The Fuzzy Onion
A proposal

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Why to build a model of quantum space?

- It is nice and instructive in some way.
- It a step towards phenomenological consequences or testable predictions.

Connecting theory with observations

Regular black holes as dark matter (Nicolini: 0807.1939), vacuum dispersion effect and modification of GZK limit (Amelino-Camelia: 0012238, 0107086, Review: 2111.05659), cosmology from matrix models (Kim, Nishimura, Tsuchiya: 1108.1540; Steinacker: 1710.11495; Brahma, Brandenberger, Laliberte: 2107.11512), ...

An example of a model of 3D quantum space:

$$[x_i, x_j] \;=\; 2i\,\lambda\,\varepsilon_{ijk}\,x_k\,,$$

$$[\mathrm{a}_\alpha, \mathrm{a}^\dagger_\beta] \;=\; \delta_{\alpha\beta}, \quad [\mathrm{a}_\alpha, \mathrm{a}_\beta] \;=\; [\mathrm{a}^\dagger_\alpha, \mathrm{a}^\dagger_\beta] \;=\; 0\,,$$

$$\mathcal{F} : |n_1, n_2\rangle \;=\; \frac{(\mathrm{a}^\dagger_1)^{n_1}\,(\mathrm{a}^\dagger_2)^{n_2}}{\sqrt{n_1!\,n_2!}}\;|0\rangle\,.$$

- NC coordinates defined using Pauli matrices as $x^i = \lambda a^\dagger \sigma^i a$.
- Sequence of fuzzy spheres of increasing radia, $x^2 = r^2 - \lambda^2$.
- $r = \lambda\,(\mathrm{N}+1)$ is quantized (N is the number operator on $\mathcal{F}$).
- Other realisations of R^3_λ more suited for QFT (Vitale, Wallet: 1212.5131)

The free Hamiltonian is defined as

$$\mathrm{H}_0\Psi = \frac{1}{2\lambda \mathrm{r}}[\mathrm{a}^+_\alpha, [\mathrm{a}_\alpha, \Psi]].$$

And we can define and solve physical problems, for example

$$\left(\mathrm{H}_0 - \frac{\alpha}{\mathrm{r}}\right)\Psi = \mathrm{E}\Psi.$$

A discrete spectrum with λ-correction has been obtained (Gáliková, SK, Prešnajder: 1309.4614).

$$E^{I}_{\lambda\, n} = \frac{1}{\lambda^2}\left(1 - \sqrt{1 + (\alpha\lambda/n)^2}\right),$$

$$E^{II}_{\lambda\, n} = \frac{2}{\lambda^2} - E^{I}_{\lambda\, n}.$$

(c.f. Nicolini: 2208.05390)

The fuzzy sphere S^2_λ

$$[x_i, x_j] = i\varepsilon_{ijk}\frac{2r}{\sqrt{N^2-1}}x_k,$$

$$x^2 = r^2, \quad x_i = \frac{2r}{\sqrt{N^2-1}}L_i.$$

$$\Phi = \sum_{l=0}^{N-1}\sum_{m=-l}^{l} c_{lm}Y_{lm},$$

$$[L_i, [L_i, Y_{lm}]] = l(l+1)Y_{lm}, \quad [L_3, Y_{lm}] = mY_{lm}.$$

The fuzzy sphere S_λ^2

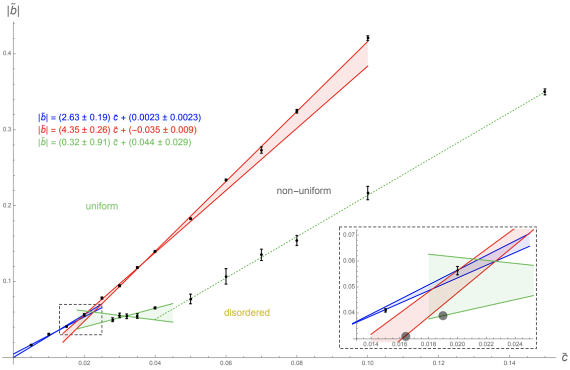
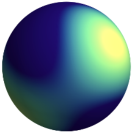


Figure: Phase diagram of the scalar field theory on S_λ^2 (O'Connor, SK: 1805.08111)

$$S_N[\Phi] = \mathrm{Tr}\ \left(a\ \Phi \mathcal{K} \Phi + b\ \Phi^2 + c\ \Phi^4\right).$$

The fuzzy sphere S^2_λ

$$\begin{pmatrix} -0.182241 & -0.356949 + 0.0169752\,i & 0.0260558 + 0.055678\,i & -0.0418167 - 0.358403\,i \\ -0.356949 - 0.0169752\,i & 0.723061 & -0.266625 - 0.323709\,i & -0.209613 - 0.250825\,i \\ 0.0260558 - 0.055678\,i & -0.266625 + 0.323709\,i & 0.93628 & 0.115833 + 0.0969497\,i \\ -0.0418167 + 0.358403\,i & -0.209613 + 0.250825\,i & 0.115833 - 0.0969497\,i & 0.30945 \end{pmatrix}$$



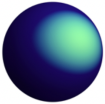
$$\Phi = \sum_{l=0}^{N-1} \sum_{m=-l}^{l} c_{lm} Y_{lm},$$

$$[L_i, [L_i, Y_{lm}]] = l(l+1) Y_{lm}, \quad [L_3, Y_{lm}] = m Y_{lm}.$$

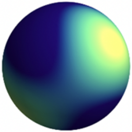
The fuzzy sphere S^2_λ

$$\begin{pmatrix} -0.906971 & -0.243605 + 0.178959\,i \\ -0.243605 - 0.178959\,i & 0.50394 \end{pmatrix}$$

$$\begin{pmatrix} 0.354273 & -0.0108495 - 0.165916\,i & 0.102162 - 0.640258\,i \\ -0.0108495 + 0.165916\,i & 0.428309 & -0.125031 + 0.00326721\,i \\ 0.102162 + 0.640258\,i & -0.125031 - 0.00326721\,i & 0.517689 \end{pmatrix}$$

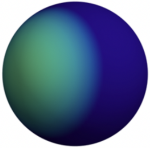


$$\begin{pmatrix} -0.182241 & -0.356949 + 0.0169752\,i & 0.0260558 + 0.055678\,i & -0.0418167 - 0.358403\,i \\ -0.356949 - 0.0169752\,i & 0.723061 & -0.266625 - 0.323709\,i & -0.209613 - 0.250825\,i \\ 0.0260558 - 0.055678\,i & -0.266625 + 0.323709\,i & 0.93628 & 0.115833 + 0.0969497\,i \\ -0.0418167 + 0.358403\,i & -0.209613 + 0.250825\,i & 0.115833 - 0.0969497\,i & 0.30945 \end{pmatrix}$$

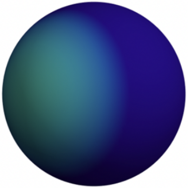


The fuzzy sphere S^2_λ

$$\begin{pmatrix} -0.366637 & -0.389114 + 0.530756\, i \\ -0.389114 - 0.530756\, i & -0.773059 \end{pmatrix}$$



$$\begin{pmatrix} -0.320966 + 0.\, i & -0.336983 + 0.459649\, i & 0. + 0.\, i \\ -0.336983 - 0.459649\, i & -0.569848 + 0.\, i & -0.336983 + 0.459649\, i \\ 0. + 0.\, i & -0.336983 - 0.459649\, i & -0.81873 + 0.\, i \end{pmatrix}$$



$$\Phi^{(N)}$$

$$\mathcal{D} \uparrow \quad \mathcal{U} \downarrow$$

$$\Phi^{(N+1)}$$

$$\Phi^{(N)} = \sum_{l=1}^{N-1} \sum_{m=-l}^{l} c_{lm}^{(N)} Y_{lm}^{(N)}$$

$$\mathcal{D}\uparrow \quad \mathcal{U}\downarrow$$

$$\Phi^{(N+1)} = \sum_{l=1}^{\color{red}{N-1}} \sum_{m=-l}^{l} c_{lm}^{(N+1)} Y_{lm}^{(N+1)}$$

The fuzzy onion $\mathcal{O}^3_\lambda$

$$
\begin{aligned}
\Phi^{(N)} &= \sum_{l=1}^{N-1} \sum_{m=-l}^{l} c_{lm}^{(N)} Y_{lm}^{(N)} \\
\mathcal{D}\uparrow \quad \mathcal{U}\downarrow & \\
\Phi^{(N+1)} &= \sum_{l=1}^{N-1} \sum_{m=-l}^{l} c_{lm}^{(N+1)} Y_{lm}^{(N+1)}
\end{aligned}
$$

$$
\begin{aligned}
c_{l,m}^{(N)} &= c_{l,m}^{(N+1)} \text{ for: } l \leq N-1 \\
c_{N,m}^{(N+1)} &= 0
\end{aligned}
$$

Scalar field theory on $\mathcal{O}_\lambda^3$

$$\Psi = \begin{pmatrix} \Phi^{(1)} & & & \\ & \Phi^{(2)} & & \\ & & \Phi^{(3)} & \\ & & & \ddots \end{pmatrix}.$$

It is easy* to define a potential

$$V(\Psi) = \mathrm{Tr}\ \left(b\ \Psi^2 + c\ \Psi^4\right),$$

and the angular part of the Laplace operator

$$\mathcal{K}_L\Psi = \begin{pmatrix} \mathcal{K}^{(1)}\Phi^{(1)} & & & \\ & \mathcal{K}^{(2)}\Phi^{(2)} & & \\ & & \mathcal{K}^{(3)}\Phi^{(3)} & \\ & & & \ddots \end{pmatrix} \to \mathrm{Tr}\ \Psi\mathcal{K}_L\Psi.$$

Scalar field theory on $\mathcal{O}^3_\lambda$

So far it is just a theory of disconnected fuzzy spheres. But since we have the $\mathcal{U}$ and $\mathcal{D}$ operators, we can compare neighbouring spheres:

$$\partial_r^2 \Phi^{(N)} = \frac{\mathcal{D}\phi^{(N+1)} + \mathcal{U}\phi^{(N-1)} - 2\phi^{(N)}}{\lambda^2}.$$

This can in turn be used to define the radial part of the kinetic term

$$\mathrm{Tr}\ \Psi \mathcal{K}_R \Psi = \sum_{n,l,m} c_{lm}^{(n)*} \frac{(n+1)c_{lm}^{(n+1)} + (n-1)c_{lm}^{(n-1)} - 2nc_{lm}^{(n)}}{4n\ \lambda^2}.$$

$$S[\Psi] = \mathrm{Tr}\ \left(a\ \Psi\mathcal{K}\Psi + b\ \Psi^2 + c\ \Psi^4\right), \quad \mathcal{K} = \mathcal{K}_R + \mathcal{K}_L.$$

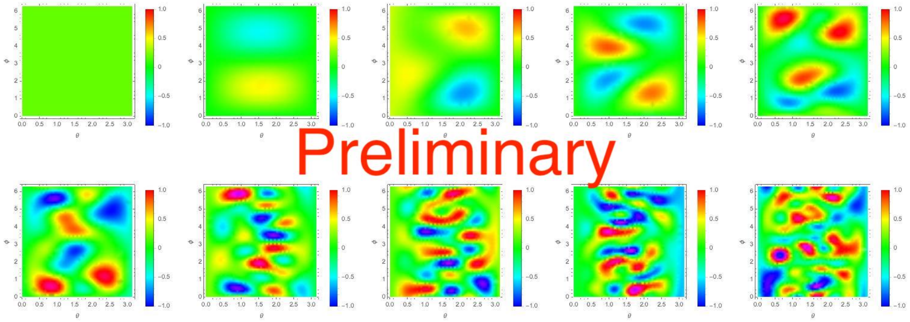


Figure: Monte Carlo simulation of $S[\Psi]$ with $a = 1, b = -13, c = 16$.

Outlook (work in progress)

- Do a proper HMC study.
- Think about a better way to define $V(\Psi)$.
- Study a quantum mechanical model.
- Analytical understanding of the Fourier picture.
- Use it to understand some aspects of quantum space phenomenology (vacuum dispersion, microscopic black holes, expansion of space).