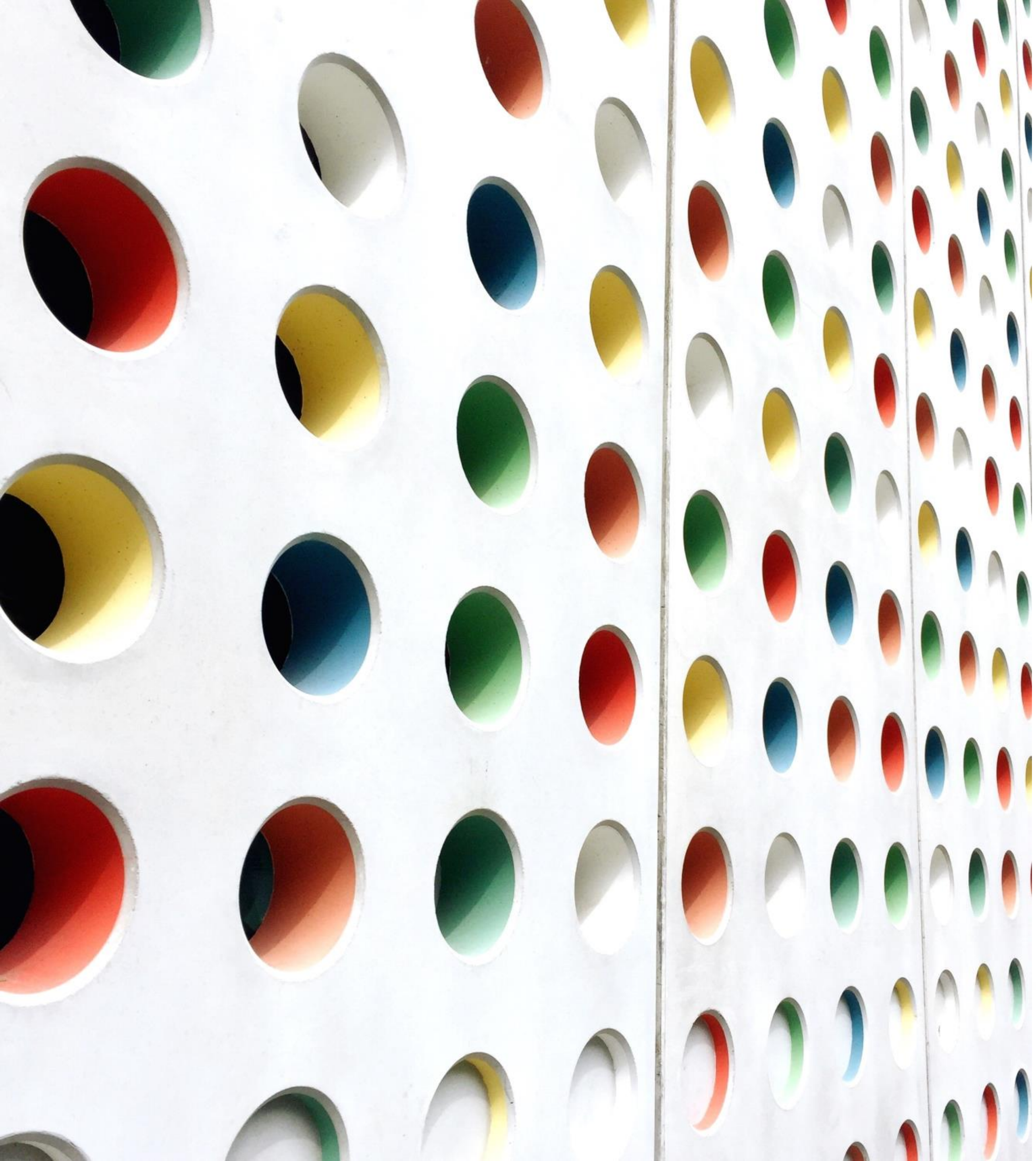

A few remarks on κ

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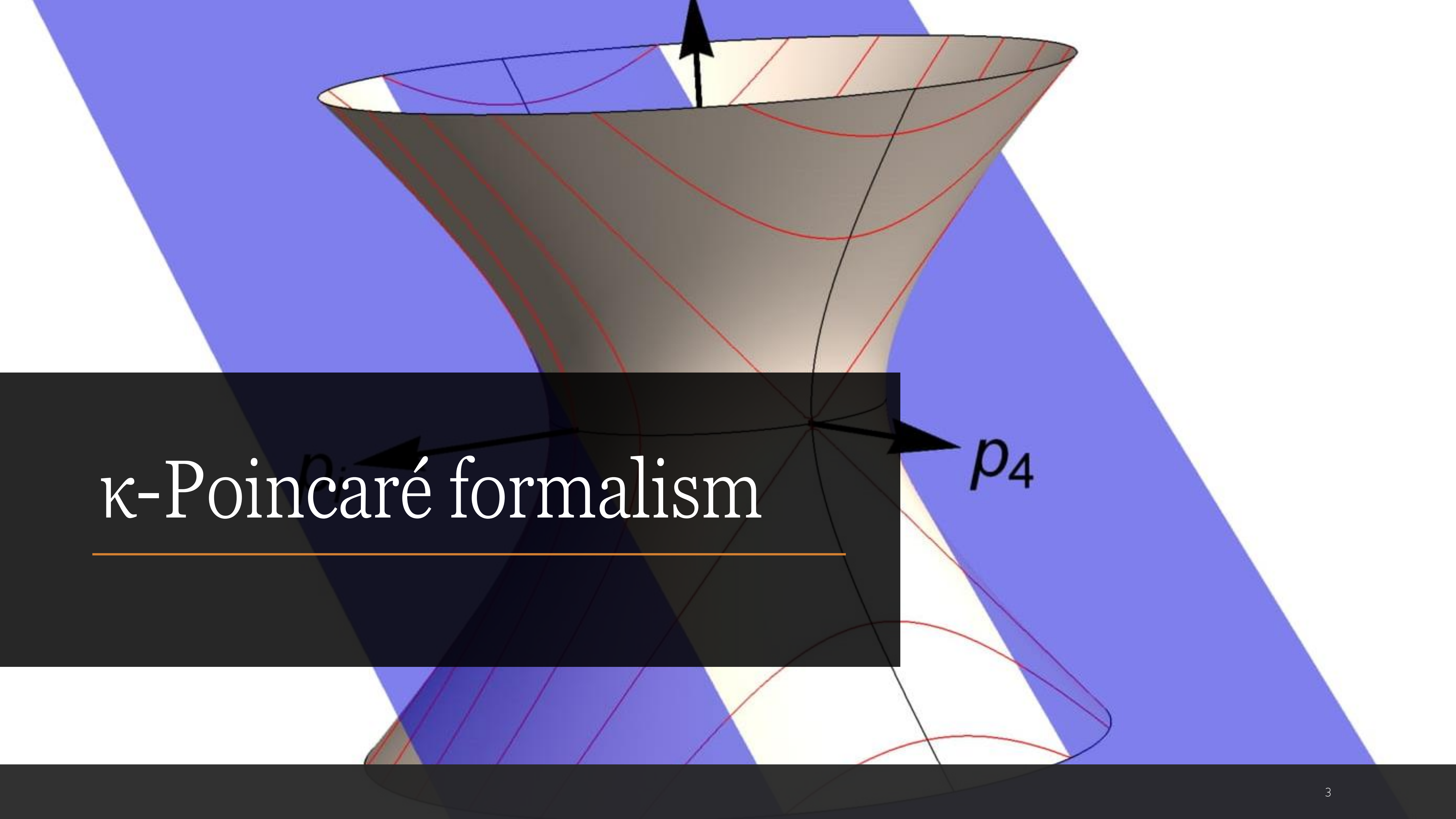
The plan

1. κ-Poincaré symmetry from de Sitter momentum space and group theory;
2. κ-Poincaré phenomenology.

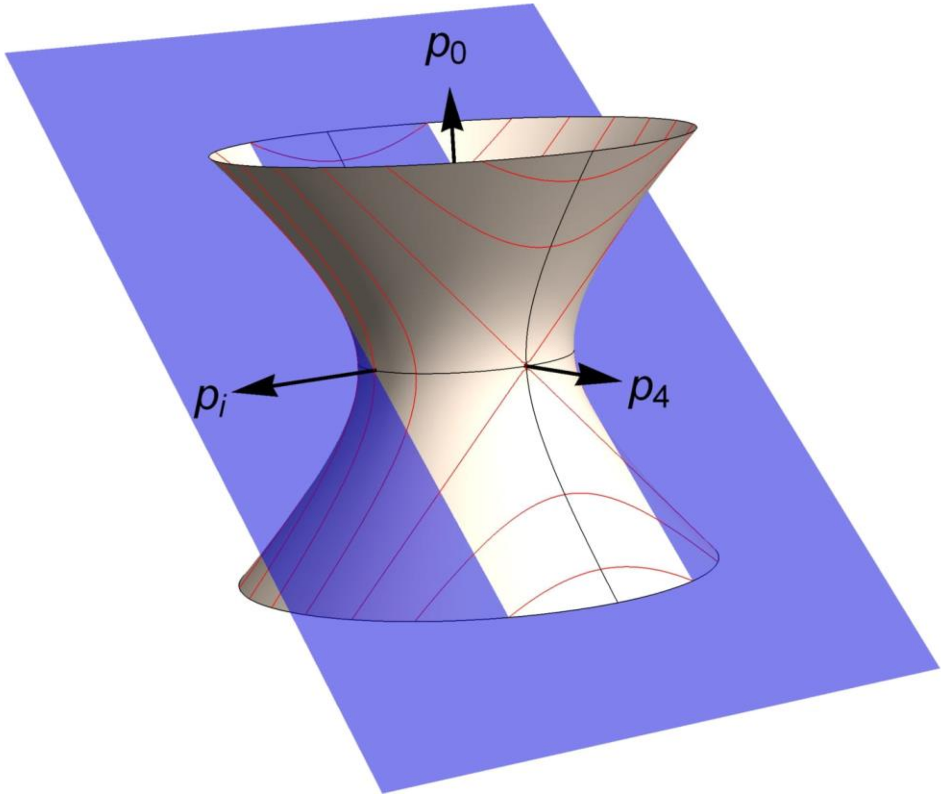
- Based on
  - M. Arzano & JKG A group theoretic description of the κ-Poincaré Hopf algebra, 2204.09394 [hep-th]
  - A. Bevilacqua, JKG, & W. Wislicki, κ-deformed complex scalar field: Conserved charges, symmetries, and their impact on physical observables, *Phys.Rev.D* 105 (2022) 10, 105004, 2201.10191 [hep-th]



κ-Poincaré formalism

Why curved momentum space?

- It naturally arises in 2+1 gravity coupled to particles, when (topological) degrees of freedom of gravity are integrated out.
- In 3+1 dimensional QG it is expected that there may exists a smooth, flat spacetime limit (the Planck length, $\ell \to 0$), characterized by the presence of the mass scale (the Planck mass, κ – finite).
- Theories with curved (dynamical) momentum space and more general Born geometries are of interest.
- Here I will be interested in the curved momentum space of the form of an(3) group manifold, with constant curvature 1/ κ^2.



AN(3) Lie algebra and an(3) Lie group

- The Lie algebra of *AN(3)* consists of 1 abelian and 3 nilpotent matrices

$$X^0 = -\frac{i}{\kappa}\left[\begin{array}{ccc} 0 & 0 & 1 \\ 0 & \mathbb{0} & 0 \\ 1 & 0 & 0 \end{array}\right], \quad X^i = \frac{i}{\kappa}\left[\begin{array}{ccc} 0 & \epsilon_i & 0 \\ (\epsilon_i)^T & \mathbb{0} & (\epsilon_i)^T \\ 0 & -\epsilon_i & 0 \end{array}\right]$$

- satisfying the commutator algebra (sometimes called κ-Minkowski noncommutative space)

$$[X^0, X^i] = \frac{i}{\kappa} X^i\,, \quad [X^i, X^j] = 0\,.$$

- An *an(3)* Lie group element can be written as

$$g = e^{ip_i X^i} e^{ip_0 X^0}\,.$$

A group element

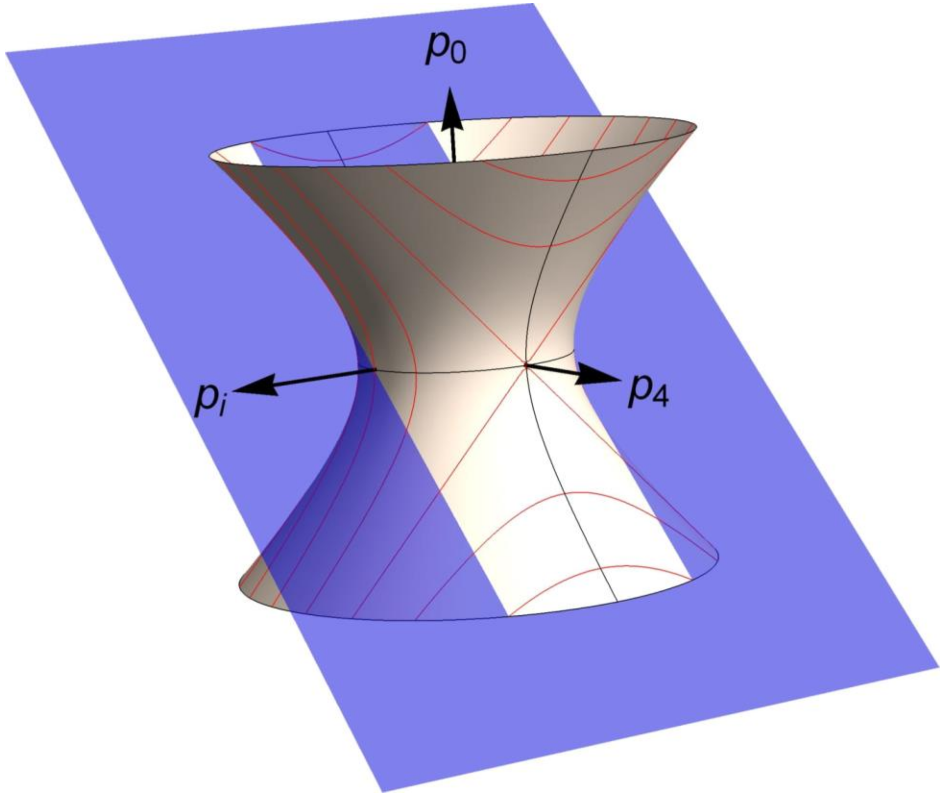
- Explicitly on finds

$$g = \frac{1}{\kappa}\begin{pmatrix} \tilde{P}_4 & \kappa\mathbf{P}/P_+ & P_0 \\ \mathbf{P} & \kappa\mathbb{1} & \mathbf{P} \\ \tilde{P}_0 & -\kappa\mathbf{P}/P_+ & P_4 \end{pmatrix}$$

$$-P_0^2 + \mathbf{P}^2 + P_4^2 = \kappa^2\,, \quad P_0 + P_4 > 0$$

- We define a (momentum eigen-)state labeled by g and momentum operators

$$\mathscr{P}_I|g\rangle = P_I(g)|g\rangle$$



Properties of momenta

- For two group elements g and h labeling two states, the most natural (unique) total momentum that can be built is $P_I(gh)$. Thus, to compute the total momentum of two states one has to compute the product of matrices and inspect the last column of the resulting matrix.

$$\boxed{P_\mu(gh) = P_\mu(g) \oplus P_\mu(h)}$$

$$gh = \frac{1}{\kappa^2}\begin{pmatrix} \tilde{P}_4 & \kappa\mathbf{P}/P_+ & P_0 \\ \mathbf{P} & \kappa\mathbb{1} & \mathbf{P} \\ \tilde{P}_0 & -\kappa\mathbf{P}/P_+ & P_4 \end{pmatrix}\begin{pmatrix} \tilde{Q}_4 & \kappa\mathbf{Q}/Q_+ & Q_0 \\ \mathbf{Q} & \kappa\mathbb{1} & \mathbf{Q} \\ \tilde{Q}_0 & -\kappa\mathbf{Q}/Q_+ & Q_4 \end{pmatrix} = \begin{pmatrix} \star & \star & P_0 \oplus Q_0 \\ \star & \star & \mathbf{P}\oplus\mathbf{Q} \\ \star & \star & \star \end{pmatrix}.$$

$$P_0 \oplus Q_0 = \frac{\kappa}{P_+}Q_0 + \frac{\mathbf{P}\cdot\mathbf{Q}}{P_+} + \frac{1}{\kappa}\,P_0\,Q_+$$

$$\mathbf{P}\oplus\mathbf{Q} = \frac{1}{\kappa}\,\mathbf{P}\,Q_+ + \mathbf{Q}$$

The co-product of momenta

- Now we have to learn how the momentum operators act on two particles states. They measure the total momentum of the two-particles system, and thus action is defined by

$$\Delta(\mathscr{P}_\mu)\,|g\rangle \otimes |h\rangle >\equiv P_\mu(gh)\,|g\rangle \otimes |h\rangle >= (P \oplus Q)_\mu\,|g\rangle \otimes |h\rangle >$$

- This operation, called co-product is a deformed counterpart of Leibniz rule

$$\Delta(\mathscr{P}_0) = \kappa\,(\mathscr{P}_+)^{-1} \otimes \mathscr{Q}_0 + \sum_i (\mathscr{P}_+)^{-1}\,\mathscr{P}_i \otimes \mathscr{Q}_i + \frac{1}{\kappa}\,\mathscr{P}_0 \otimes \mathscr{Q}_+$$

$$\Delta(\mathscr{P}_i) = \frac{1}{\kappa}\,\mathscr{P}_i \otimes \mathscr{Q}_+ + \mathscr{Q}_i \otimes \mathbb{1}\,.$$

The antipode of momenta

- We also have to learn how momentum operator acts on a state labeled by an inverse group element. This action is defined by the antipode and is a deformed generalization of the minus.

$$P_I(g\, g^{-1}) = 0 = P_I(g) \oplus P_I(g^{-1}) \equiv P_I(g) \ominus P_I(g)$$

$$\begin{aligned} \ominus P_0 &= -P_0 + \mathbf{P}^2/P_+ \\ \ominus \mathbf{P} &= -\kappa \mathbf{P}/P_+ \,. \end{aligned}$$

$$S(\mathcal{P}_I)|g\rangle = \mathcal{P}_I|g^{-1}\rangle = \ominus P_I|g^{-1}\rangle$$

$$\begin{aligned} S(\mathcal{P})_0 &= -\mathcal{P}_0 + (\mathcal{P}_i)^2\,(\mathcal{P}_+)^{-1} \\ S(\mathcal{P})_i &= -\kappa\mathcal{P}_i\,(\mathcal{P}_+)^{-1} \quad . \end{aligned}$$

Lorentz transformations

- In order to understand Lorentz transformation, we recall Iwasawa decomposition of the group $so(4,1) = so(3,1)\, an(3)$. Given an element of $so(4,1)$, it can be uniquely decomposed into a product of a Lorentz transformation Λ and an element g of $an(3)$ describing a point in momentum space

$$\Lambda g = g' \Lambda_g$$

- We define g' to be the Lorentz transformation of g.

Lorentz transformation

$$\Lambda g = g' \Lambda_g$$

- Consider an infinitesimal boost along the first axis, $\Lambda = 1 + \xi N_1$. We compute

$$\begin{pmatrix} 1 & \xi & 0 & 0 & 0 \\ \xi & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \star & \star & \star & \star & P_0 \\ \star & \kappa & 0 & 0 & P_1 \\ \star & 0 & \kappa & 0 & P_2 \\ \star & 0 & 0 & \kappa & P_3 \\ \star & \star & \star & \star & P_4 \end{pmatrix} = \begin{pmatrix} \star & \star & \star & \star & P_0 + \xi P_1 \\ \star & \kappa & 0 & 0 & P_1 + \xi P_0 \\ \star & 0 & \kappa & 0 & P_2 \\ \star & 0 & 0 & \kappa & P_3 \\ \star & \star & \star & \star & P_4 \end{pmatrix} \begin{pmatrix} 1 & \bar{\xi}_1 & 0 & 0 & 0 \\ \bar{\xi}_1 & 1 & \bar{\rho}_3 & -\bar{\rho}_2 & 0 \\ 0 & -\bar{\rho}_3 & 1 & 0 & 0 \\ 0 & \bar{\rho}_2 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\bar{\rho}_3 = \xi \frac{P_2}{P_+}, \quad \bar{\rho}_2 = -\xi \frac{P_3}{P_+}$$

$$\bar{\xi} = \frac{\kappa}{P_+}$$

- Therefore, P_μ form a Lorentz vector, while P_4 is a Lorentz scalar related to the Casimir.

- The last column of g' does not change when multiplied by Λ_g

Lorentz transformations

- In general, if $\Lambda = \mathbb{1} + \xi^i N_i$

$$\Lambda_g = \mathbb{1} + \xi^i h_i^a(g)\iota_a = \mathbb{1} + \xi^i \frac{\kappa}{P_+}\left(N_i + \frac{1}{\kappa}\epsilon_{ijk}P_j M_k\right)$$

Lorentz symmetry

- We define Lorentz action on one-particle state as

$$\mathscr{P}_\mu \mathscr{L}|g\rangle = \mathscr{P}_\mu|\Lambda g\rangle = P_\mu(g')|g'\rangle$$

- To find the action on two-particles state, i.e., the co-product of *Λ*, we must understand the action of Lorentz matrix on a product of two ***an(3)*** group elements. From Iwasawa decomposition

$$\Lambda(gh) = (gh)'\Lambda_{gh}$$

$$(gh)' = \Lambda gh\Lambda_{gh}^{-1} = (g')(\Lambda_g h)\Lambda_{gh}^{-1}$$

$$\boxed{P_\mu\left((gh)'\right) = P_\mu\left(g'\Lambda_g h\right) \neq P_\mu\left(g'h'\right)}$$

Lorentz generators co-product

- In the matrix form the Lorentz action can be written as

$$(gh)' = \frac{1}{\kappa^2}\begin{pmatrix} \tilde{P}_4' & \kappa\left(\mathbf{P}/P_+\right)' & P_0' \\ \mathbf{P}' & \kappa\mathbb{1} & \mathbf{P}' \\ \tilde{P}_0' & -\kappa\left(\mathbf{P}/P_+\right)' & P_4' \end{pmatrix}\begin{pmatrix} 1 & \frac{\kappa}{P_+}\xi & 0 & 0 & 0 \\ \frac{\kappa}{P_+}\xi & 1 & \frac{P_2}{P_+}\xi & \frac{P_3}{P_+}\xi & 0 \\ 0 & -\frac{P_2}{P_+}\xi & 1 & 0 & 0 \\ 0 & -\frac{P_3}{P_+}\xi & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} \tilde{Q}_4 & \kappa\mathbf{Q}/Q_+ & Q_0 \\ \mathbf{Q} & \kappa\mathbb{1} & \mathbf{Q} \\ \tilde{Q}_0 & -\kappa\mathbf{Q}/Q_+ & Q_4 \end{pmatrix}\Lambda'^{-1}_{gh}$$

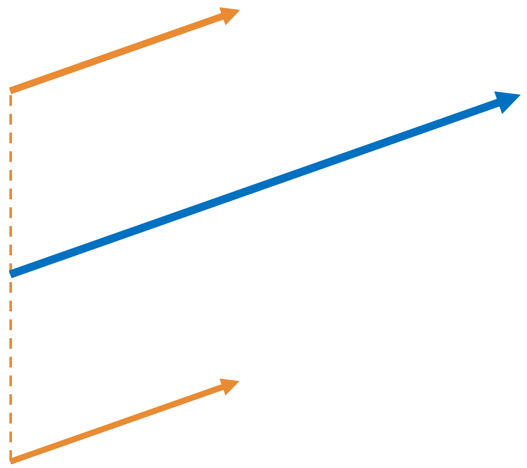
$$(gh)' = g\,h + \xi^i\left(N_i g\right)\,h + \xi^i\left(\frac{\kappa}{P_+}g\right)\left(N_i h\right) + \xi^i\epsilon_{ijk}\left(\frac{P_j}{P_+}g\right)\left(M_k h\right)$$

- From this we can read off the action of boosts on a two-particle state $\Delta\mathscr{N}_i\,|g\rangle\otimes|h\rangle$ with

$$\Delta\mathscr{N}_i = \mathscr{N}_i\otimes\mathbb{1} + \frac{\kappa}{\mathscr{P}_+}\otimes\mathscr{N}_i + \epsilon_i{}^{jk}\,\frac{\mathscr{P}_j}{\mathscr{P}_+}\otimes\mathscr{M}_k$$

Lorentz generators antipode

If we boost a two-particle system, each having the same momentum, its total momentum changes



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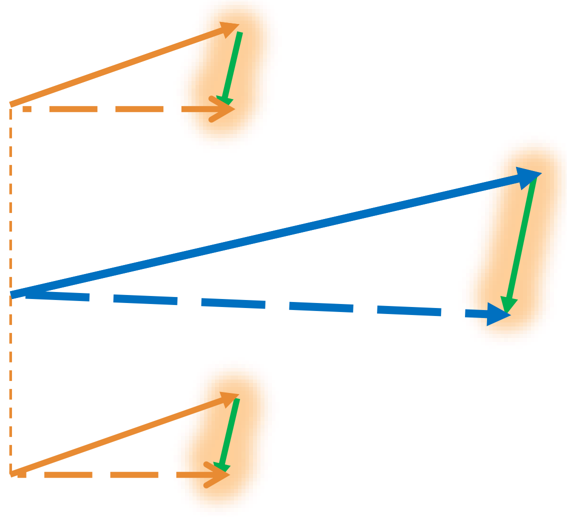
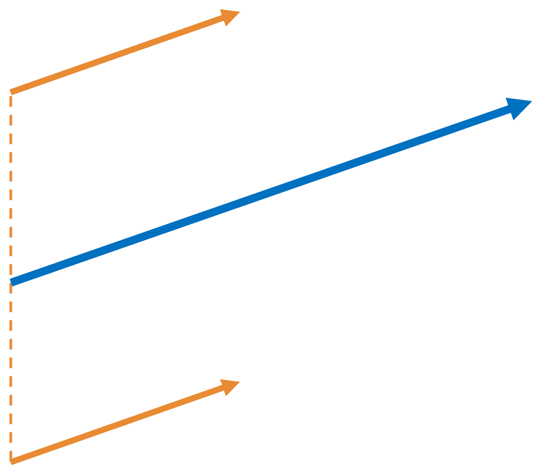


Lorentz generators antipode

If instead we replace one of the boosts with its antipode, the total momentum does not change



$$S(\mathscr{N}_i) = -\frac{\kappa}{\mathscr{P}_+}\left(\mathscr{N}_i + \epsilon_{ijk}\mathscr{P}_j\mathscr{M}_k\right)$$

Lorentz generators antipode

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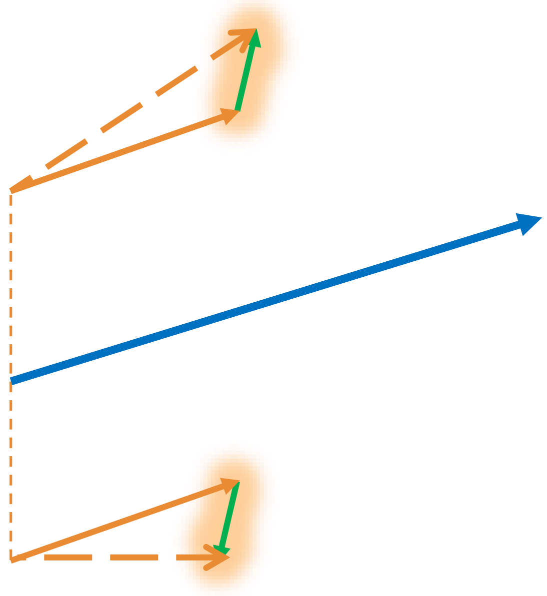

$$S(\mathscr{N}_i) = -\frac{\kappa}{\mathscr{P}_+}\left(\mathscr{N}_i + \epsilon_{ijk}\mathscr{P}_j\mathscr{M}_k\right)$$

Summary

- We see that the whole of the κ-Poincaré algebra can be derived just by using group theory.
- The notions of co-product and antipode are necessary to construct a field theoretical models with curved momentum space (noncommutative spacetime)
- In turn, the field theory has phenomenological consequences that might be tested experimentally.

κ-Poincaré phenomenology

Field theory

In application to field theory, we regard the group element g as a plane wave, intertwining between curved momentum space and non-commutative (κ-Minkowski) spacetime. Instead of working in non-commutative spacetime we choose to use the standard commutative Minkowski space with star product, such that

$$g = e^{ip_i X^i} e^{ip_0 X^0} \to e^{ip_\mu x^\mu}$$

$$e^{ip_\mu x^\mu} \star e^{iq_\mu x^\mu} = e^{i(p\oplus q)_\mu x^\mu}$$

$$(p \oplus q)_0 = p_0 + q_0 \,, \quad (p \oplus q)_i = p_i + e^{-p_0/\kappa}\, q_0$$

$$S(p)_0 \equiv \ominus p_0 = -p_0 \,, \quad S(p)_i = \ominus p_i = -p_i e^{p_0/\kappa}$$

Scalar field and action

- The on-shell field

$$\phi(x) = \int \frac{d^3p}{\sqrt{2\omega_{\mathbf{p}}}}\ \zeta(p)\, a_{\mathbf{p}}\, e^{-i(\omega_{\mathbf{p}}t - \mathbf{p}\mathbf{x})} + \int \frac{d^3p}{\sqrt{2\omega_{\mathbf{p}}}}\ \zeta(p) b^\dagger_{\mathbf{p}}\, e^{-i(S(\omega_{\mathbf{p}})t - S(\mathbf{p})\mathbf{x})}$$

$$\phi^\dagger(x) = \int \frac{d^3p}{\sqrt{2\omega_{\mathbf{p}}}}\ \zeta(p)\, a^\dagger_{\mathbf{p}}\, e^{-i(S(\omega_{\mathbf{p}})t - S(\mathbf{p})\mathbf{x})} + \int \frac{d^3p}{\sqrt{2\omega_{\mathbf{p}}}}\ \zeta(p) b_{\mathbf{p}}\, e^{-i(\omega_{\mathbf{p}}t - \mathbf{p})\mathbf{x}}$$

- The (on-shell) action

$$\begin{aligned} S &= -\frac{1}{2}\int_{\mathbb{R}^4} d^4x\ \left[(\partial_\mu\phi)^\dagger \star \partial^\mu\phi + (\partial_\mu\phi)\star(\partial^\mu\phi)^\dagger + m^2(\phi^\dagger\star\phi + \phi\star\phi^\dagger)\right] \\ &= \frac{1}{2}\int \frac{d^3p}{2\omega_{\mathbf{p}}}\,\zeta(p)^2\left(1 + \frac{|p_+|^3}{\kappa^3}\right)(p_\mu p^\mu + m^2)\left[a^\dagger_{\mathbf{p}}a_{\mathbf{p}} + b^\dagger_{\mathbf{p}}b_{\mathbf{p}}\right] \end{aligned}$$

Symplectic structure and Poisson bracket

- Having the action, and fixing an appropriate value of $\zeta(\mathbf{p})$ we can compute the symplectic form

$$\Omega = -i \int d^3p \, \left(\delta a^\dagger_{\mathbf{p}} \wedge \delta a_{\mathbf{p}} - \delta b_{\mathbf{p}} \wedge \delta b^\dagger_{\mathbf{p}}\right)$$

- The Poisson brackets/commutators

$$[\hat{a}_{\mathbf{p}}, \hat{a}^\dagger_{\mathbf{q}}] = \delta^3(\mathbf{p} - \mathbf{q}),$$
$$[\hat{b}_{\mathbf{p}}, \hat{b}^\dagger_{\mathbf{q}}] = \delta^3(\mathbf{p} - \mathbf{q}).$$

Continuous symmetries charges

- There is a straightforward construction of conserved Noether charges from the symplectic structure, called the "covariant phase space method". We use it to compute the charges associated with ten (**κ-**) Poincaré symmetries.

$$\mathscr{P}_\mu = \int d^3p\,[-S(p)_\mu a^\dagger_{\mathbf{p}} a_{\mathbf{p}} + p_\mu b^\dagger_{\mathbf{p}} b_{\mathbf{p}}]$$

$$\mathscr{N}_i = -\frac{1}{2}\int d^3p\left\{S(\omega_p)\left[\frac{\partial a^\dagger_{\mathbf{p}}}{\partial S(\mathbf{p})^i}a_{\mathbf{p}} - a^\dagger_{\mathbf{p}}\frac{\partial a_{\mathbf{p}}}{\partial S(\mathbf{p})^i}\right] + \omega_p\left[b_{\mathbf{p}}\frac{\partial b^\dagger_{\mathbf{p}}}{\partial \mathbf{p}^i} - \frac{\partial b_{\mathbf{p}}}{\partial \mathbf{p}^i}b^\dagger_{\mathbf{p}}\right]\right\}$$

$$\mathscr{M}_i = -\epsilon_i{}^{jk}\frac{1}{8}\int d^3q\left(S(\mathbf{q})_j\frac{\partial a^\dagger_{\mathbf{q}}}{\partial S(\mathbf{q})^k}a_{\mathbf{q}} - a^\dagger_{\mathbf{q}}S(\mathbf{q})_j\frac{\partial a_{\mathbf{q}}}{\partial S(\mathbf{q})^k} + b_{\mathbf{q}}\mathbf{q}_j\frac{\partial b^\dagger_{\mathbf{q}}}{\partial \mathbf{q}^k} - \mathbf{q}_j\frac{\partial b_{\mathbf{q}}}{\partial \mathbf{q}^k}b^\dagger_{\mathbf{q}}\right)$$

- The Poisson brackets of the charges form a representation of the standard Poincaré algebra. In this sense we can say that the theory is Poincaré-invariant.

- The charges are generators of the infinitesimal translations/boosts/rotations.

DISCRETE SYMMETRIES GENERATORS

Of three discrete symmetries the most interesting is charge conjugation. The associated operators transforms a (one-) particle state into the (one-) antiparticle state.

$$\mathscr{C} = \int d^3p \, (b^\dagger_{\mathbf{p}} a_{\mathbf{p}} + a^\dagger_{\mathbf{p}} b_{\mathbf{p}})$$

$$\mathscr{C} a^\dagger_{\mathbf{p}} |0\rangle = b^\dagger_{\mathbf{p}} |0\rangle \qquad \mathscr{C} b^\dagger_{\mathbf{p}} |0\rangle = a^\dagger_{\mathbf{p}} |0\rangle$$

It is remarkable, however that the charge operator does not commute with the boost generator

$$\boxed{[\mathcal{N}_j, \mathscr{C}] \neq 0}$$

This leads to non-trivial CPT deformation (violation).

Aside: Jost—Greenberg theorem

- The famous Jost–Greenberg theorem states that there is a one-two-one correspondence between Poincaré symmetry and CPT symmetry (so that the violation of CPT inevitably leads to Poincaré Invariance Violation, PIV).
- The status of κ - Poincaré in this respect is not completely clear. The Jost-Greenberg theorem failure can be traced back to the fact that the (naively) deformed Wightman functions are no longer Lorentz-invariant. On the other hand, the theory possesses 10 conserved charges satisfying Lorentz algebra. There is a puzzle here that must be understood.

Phenomenology

- Particles and anti-particles at rest have the same mass.
- But things change when the (anti-) particle is boosted. If we boost them with rapidity ξ, they have energy/momentum

$$\mathscr{P}_1|M\cosh\xi, M\sinh\xi, 0, 0\rangle_a = -S(M\sinh\xi)|M\cosh\xi, M\sinh\xi, 0, 0\rangle_a$$
$$\mathscr{P}_0|M\cosh\xi, M\sinh\xi, 0, 0\rangle_a = -S(M\cosh\xi)|M\cosh\xi, M\sinh\xi, 0, 0\rangle_a$$

$$\mathscr{P}_1|M\cosh\xi, M\sinh\xi, 0, 0\rangle_b = M\sinh\xi|M\cosh\xi, M\sinh\xi, 0, 0\rangle_b$$
$$\mathscr{P}_0|M\cosh\xi, M\sinh\xi, 0, 0\rangle_b = M\cosh\xi|M\cosh\xi, M\sinh\xi, 0, 0\rangle_b$$

- The difference between energies between particles and antiparticles boosted by the same boost is of order of p^2/κ

Phenomenology

- This translates into different decay times of moving unstable particles and antiparticles.
- The most promising are muons, for which the current data already makes it possible to constrain $\kappa \geq 10^{14}$ GeV, in the case LHC muons, with a couple of orders of magnitude improvement in the future collider. It might be further improved in some dedicated machines.