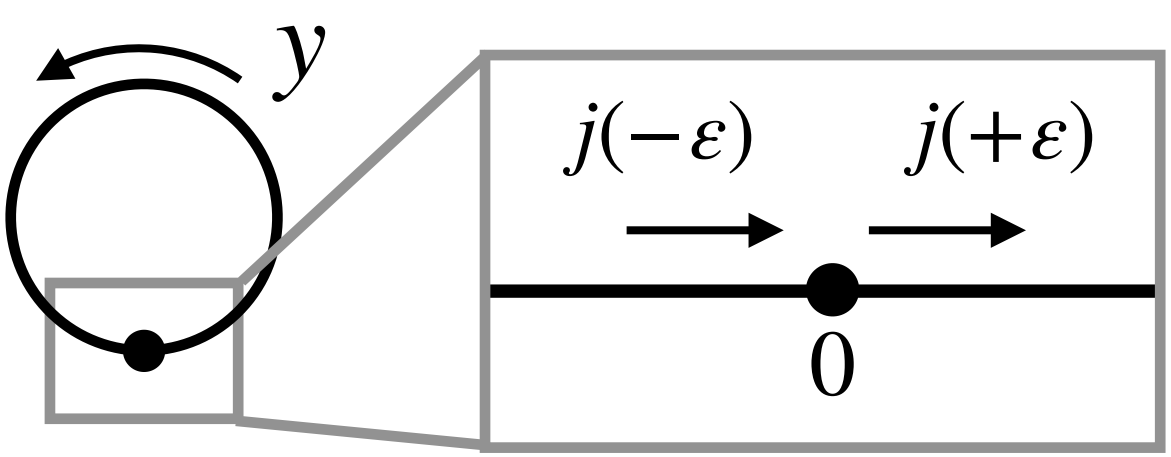
$$H = Q^2,$$

$$[(-1)^F]^2 = 1,$$

$$\{Q, (-1)^F\} = 0$$



B.C. in simple example



Quantum System

Free scaler particle in one-dim.

Current conservation $\longleftrightarrow$ **Boundary conditions (B.C.)**

$$j(-\varepsilon) = j(+\varepsilon)$$

$$j(y) = -i\left[\varphi^* \varphi'(y) - \varphi'^* \varphi(y)\right]$$

$$(1_2 - U)\Phi + iL_0(1_2 + U)\Phi' = 0$$

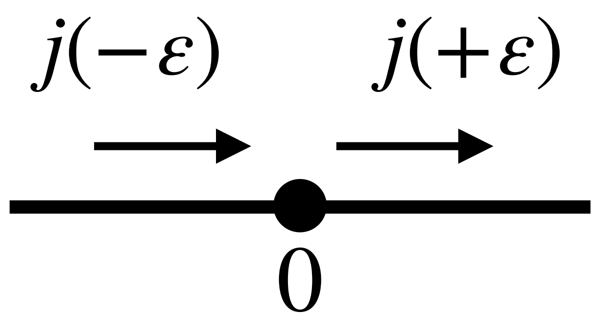
$$\Phi \equiv \begin{pmatrix} \varphi(+\varepsilon) \\ \varphi(-\varepsilon) \end{pmatrix}, \quad \Phi' \equiv \begin{pmatrix} \varphi'(+\varepsilon) \\ -\varphi'(-\varepsilon) \end{pmatrix}$$

$$U = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in U(2)$$

$\downarrow$

The boundary conditions are characterized by the parameters of U(2).

Derivation of B.C.



$$j(y) = -\,i\left[\varphi^{*}\varphi'(y) - \varphi'^{*}\varphi(y)\right]$$

$$\Phi \equiv \begin{pmatrix} \varphi(+\varepsilon) \\ \varphi(-\varepsilon) \end{pmatrix}, \quad \Phi' \equiv \begin{pmatrix} \varphi'(+\varepsilon) \\ -\varphi'(-\varepsilon) \end{pmatrix}$$

Rewriting the current conservation raw

$$j(-\varepsilon) = j(+\varepsilon)$$

$$\Longleftrightarrow |\,\Phi - iL_0\Phi'\,|^2 = |\,\Phi + iL_0\Phi'\,|^2$$

(L_0: parameter for dimension matching)

Two complex vectors with equal length can be connected by a unitary transformation.

$$\Longleftrightarrow \Phi - iL_0\Phi' = U(\Phi + iL_0\Phi')$$

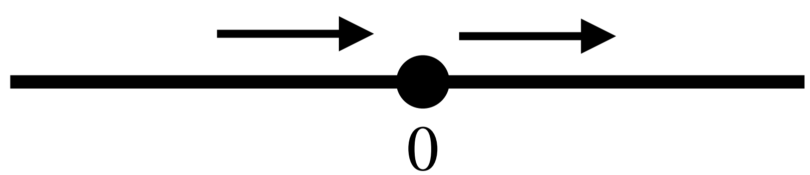
$$\boxed{(1_2 - U)\Phi + iL_0(1_2 + U)\Phi' = 0}$$

(*)The boundary conditions can be parameterized by $U \in U(2)$.

Specific B.C. (1)

The boundary conditions are characterized by the parameters of U(2).

$$j(y) = -i\left[\varphi^{*}\varphi'(y) - \varphi'^{*}\varphi(y)\right]$$



$$U = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in U(2)$$

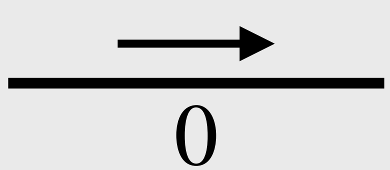
Connected

$$U = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

Periodic B.C.

$$\varphi(-\varepsilon) = \varphi(+\varepsilon)$$

$$\varphi'(-\varepsilon) = \varphi'(+\varepsilon)$$

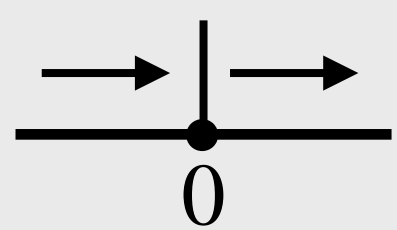


$$U = \frac{1}{2}\begin{pmatrix} e^{i\theta} - 1 & e^{i\theta} + 1 \\ e^{i\theta} + 1 & e^{i\theta} - 1 \end{pmatrix}$$

δ-function like B.C.

$$\varphi(-\varepsilon) = \varphi(+\varepsilon)$$

$$\varphi'(-\varepsilon) \neq \varphi'(+\varepsilon)$$

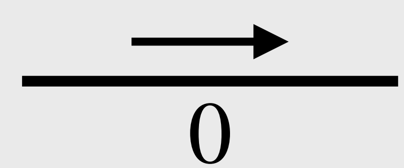


$$U = \begin{pmatrix} 0 & e^{-i\theta} \\ e^{i\theta} & 0 \end{pmatrix}$$

twisted B.C.

$$\varphi(-\varepsilon) = e^{i\theta}\varphi(+\varepsilon)$$

$$\varphi'(-\varepsilon) = e^{i\theta}\varphi'(+\varepsilon)$$



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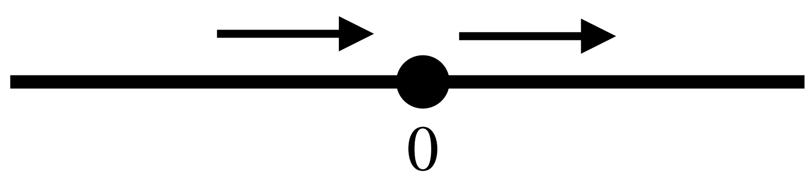
Specific B.C. (2)

The boundary conditions are characterized by the parameters of U(2).

$$j(y) = -\, i\left[\varphi^{*}\varphi'(y) - \varphi'^{*}\varphi(y)\right]$$



$$U = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in U(2)$$

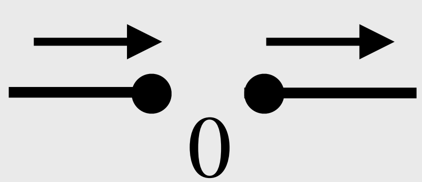
Disconnected

$$U = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$$

Dirichlet B.C.

$$\varphi(-\varepsilon) = 0$$

$$\varphi(+\varepsilon) = 0$$

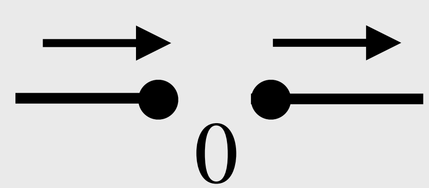


$$U = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Neumann B.C.

$$\varphi'(-\varepsilon) = 0$$

$$\varphi'(+\varepsilon) = 0$$

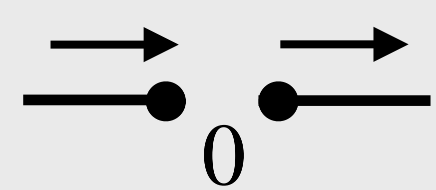


$$U = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$$

Robin B.C.

$$a\varphi(-\varepsilon) + b\varphi'(-\varepsilon) = 0$$

$$c\varphi(+\varepsilon) + d\varphi'(+\varepsilon) = 0$$



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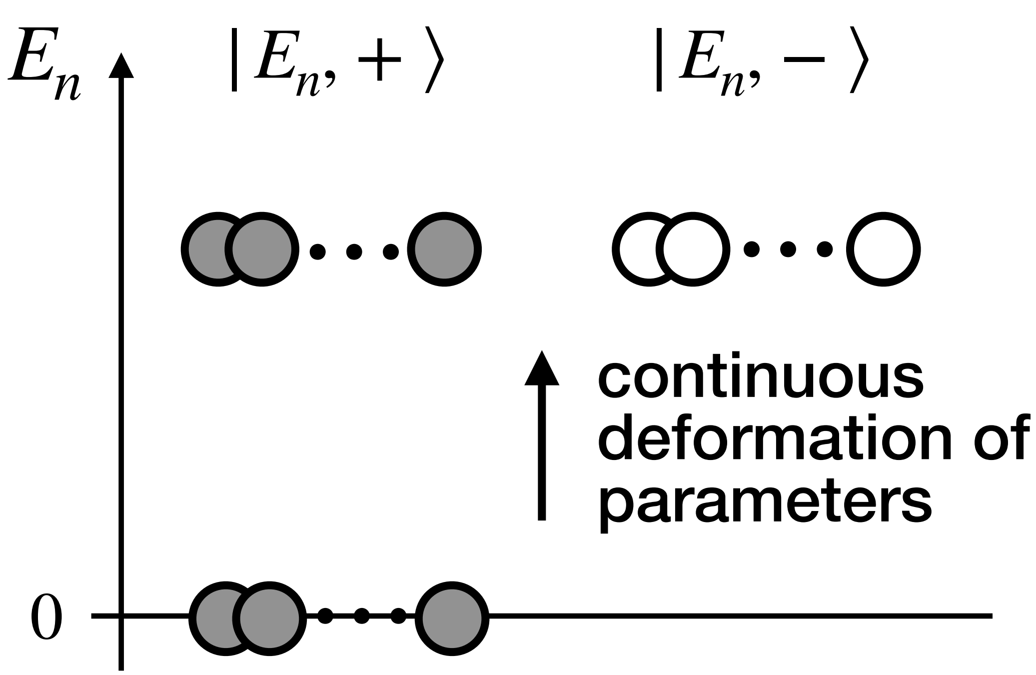
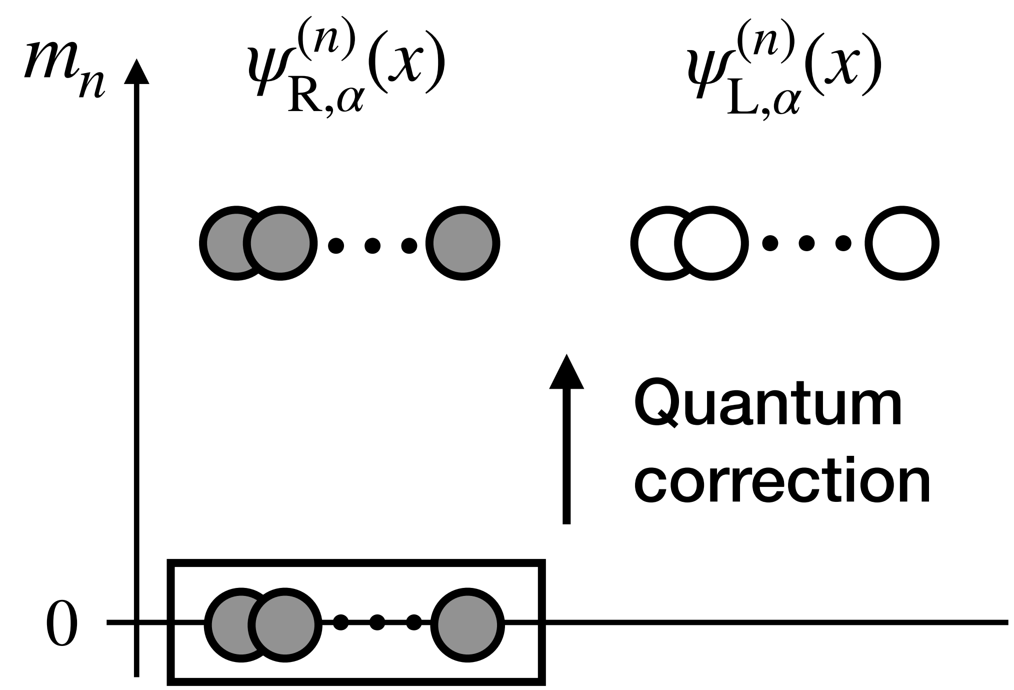
Perspective on SUSY quantum mechanics



Zero-mode in SUSY QM



Zero-mode of 5d Dirac fermion

Witten index

$$\Delta_W = \Big(\text{ \# of } |0,+\rangle\Big) - \Big(\text{ \# of } |0,-\rangle\Big)$$

Observable particle at SM energy scale

=generation of fermion flavor

Details of calculation

T (T²=+1) symmetric class

$$\Psi(x,y) \xrightarrow{T} \Psi^T(x,y) = U_T \Psi^*(x,y)$$

$$U_T \gamma^A U_T^{-1} = \begin{cases} (\gamma^0)^* & (A=0) \\ -(\gamma^i)^* & (i \neq 0) \end{cases}$$

substituting KK expansion

$$\sum_{n=0}^{\infty} [\psi^{(i)}_{\mathrm{R},n}(x) f^{(i)}_n(y) + \psi^{(i)}_{\mathrm{L},n}(x) g^{(i)}_n(y)]$$

$$\xrightarrow{T} U_T K \sum_{n=0}^{\infty} [\psi^{(i)}_{\mathrm{R},n}(x) f^{(i)}_n(y) + \psi^{(i)}_{\mathrm{L},n}(x) g^{(i)}_n(y)]$$

$$= \sum_{n=0}^{\infty} [U_T \psi^{(i)*}_{\mathrm{R},n}(x) f^{(i)*}_n(y) + U_T \psi^{(i)}_{\mathrm{L},n}(x) g^{(i)*}_n(y)]$$

$$\longrightarrow \begin{cases} \psi^{(i)}_{\mathrm{R/L},n}(x) \xrightarrow{T} U_T \psi^{(i)*}_{\mathrm{R/L},n}(x) \\ f^{(i)}_n(y) \xrightarrow{T} f^{(i)*}_n(y), \quad g^{(i)}_n(y) \xrightarrow{T} g^{(i)*}_n(y) \end{cases}$$

transformation of mode function

$$\vec{F} \xrightarrow{T} \vec{F}^*, \quad \vec{G} \xrightarrow{T} \vec{G}^*$$

Boundary conditions to be satisfied by the transformed mode function

$$(1_{2N} - U_{\mathrm{B}})\vec{F}^* = \vec{0}$$
$$(1_{2N} + U_{\mathrm{B}})\vec{G}^* = \vec{0}$$

Boundary conditions before transformation

$$(1_{2N} - U_{\mathrm{B}})\vec{F} = \vec{0}$$
$$(1_{2N} + U_{\mathrm{B}})\vec{G} = \vec{0}$$
$$(U_{\mathrm{B}} \in U(2N), \quad U_{\mathrm{B}}^2 = 1_{2N})$$

$$\boxed{\therefore U_{\mathrm{B}}^* = U_{\mathrm{B}}, \quad K U_{\mathrm{B}} K^{-1} = U_{\mathrm{B}}}$$

AZ $THT^{-1} = H \quad (H^2 = 1)$

Details of “Z_2 = parity of N_R”

- **Correspondence of Z_2 class (BDI, D)**

  If C^2=+1 symmetry is present, the massive modes need to be 4-fold degenerate.