Non-commutative coordinates from quantum gravity

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Perturbative quantum gravity

Perturbative quantum gravity (1/2)

- Study gravity $S = S_{\mathrm{G}} + S_{\mathrm{M}}$ with $S_{\mathrm{G}} = \frac{1}{16\pi G_{\mathrm{N}}} \int (R - 2\Lambda)\sqrt{-g}\, \mathrm{d}^4 x$ and S_{M} matter action perturbatively around a given background
- Metric decomposition $g_{\mu\nu} = g^{(0)}_{\mu\nu} + h_{\mu\nu}$
- Linear approximation: Gravitational action S_{G} to second order in perturbation $h_{\mu\nu}$, matter action S_{M} to linear order
- Straightforward example: One-loop quantum corrections to Newtonian potential of free particle in flat space background
- $V(r) = -\frac{Gm}{r}\left[1 + \frac{[1+\frac{5}{4}(1-6\xi)^2]N_0 + 6N_{1/2} + 12N_1}{45\pi}\frac{\ell_{\mathrm{Pl}}^2}{r^2}\right]$
- $\ell_{\mathrm{Pl}} = \sqrt{\hbar G_{\mathrm{N}}/c^3}$: Planck length, N_s: number of particles of spin s, ξ: non-minimal scalar coupling
- Various methods for calculation, via effective action: MBF 1607.03129

Perturbative quantum gravity (2/2)

- Infinitesimal coordinate transformation $x^\mu \to \tilde{x}^\mu = x^\mu + \xi^\mu$ ($\delta x^\mu = \xi^\mu$) gives gauge transformation $\delta A = \mathcal{L}_\xi A$
- If $A^{(0)} = 0$, then at linear order $\delta A^{(1)} = \mathcal{L}_\xi A^{(0)} = 0$, but at higher orders gauge-dependent since $A^{(1)} \neq 0$ in general
- Stewart–Walker lemma: if $\delta A^{(n+1)} = 0$, then $A^{(n)}$ is a linear combination of δ's with constant coefficients
- Perturbative diffeo's lead to gauge transformations of $h_{\mu\nu}$, which can be seen as spin-2 field on the background: $\delta_\xi h_{\mu\nu} = \nabla^{(0)}_\mu \xi_\nu + \nabla^{(0)}_\nu \xi_\mu + \cdots$, with $\nabla^{(0)}$ background derivative
- (Non-local) decomposition of metric perturbation at linear order: $h_{\mu\nu} = h^{\text{inv}}_{\mu\nu} + \mathcal{L}_Z g^{(0)}_{\mu\nu}$ with $\delta h^{\text{inv}}_{\mu\nu} = 0$ and $\delta Z^\mu = \xi^\mu$ (e.g., Bardeen variables)
- Newtonian potential $V(r) = \frac{1}{2} \left\langle h^{\text{inv}}_{00}(x) \right\rangle$

Relational observables

Relational observables (1/2)

- Let there be given 4 fields $X^{(\mu)}[g, \phi, \ldots]$ depending on field content, transforming under diffeo's as scalars: $\delta X^{(\mu)} = \xi^\rho \partial_\rho X^{(\mu)}$, and their background value $X_0^{(\mu)}$
- Expand $X^{(\mu)} = X_0^{(\mu)} + X_1^{(\mu)} + \ldots$ in perturbation theory and invert to obtain $X_0^{(\mu)}[X] \Rightarrow$ transforms inversely to a scalar
- Invariant observable $\mathcal{A}(\chi)$ is given by evaluating a field A at the position $X_0^{(\mu)}$, holding $X^{(\mu)}$ fixed
- $\Rightarrow$ Relational observables: the $X^{(\mu)}$ are configuration-dependent coordinates, $\mathcal{A}(\chi)$ is the value of A provided that $\chi^\mu = X^{(\mu)}$, and by evaluating at $X_0^{(\mu)}$ we interpret $\mathcal{A}$ as field on background
- Generic spacetime: use curvature scalars for the $X^{(\mu)}$
- Add scalar fields by hand (Brown–Kuchař dust), but this changes the dynamics

Giesel et al. 2003.13729

Relational observables (2/2)

- Cosmology (FLRW): only 1 scalar field (inflaton ϕ), but we need 4
- Minkowski/de Sitter: no scalar field at all
- First complete solution in cosmology: Since background spatial coordinates are harmonic $\triangle x^i = 0$, define $X^{(i)}(x)$ as harmonic coordinates for the full Laplacian $\triangle_{g,\phi}$ on constant-inflaton hypersurfaces $\phi = \text{const}$ Brunetti et al. 1605.02573
- Generalisations to flat space, to causal harmonic coordinates in cosmology, to geodesic lightcone coordinates MBF, Lima 2108.11960 and references therein
- Invariant metric perturbation $\mathcal{H}_{\mu\nu}(X) = \frac{\partial x^\alpha}{\partial X^\mu} \frac{\partial x^\beta}{\partial X^\nu} g_{\alpha\beta}(x(X)) - \eta_{\mu\nu}$, holding X fixed. First order: $\mathcal{H}_{\mu\nu} = h_{\mu\nu} - 2g^{(0)}_{\rho(\mu} \partial_{\nu)} X^{(\rho)}_{(1)} + \ldots$
- Gauge-invariant graviton corrections to Newton potential, to Hubble rate in inflation, ... MBF et al. 2109.09753, MBF 1806.11124, Lima 2007.04995, ...

Non-commutative coordinates

Non-commutative coordinates (1/4)

- In flat space: generalised harmonic coordinates $\nabla^2 X^{(\mu)} = 0$
- First order: $X^{(\mu)}_{(1)}(x) = \int G(x,y) \left[\partial_\nu h^{\mu\nu} - \frac{1}{2} \partial^\mu h \right] (y) \, \mathrm{d}^4 y$
- Green's function of the flat d'Alembertian $\partial^2 G(x,y) = \delta^4(x-y)$
- Classical theory: retarded Green's function, quantum theory: Feynman propagator
- Since quantised $h_{\mu\nu}$ has non-trivial commutator, also the field-dependent coordinates $X^{(\mu)}$ have!
- Linear theory: $\left[X^{(\mu)}_1, X^{(\nu)}_1 \right] = \left\langle \left[X^{(\mu)}_1, X^{(\nu)}_1 \right] \right\rangle \mathbb{1}$
- Graviton propagator: $G^{\mathrm{F}}_{\mu\nu\rho\sigma}(x,x') = \left(2\eta_{\mu(\rho}\eta_{\sigma)\nu} - \eta_{\mu\nu}\eta_{\rho\sigma} \right) G^{\mathrm{F}}(x,x')$
- Scalar propagator: $G^{\mathrm{F}}(x,x') = - \int \frac{\mathrm{e}^{\mathrm{i}p(x-x')}}{p^2 - \mathrm{i}0} \frac{\mathrm{d}^4 p}{(2\pi)^4}$

Non-commutative coordinates (2/4)

- Possible IR divergence due to integration over infinite space-time
  related: Higuchi 0809.1255, Higuchi Lee 0903.3881
- Solution: in-in formalism (= Schwinger–Keldysh formalism = closed-time-path formalism) computes true expectation values
- Instead of simple time integration computing in-out matrix elements, add backward part of time contour
- If all fields are on forward ("+") contour: time-ordered (Feynman) expectation values, all fields on backward ("−") contour: anti-time-ordered (Dyson), mixed: Wightman functions
- Change also coordinates ($H^\mu = \partial_\nu h^{\mu\nu} - \frac{1}{2}\partial^\mu h$):
  $X_1^{+(\mu)}(x) = \int G^{++}(x,y) H_+^\mu(y) \, \mathrm{d}^4 y - \int G^{+-}(x,y) H_-^\mu(y) \, \mathrm{d}^4 y,$
  $X_1^{-(\mu)}(x) = \int G^{-+}(x,y) H_+^\mu(y) \, \mathrm{d}^4 y - \int G^{--}(x,y) H_-^\mu(y) \, \mathrm{d}^4 y$

Non-commutative coordinates (3/4)

- Classical limit: both time contours coincide, and both $X_1^{+(\mu)}(x)$ and $X_1^{-(\mu)}(x) \to X_1^{(\mu)}(x) = \int G_{\text{ret}}(x, y) H^\mu(y) \, \mathrm{d}^4 y$
- Quantum theory: IR divergence is avoided by choosing adiabatic interacting vacuum $|\Omega\rangle$
- Practically: choose initial time $t^\pm \to -\infty(1 \pm \mathrm{i}\epsilon)$ with $\epsilon > 0$ and take limit $\epsilon \to 0$ after integration Peskin/Schroeder QFT
- $\left[X_1^{(\mu)}(x), X_1^{(\nu)}(x')\right] =$
$\int \frac{\mathrm{i}\eta^{\mu\nu}}{2|\mathbf{p}|^3} \Big[\cos\left[|\mathbf{p}| (t - t')\right] |\mathbf{p}| (t - t') - \sin\left[|\mathbf{p}| (t - t')\right] \Big] \mathrm{e}^{\mathrm{i}\mathbf{p}(\mathbf{x}-\mathbf{x}')} \frac{\mathrm{d}^3\mathbf{p}}{(2\pi)^3} \mathbb{1}$
$= -\mathrm{i} \frac{\eta^{\mu\nu}}{8\pi} \operatorname{sgn}(t - t') \Theta[-(x - x')^2] \mathbb{1}$
- Fully Lorentz-invariant result: commutator vanishes for spacelike separations, is constant for timelike separations

Non-commutative coordinates (4/4)

- Since background coordinates x^μ commute, we have for the full field-dependent coordinates $X^\mu = x^\mu + \sqrt{16\pi}\ell_{\mathrm{Pl}} X_1^{(\mu)}(x) + \mathcal{O}(\ell_{\mathrm{Pl}}^2)$ the commutator
- $[X^\mu, Y^\nu] = -2\mathrm{i}\ell_{\mathrm{Pl}}^2 \eta^{\mu\nu} \operatorname{sgn}(X^0 - Y^0)\Theta[-(X-Y)^2] + \mathcal{O}(\ell_{\mathrm{Pl}}^3)$
- Most general form of possible commutator: $[X^\mu, Y^\nu] = \mathrm{i}\theta^{\mu\nu}(X, Y)$ with $\theta^{\mu\nu}(X, Y) = -\theta^{\nu\mu}(Y, X)$
- Well-known NC: $\theta^{\mu\nu} = -\theta^{\nu\mu} =$ const, our NC: symmetry in indices, antisymmetry from sgn function
- Standard deviation formula $\Delta_A \Delta_B \geq \frac{1}{2} \left|\langle [A, B] \rangle\right|$ leads to generalised uncertainty principle $\Delta_X \Delta_Y \geq \ell_{\mathrm{Pl}}^2 \, \Theta[-(X-Y)^2]$: measurements of coordinates with timelike separation are uncertain with standard deviation of Planck length, measurement with spacelike separation can be exact

Conclusion and outlook

- To define invariant observables in (perturbative) quantum gravity, one needs field-dependent coordinates
- These coordinates are non-commutative, and their commutator follows from standard effective quantum field theory techniques
- Non-commutativity is not constant and Lorentz-invariant, standard deviation for coordinate measurements is Planck length $\ell_{\mathsf{Pl}} = \sqrt{G_{\mathsf{N}}}$
- Future work: higher orders in ℓ_{Pl}, different backgrounds (de Sitter space, ...), observational signatures

Thank you for your attention

Questions?

Reference: M. B. Fröb, A. Much, K. Papadopoulos, Non-commutative Geometry from Perturbative Quantum Gravity, arXiv:2207.03345

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