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The Jacobi sigma model

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Outline

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Motivations

- It is natural generalisation of the Poisson sigma model; PSM is a topological 2-dim field theory, important for many reasons:
  - First introduced [Ikeda '94, Scallet-Strobl '94] in relation with two-dimensional field theories with non-trivial target space, e.g. gauge and gravity models, gauged WZW models
  - It provides a relation with deformation quantization and path-integral approach to Kontsevich star product [Cattaneo-Felder '99]
  - it provides interesting strings backgrounds
  - it exhibits QM dynamics on the boundary
  - it possesses dynamical generalizations by adding a metric tensor on the target space
- Jacobi manifolds share some features of twisted Poisson manifolds
  - Twisted PSM are relevant for the same reasons as above; specifically, they provide non-associative generalization of deformation quantization and entail a WZW extension of the action functional
- It is interesting to compare twisted PSM with the natural JSM induced by the Jacobi structure: they result to be different models
  - JSM may bring to deformation quantisation of Jacobi manifolds

  Noteworthy cases: Contact and Locally Conformal Symplectic (LCS) manifolds are two main classes of Jacobi manifolds, with Poisson, symplectic and GCS as special cases;

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The PSM action

- It is a two-dimensional, topological field theory with target space a Poisson manifold (M, Π)

$$S = \int_\Sigma \left[\eta_i \wedge dX^i + \frac{1}{2} \Pi^{ij}(X) \eta_i \wedge \eta_j \right], \quad i,j = 1, \dots, \dim M$$

the fields $X : \Sigma \to M$, $\eta \in \Omega^1(\Sigma, X^*(T^*M))$, $dX \in \Omega^1(\Sigma, X^*(TM))$
In local coordinates (t, u) on Σ $\quad dX^i = \partial_\mu X^i du^\mu$, $\eta_i = \eta_{i\mu} du^\mu$

- EOM's

$$dX^i + \Pi^{ij}(X)\eta_j = 0,$$

$$d\eta_i + \frac{1}{2}\partial_i \Pi^{jk}(X)\eta_j \wedge \eta_k = 0$$

consistency of the two requires $[\Pi, \Pi]_S^{ijk} = \Pi^{il}\partial_l \Pi^{jk} + cycl\{ijk\} = 0$
if $\partial\Sigma \neq 0$ b.c. needed, $\eta|_{\partial\Sigma} = 0$

- the model is invariant under diffeos of the source space
- If the target space is symplectic, the auxiliary fields η_i can be integrated away, resulting in a second order action, $S = \int_\Sigma \omega_{ij} dX^i \wedge dX^j$
This corresponds to a topological action with B-field coinciding with the symplectic two-form

The Hamiltonian setting

From the Lagrangian,

$$L(X,\zeta;\beta) = \int_I du\left[-\zeta_i\dot{X}^i + \beta_i\left(X'^i + \Pi^{ij}(X)\zeta_j\right)\right]$$

with $\Sigma = \mathbb{R}\times[0,1]$ $t\in\mathbb{R}$, $u\in[0,1]$. $\beta_i = \eta_{ti}$, $\zeta_i = \eta_{ui}$, $\dot{X} = \partial_t X$, $X' = \partial_u X$

- X^i and $-\zeta_i$ are conjugate variables with PB $\{X^i,\zeta_j\} = \delta^i_j\delta(u-u')$
- $\pi_\beta = 0$ primary constraint
- $X'^i + \Pi^{ij}(X)\zeta_j = 0$ secondary constraint
- the Hamiltonian
$$H_\beta = -\int_I du\,\beta_i\left[X'^i + \Pi^{ij}(X)\zeta_j\right]$$
is itself a pure constraint

Gauge invariance

- The model is Diff invariant. The infinitesimal generators are the Hamiltonian vector fields associated with H_β by the canonical Poisson bracket

$$\xi_\beta = \{H_\beta, \cdot\} = \int du \left(\dot{X}^i \frac{\delta}{\delta X^i} + \dot{\zeta}_i \frac{\delta}{\delta \zeta_i} \right),$$

with $\dot{X}^i = -\Pi^{ij}\beta_j, \quad \dot{\zeta}_i = \partial_u \beta_i - \partial_i \Pi^{jk}\zeta_j\beta_k$. But their algebra only closes if one allows $\beta = \beta(u, X(u)) \Longrightarrow$

- The algebra $\{H_\beta\}$ closes under PB's

$$\{H_\beta, H_{\tilde{\beta}}\} = H_{[\beta,\tilde{\beta}]}$$

  - with $[\beta,\tilde{\beta}] = d\langle\beta, \Pi(\tilde{\beta})\rangle - i_{\Pi(\beta)}d\tilde{\beta} + i_{\Pi(\tilde{\beta})}d\beta$ the Koszul bracket [Koszul '85] (or Gerstenhaber) for $\beta \in \Omega^1(M)$–> $\Omega(M)$ is BV algebra
  - it satisfies Jacobi identity provided Π is a Poisson tensor. $\langle\ ,\ \rangle$ is the natural pairing between T^*M and TM. $\Rightarrow$

- the map $\beta \to H_\beta$ is a Lie algebra homomorphism, the Hamiltonian constraints are first class and the Hamiltonian vector fields generate gauge transformations.
- The reduced phase space of the model is defined in the usual way as the quotient $C = P/\mathscr{G}$, where $\mathscr{G}$ is the gauge group, P the (infinite-dim) constrained phase space
- It can be proven [Cattaneo'01] that the reduced phase space is a finite-dimensional manifold of dimension 2dim(M).

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H-twisted Poisson structure [Mylonas-Schupp-Szabo '12, Ikeda-Strobl '20]

It is defined by a skew-symmetric bi-vector field Π with non-vanishing Schouten brackets

$$[\Pi,\Pi]_S := 2\langle \Pi\otimes\Pi\otimes\Pi, H\rangle \quad (\star)$$

This defines the three-form H as a kind of inverse of the three-vector field $[\Pi,\Pi]_S$ if Π is non degenerate

The H-twisted Poisson Sigma model is a generalization of PSM by adding a WZW term:

$$S=\int_{\Sigma=\partial N}\left[\eta_i\wedge dX^i+\frac{1}{2}\Pi^{ij}(X)\eta_i\wedge\eta_j\right]+\int_N X^*(H)$$

$i,j=1,...dimM$ and X is extended to $X:N\to M$. The EOM's

$$dX^i+\Pi^{ij}(X)\eta_j=0,$$

$$d\eta_i+\frac{1}{2}\left(\partial_i\Pi^{jk}(X)\eta_j\wedge\eta_k+H_{ijk}dX^j\wedge dX^k\right)=0$$

Consistency of the two now is assured by $(\star)$ being satisfied

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Jacobi brackets

Jacobi brackets [Lichnerowicz'78] on $\mathscr{F}(M)$ generalise Poisson brackets

- they are defined by (Λ, E), Λ a bi-vector field, E Reeb vector field

$$\{f,g\}_J = \Lambda(df,dg) + f(Eg) - g(Ef), \quad with$$

  - $[\Lambda,\Lambda]_S = 2E\wedge\Lambda, \qquad \mathscr{L}_E\Lambda = 0,$
  - Jacobi identity holds for J, not for Λ
  - Leibniz rule is violated $\quad\longrightarrow \{f,gh\}_J = \{f,g\}_J h + g\{f,h\}_J + gh(Ef)$
  - $E = 0$ corresponds to Poisson brackets
- two main classes of Jacobi manifolds: contact (odd dims) and LCS (even dims)
- generic ones admit foliations with contact and/or LCS leaves [Vaisman]

LCS and Contact manifolds [Marle, Vaisman]

- LCS manifolds are even-dim manifolds with a non-degenerate two-form $\omega \in \Omega^2(M)$ locally equivalent to a symplectic form ξ; i.e.
  - $\omega = e^{-f}\xi, \; f \in C^\infty(U_i)$
  - $d\omega = -\alpha \wedge \omega \quad (\alpha = df$ locally)
  - The global structures (Λ, E), are uniquely defined in terms of (α, ω)

$$\iota_E \omega = -\alpha, \quad \iota_{\Lambda(\gamma)}\omega = -\gamma, \qquad \gamma \in \Omega^1(M)$$

  [Globally conformal symplectic and symplectic manifolds are particular cases of LCS]

- Contact manifolds are odd-dim manifolds, $dimM = 2n+1$, with a contact one-form, ϑ and a volume form Ω s.t.

$$\vartheta \wedge (d\vartheta)^n = \Omega$$

  - The global structure (Λ, E) is uniquely fixed by

$$\iota_E(\vartheta \wedge (d\vartheta)^n) = (d\vartheta)^n \quad \iota_\Lambda(\vartheta \wedge (d\vartheta)^n) = n\vartheta \wedge (d\vartheta)^{n-1}$$

[In $dimM = 3 \Longrightarrow \iota_E\vartheta = 1, \;\; \iota_E d\vartheta = 0, \;\; \Lambda(\vartheta) = 0, \;\; \Lambda(d\vartheta) = 1$]

Poissonization [Lichnerowicz]

Given a Jacobi manifold (M, Λ, E) the manifold $M \times \mathbb{R}$ may be given a one-parameter family of homogeneous Poisson structures

$$\Pi = e^{-\tau}(\Lambda + \frac{\partial}{\partial \tau} \wedge E), \quad \tau \in \mathbb{R}$$

(homogeneous: $L_{\partial_\tau}\Pi = -\Pi$) This allows for definition of Hamiltonian vector fields:

$$X_f := \pi_*(X^{\Pi}_{e^\tau f})|_{\tau=0}$$

where $X^{\Pi}_{e^\tau f}$ is the Hamiltonian vector field associated with the Poisson bracket on $M \times \mathbb{R}$ and $\pi : M \times \mathbb{R} \to M$ the projection map
This yields, for any function $f \in \mathscr{F}(M)$

$$X_f = \Lambda(df, \cdot) + fE$$

The map $f \to X_f$ is homomorphism of Lie algebras, $[X_f, X_g] = X_{\{f,g\}_J}$, where the bracket $[\cdot, \cdot]$ is the standard Lie bracket of vector fields

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The action

The Jacobi sigma model with source space a two-dimensional manifold Σ with boundary $\partial\Sigma$ and target space (M, Λ, E) is defined by the action functional

$$S[X,(\eta,\lambda)] = \int_\Sigma \left[\eta_i \wedge dX^i + \frac{1}{2}\Lambda^{ij}(X)\eta_i \wedge \eta_j - E^i(X)\eta_i \wedge \lambda\right]$$

with boundary condition $\eta_{|_{\partial\Sigma}} = 0$

- field configurations: $(X,(\eta,\lambda))$ $X : \Sigma \to M$ the base map, $(\eta,\lambda) \in \Omega^1(\Sigma, X^*(J^1 M))$
  [$J^1 M = T^* M \oplus \mathbb{R}$ the 1-jet bundle of real functions on M]
- Sections of $J^1 M$ are isomorphic to the algebra of one-forms

$$\{e^\tau(\alpha + f d\tau) | \alpha \in \Omega^1(M), f \in C^\infty(M), \tau \in \mathbb{R}\} \subseteq \Omega^1(M \times \mathbb{R})$$

closed with respect to the Koszul bracket of the Poissonised manifold
Vaisman'00

Important difference: new auxiliary field, λ, which is a one-form on the source manifold Σ but a scalar on the Jacobi manifold. It is needed in order to take into account the Reeb field E [J is not a bi-vector field]

Hamiltonian approach

In local coordinates $t \in \mathbb{R}$, $u \in [0,1]$ for $\Sigma = \mathbb{R} \times [0,1]$

$$dX = \dot{X}dt + X'du, \qquad \eta = \beta dt + \zeta du, \quad \lambda = \lambda_t dt + \lambda_u du,$$

the Lagrangian becomes

$$L = \int_I du \left[-\dot{X}^i \zeta_i + \beta_i \left(X'^i + \Lambda^{ij}\zeta_j - E^i \lambda_u \right) + \lambda_t \left(E^i \zeta_i \right) \right]$$

λ_t, λ_u scalar fields, $\dot{X}, X'$ and β, ζ carrying and extra index on (the pull-back of) M

The b.c. $\eta_{\partial\Sigma} = 0$ results in $\beta_{\partial\Sigma} = 0$ and no b.c. for $\lambda \Longrightarrow$

$$H = -\int_I du\, \beta_i \left(X'^i + \Lambda^{ij}\zeta_j - E^i \lambda_u \right) + \lambda_t \left(E^i \zeta_i \right)$$

- $-\zeta_i, X^i$ conjugate variables with canonical PB
- $\pi_{\beta_i}, \pi_{\lambda_t}, \pi_{\lambda_u}$ primary constraints
- $\mathscr{G}_{\beta_i} = X'^i + \Lambda^{ij}\zeta_j - E^i \lambda_u, \;\; \mathscr{G}_{\lambda_t} = E^i \zeta_i, \;\; \mathscr{G}_{\lambda_u} = E^i \beta_i$ secondary constraints

The Poisson algebra of Constraints

- The matrix $\{\phi_k, \phi_{k'}\}$ has finite, non-zero rank $\Longrightarrow$ there are second class constraints

- The rank is not maximal $\Longrightarrow$ there are first class constraints

  Second class: $\pi_{\lambda_u}, \pi_{\beta_m}, \mathscr{G}_{\lambda_u}, \mathscr{G}_{\beta_m}$ [$E = \mathscr{E}\partial_m$ has been chosen]

  All other are first class $\Longrightarrow$ A combination of these yields gauge transformations

$$K(\beta_a, \lambda_t, a_t, a_{\beta_a}) = \int du\, \lambda_t \mathscr{G}_{\lambda_t} + \beta_a \mathscr{G}_{\beta_a} + a_t \pi_{\lambda_t} + a_{\beta_a} \pi_{\beta_a}, \quad a = 1, \ldots, m-1$$

and $\beta_a, \lambda_t, a_t, a_{\beta_a}$ are gauge parameters

- Primary constraints may be ignored in computing the algebra of gauge generators because their PB's are strongly zero.

  Similarly to the Poisson sigma model, the algebra will only close on-shell

$\Longrightarrow$ In order to obtain a closed algebra off-shell, gauge parameters have to be functions of the fields $\beta_a = \beta_a(u, X(u)), \lambda_t = \lambda_t(u, X(u))$

The Poisson algebra of gauge generators

One gets for $K(\beta,\lambda_t) = \int du\, \beta_c \mathscr{G}_{\beta_c} + \lambda_t \mathscr{G}_{\lambda_t} \qquad c = 1, ..., m-1$

$$\{K(\beta,\lambda_t), K(\tilde\beta,\tilde\lambda_t)\} = \int du du' \Big[\mathscr{G}_{\beta_c} \Big(\beta_a \tilde\beta_b \partial_c \Lambda^{ba} + \Lambda^{aj} (\tilde\beta_a \partial_j \beta_c - \beta_a \partial_j \tilde\beta_c)$$

$$+ \mathscr{E} (\tilde\lambda_t \partial_m \beta_c - \lambda_t \partial_m \tilde\beta_c) \Big)$$

$$+ \mathscr{G}_{\lambda_t} \Big(\beta_a \tilde\beta_b \Lambda^{ab} + \Lambda^{aj} (\tilde\beta_a \partial_j \lambda_t - \beta_a \partial_j \tilde\lambda_t) + \mathscr{E} (\tilde\lambda_t \partial_m \lambda_t - \tilde\lambda_t \partial_m \tilde\lambda_t) \Big) \Big]$$

which is possible to rearrange as

$$\{K_{(\beta,\lambda_t)}, K_{(\tilde\beta,\tilde\lambda_t)}\} = -K_{[(\beta,\lambda_t),(\tilde\beta,\tilde\lambda_t)]}$$

with $[(\beta,\lambda_t),(\tilde\beta,\tilde\lambda_t)]$ a generalisation of the Koszul bracket to Jacobi manifolds [Kerbrat'93,Vaisman'00] $[T^*M$ now replaced by $J^1M]$

$$[(\alpha, f), (\beta, g)] = (\gamma, h)$$

$\gamma = L_{\Lambda(\alpha)}\beta - L_{\Lambda(\beta)}\alpha - d(\Lambda(\alpha,\beta)) - d\Big(f L_E \beta - g L_E \alpha - \alpha(E)\beta + \beta(E)\alpha \Big)$
$h = \{f, g\}_J - \Lambda(df - \alpha, dg - \beta)$
Jacobi identity holds, provided the manifold is Jacobi.

Results

There are first and second class constraints

First class constraints generate gauge transformations

The algebra of gauge transformations is closed if we allow for a generalisation of the Koszul bracket for gauge parameters

Further results and perspectives

The reduced phase space C is finite-dim, with $dimC = 2dimM - 2$

The auxiliary fields may be integrated out

A dynamical extension of the model is possible, yielding a string sigma model action with metric and B field

Quantization along the lines of H-twisted Poisson structures

Analyse the geometry of reduced phase space

Models building for specific target spaces, both LCS and contact mflds

Work in collaboration with Francesco Bascone and Franco Pezzella
JHEP 03 (2021) 110 e-Print: 2007.12543 [hep-th]
Symmetry 13 (2021) 7, 1205 e-Print: 2105.09780 [hep-th]

Thank you