Black Holes in Klein Space

Erin Crawley

Harvard University

Based on: arXiv: 2112.03954 w/ A. Guevara, N. Miller, A. Strominger

Corfu Celestial Holography Workshop

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- Part I: Global Kleinian Spacetimes

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- Part I: Global Kleinian Spacetimes
- Part II: Classical Linearized Metrics from 3-pt Scattering Amplitudes
- Part III: Future Directions: Mapping to the Celestial Torus

Part 1: Global Spacetimes

Lorentzian Taub-NUT in (r, θ, ϕ, t) coordinates with parameters M, N

$$ds^2_{\text{TN, L}} = -f_{\text{L}}(r)\left(dt - 2N\cos\theta d\phi\right)^2 + \frac{dr^2}{f_{\text{L}}(r)} + (r^2 + N^2)(d\theta^2 + \sin^2\theta d\phi^2)$$

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Self-dual $(M = N)$ Kleinian Taub-NUT:

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$\text{SL}(2, \mathbb{R}) \times U(1)$ Killing symmetry group

Toric Penrose Diagrams

- Flat Klein Space

$$
\begin{aligned}
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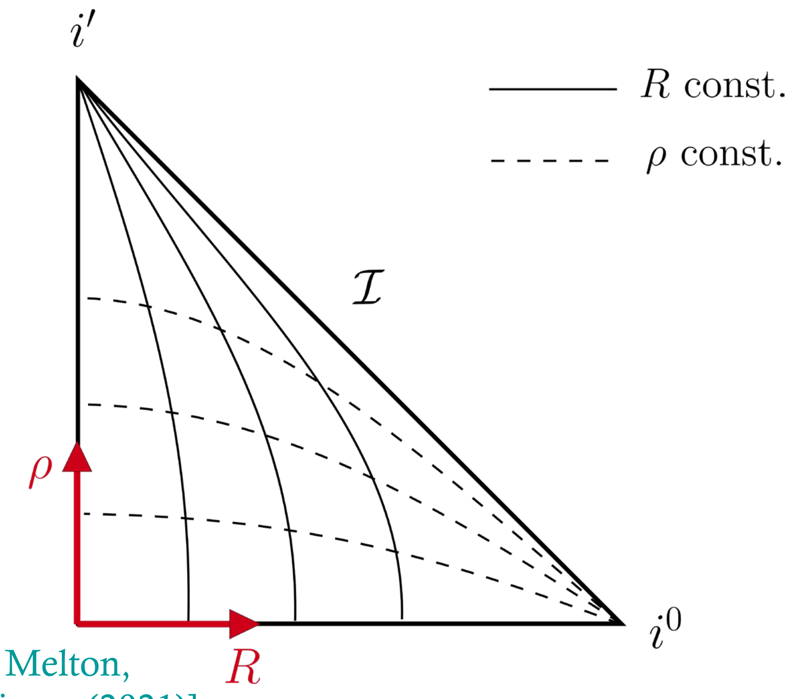

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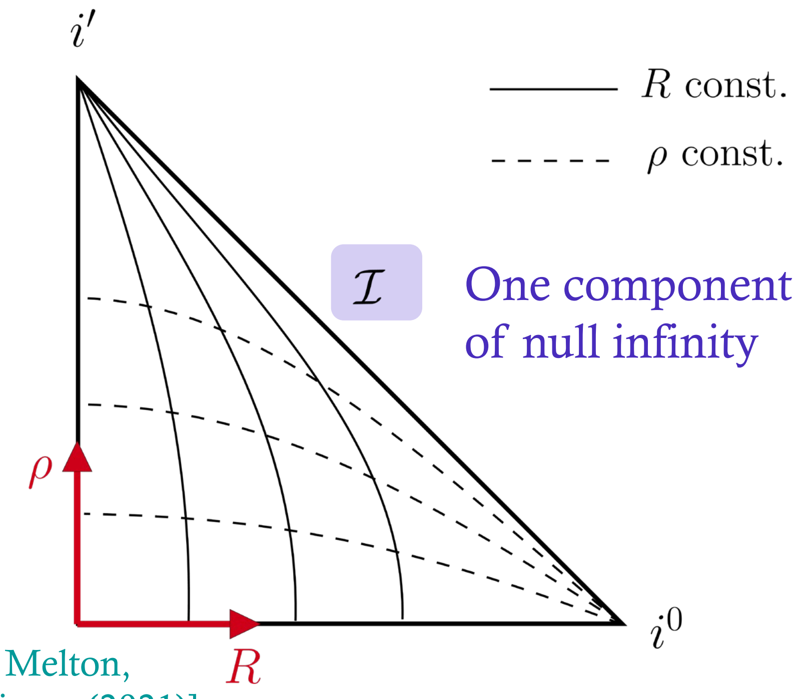

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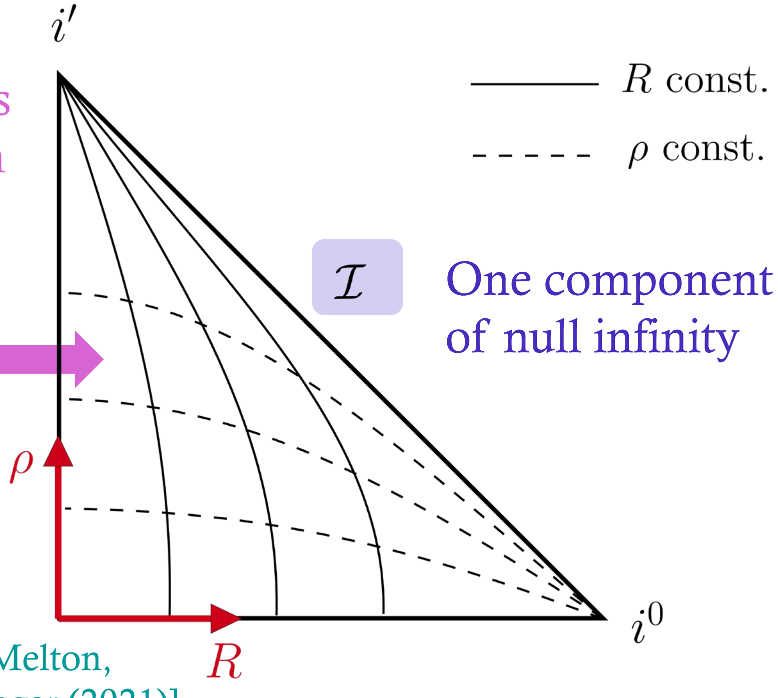

One component of null infinity

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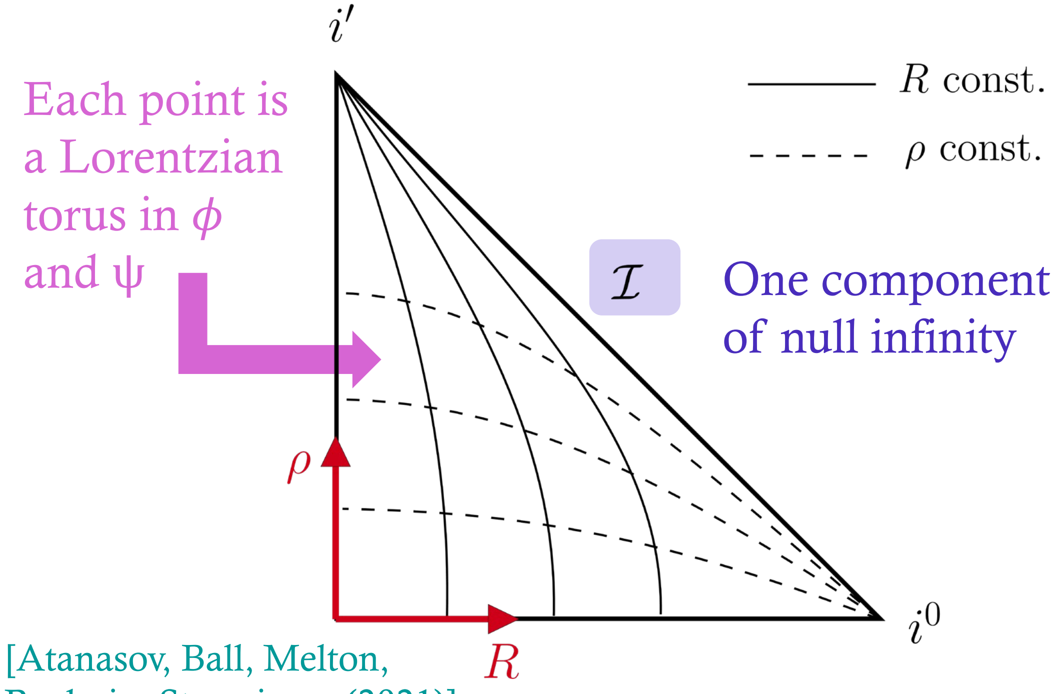

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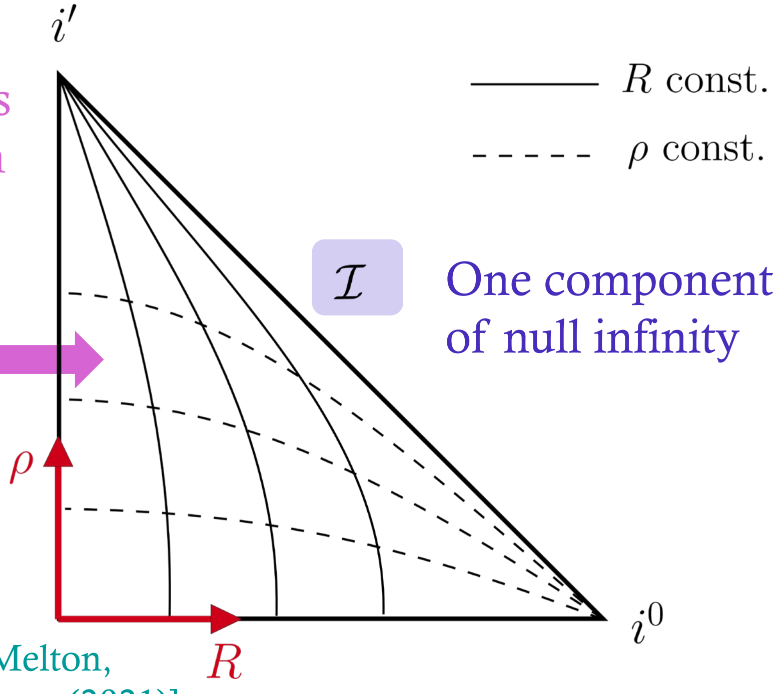

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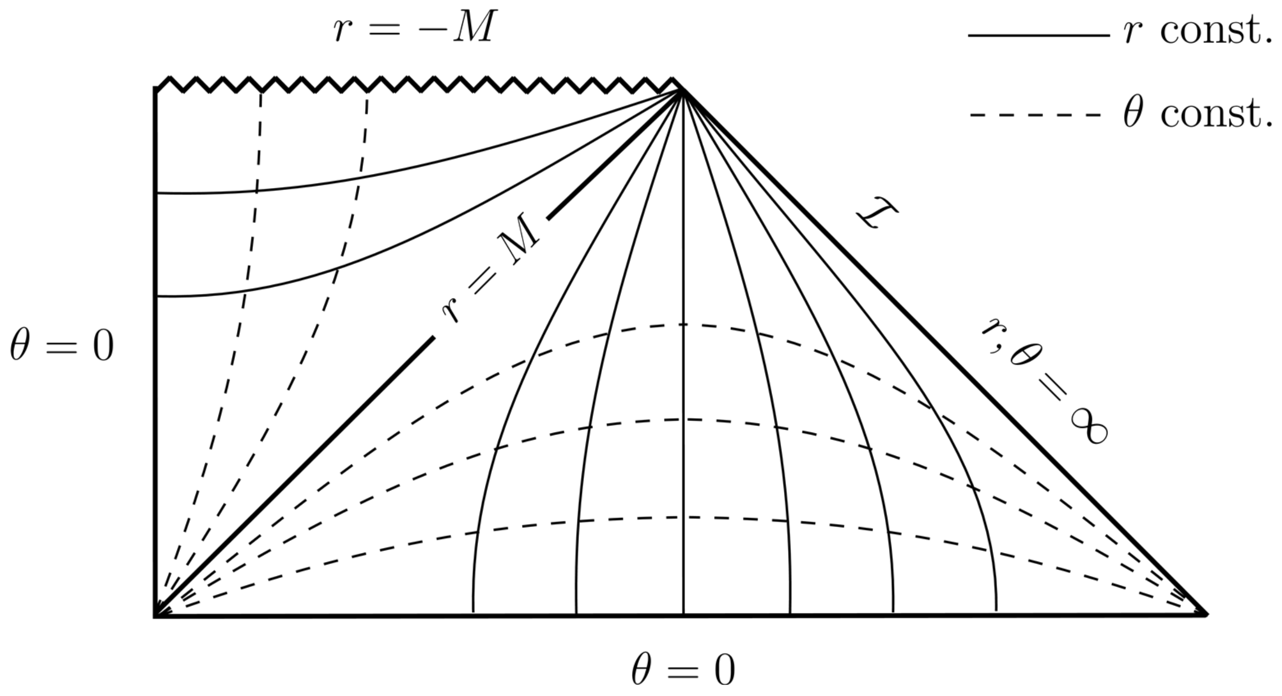

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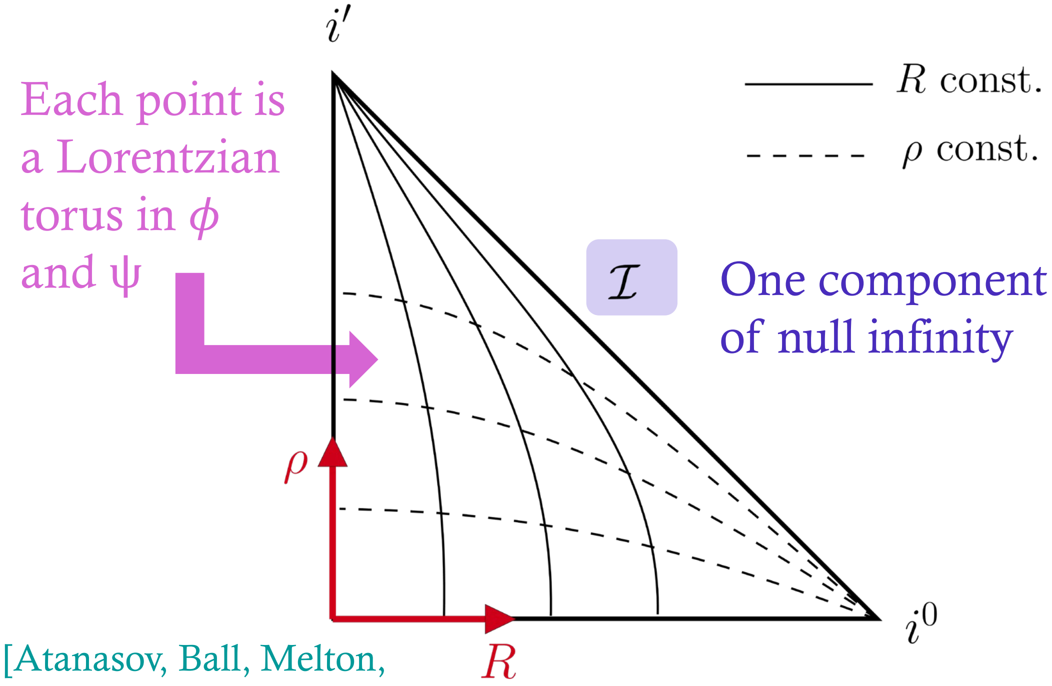

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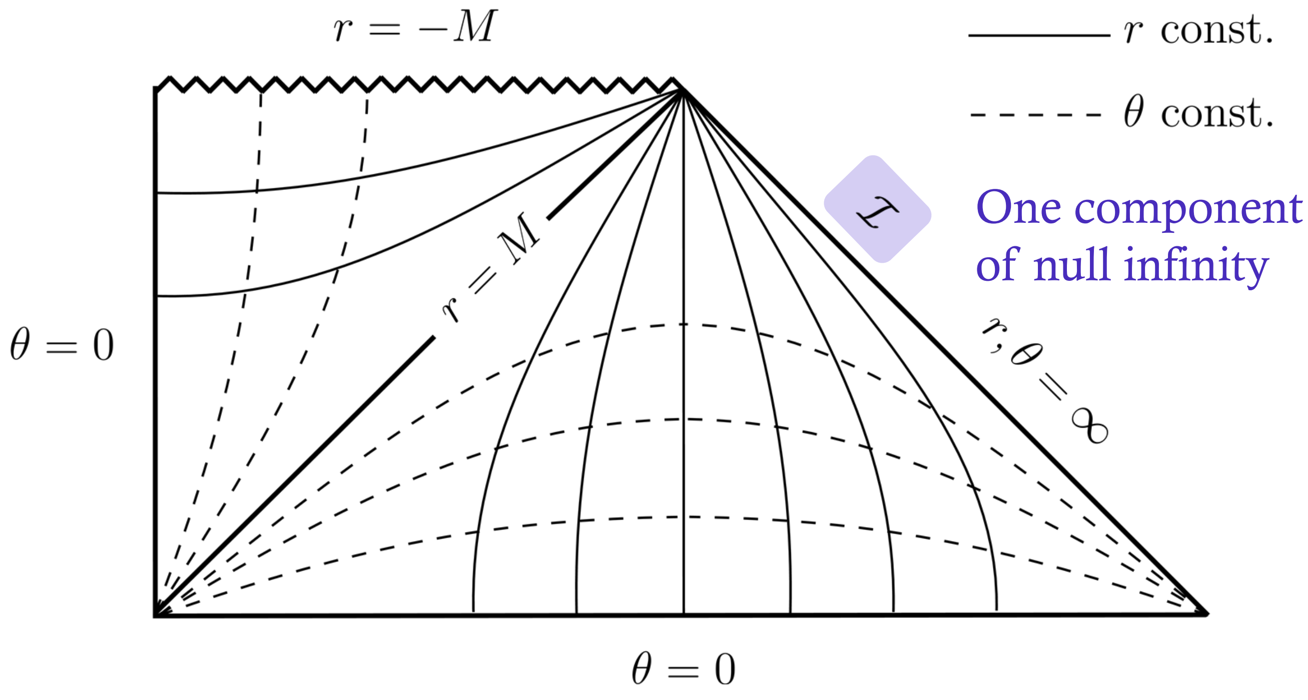

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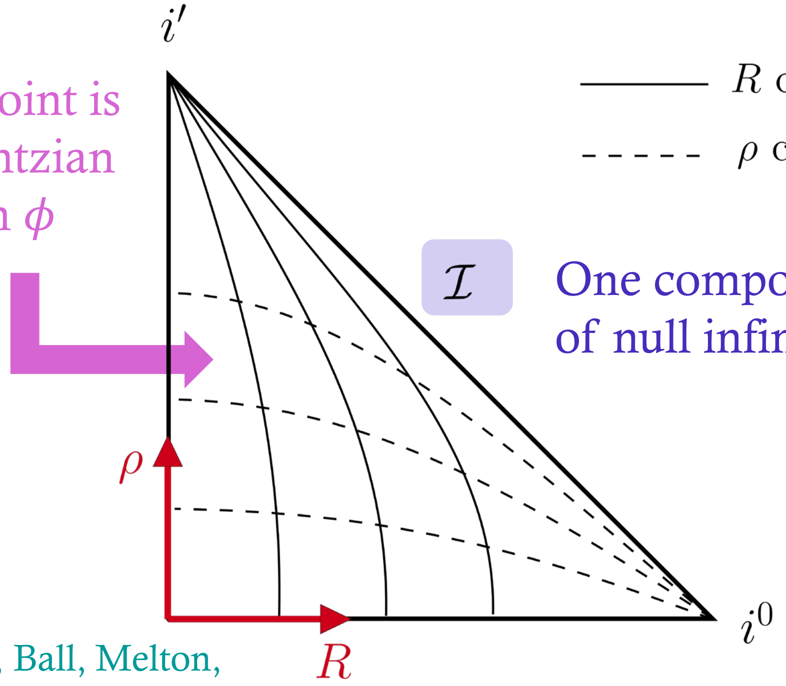

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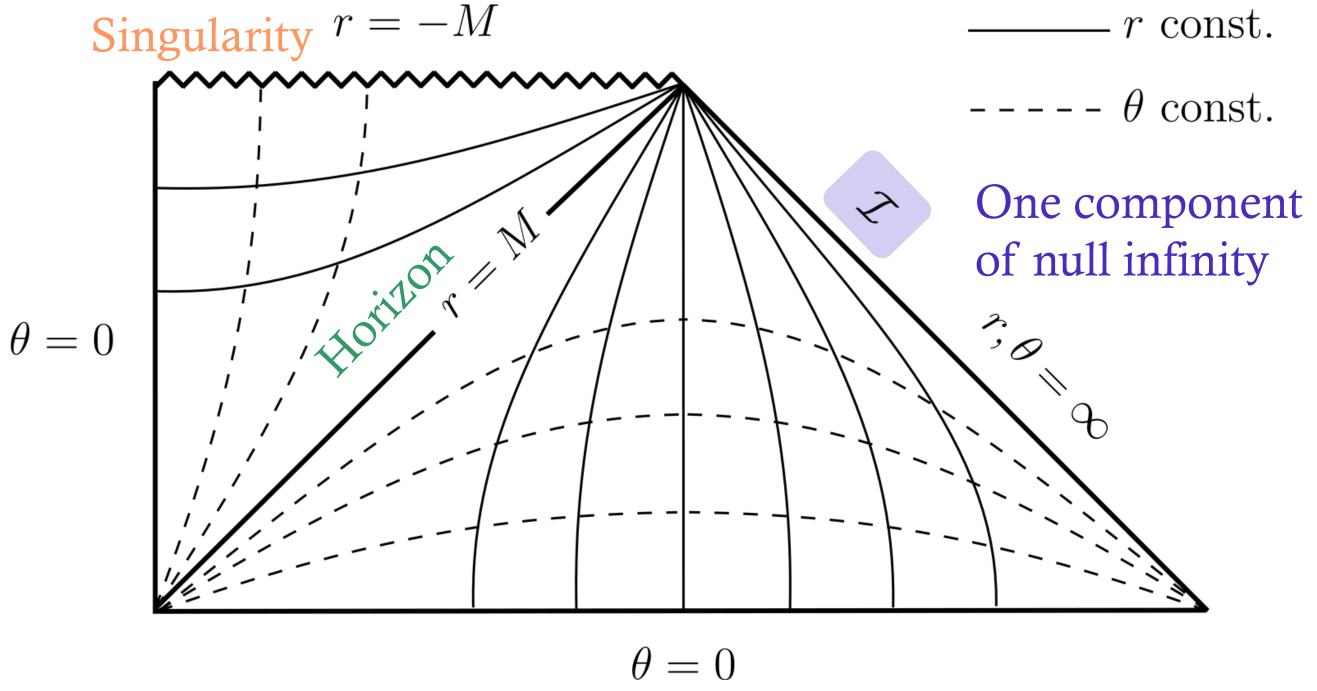

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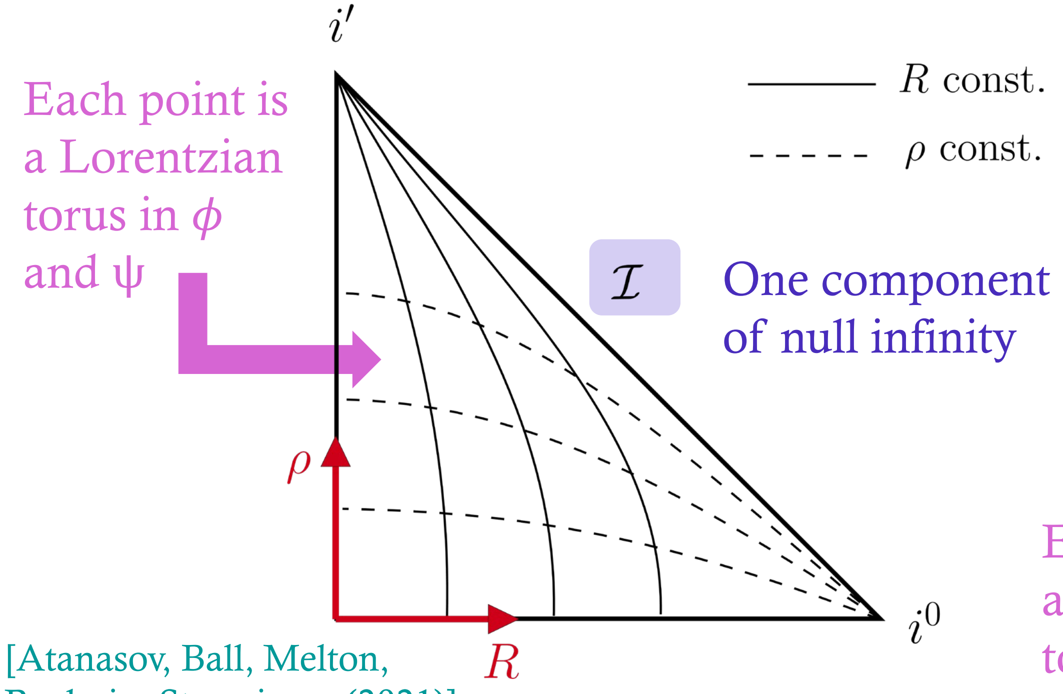

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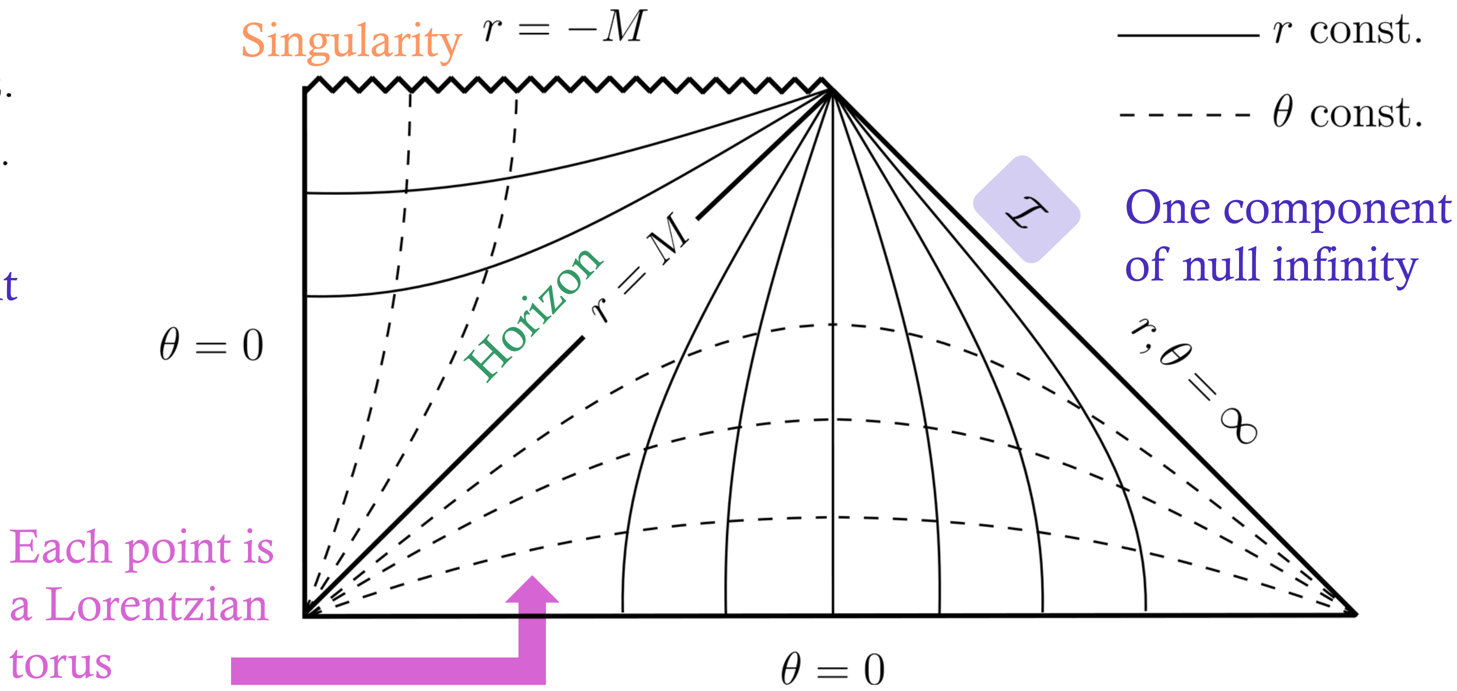

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Introduce rectangular coordinates (t, x, y, z)

$$z_{TN} = z_{KTN} + a$$

t, x, y unchanged

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- The toric Penrose diagram for self-dual TN has expected black hole features
- There exists a real diffeomorphism mapping Taub-NUT ←→ Kerr-Taub-NUT
  - In (t, x, y, z) coordinates, this takes the form $z \to z + a$

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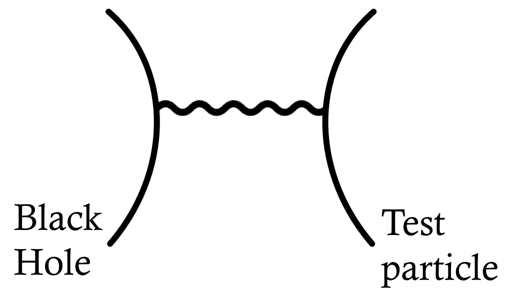

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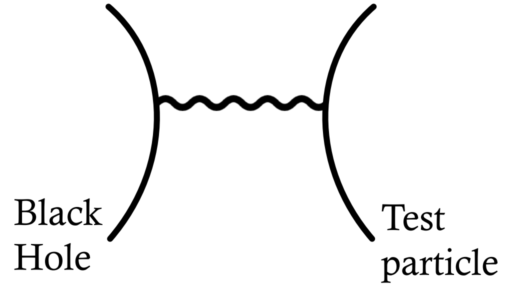

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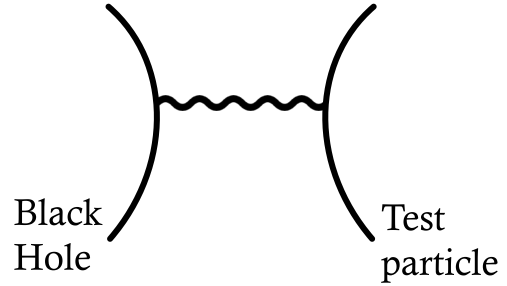

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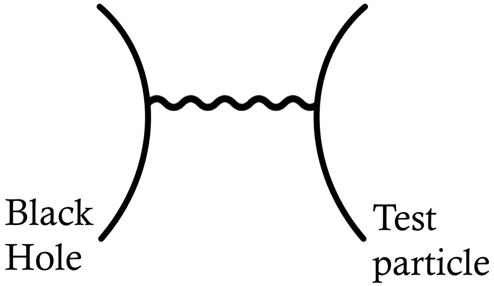

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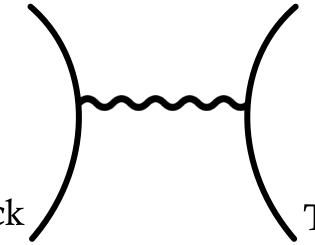

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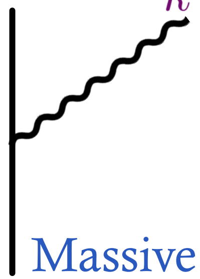

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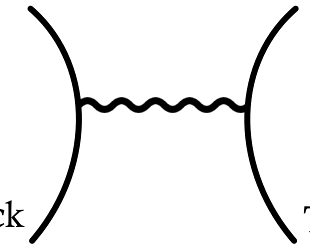

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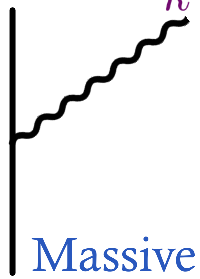

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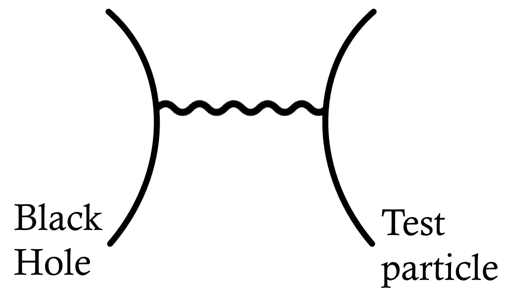

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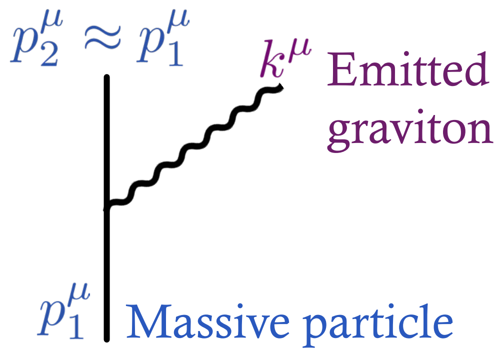
- Motivate analytic continuation, diffeomorphism from previous part



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Since $h_{\mu\nu} \sim \mathcal{O}(G)$, it can be changed by linearized diffeomorphisms

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Current

Ex. Schwarzschild:

$\mathcal{T}_{\mu\nu}(k) \propto M u_\mu u_\nu$

with $u^\mu = (1, 0, 0, 0)$

Connecting Amplitudes and Spacetimes in (1,3)

In (1, 3):

$$\mathcal{M}^{\pm}_{n,L} = \epsilon^{\pm}_{\mu\nu}(k)\mathcal{T}^{\mu\nu}_{L}(k) \quad \text{at } k_0^2 - \vec{k}^2 = 0 \ \ , \ n > 3$$

[Bautista, Guevara, Kavanaugh, Vines (2021) ;
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→ However, both sides can be nonzero in Klein Space!

Moving to (2,2) Signature:

Beginning with

In (1, 3): $$\bar{h}^L_{\mu\nu} = -16\pi G \int \frac{d^4k}{(2\pi)^4} \frac{e^{ik\cdot x_L}}{(k^0+i\epsilon)^2 - \vec{k}^2} \mathcal{T}^L_{\mu\nu}(k)$$

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Insert stationary condition

$$\mathcal{T}^L_{\mu\nu}(k) = -2\pi\delta(k^0) T^L_{\mu\nu}(k)$$

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Wick rotate

$t \to it$

$x \to ix$

$y \to iy$

See [Monteiro, O'Connell, Veiga, Sergola (2021)] for $z \to iz$

In (2, 2):

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Reparametrize k_1 , k_2 in terms of new variables

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$$h_{\mu\nu}(x) \propto G \int d\begin{pmatrix}\text{on-shell} \\ \text{stationary} \\ \text{k space}\end{pmatrix} e^{-k\cdot x} \, T_{\mu\nu}(k) \text{ with } k^\mu k_\mu = 0$$

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So, $T_{\mu\nu}(k)$ is evaluated on on-shell momenta

Equivalently: $\partial^2 \bar{h}_{\mu\nu} = 0$, so $\bar{h}_{\mu\nu}$ is free everywhere!

Connecting to (2, 2) Amplitudes

Since $T_{\mu\nu}(k)$ is evaluated on on-shell momenta, we can use

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and + corresponds to a self-dual source, – an anti-self-dual

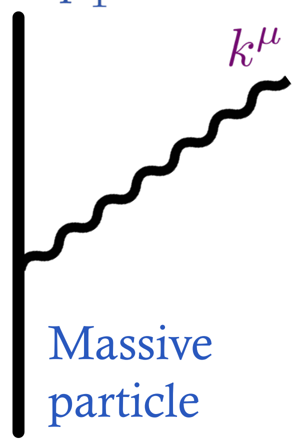
Putting Everything Together

We have $h_{\mu\nu}(x) = h^{+}_{\mu\nu}(x) + h^{-}_{\mu\nu}(x)$

where $h^{\pm}_{\mu\nu}(x) \propto G \int d\begin{pmatrix}\text{on-shell} \\ \text{stationary} \\ \text{k space}\end{pmatrix} e^{-k\cdot x} \epsilon^{\mp}_{\mu} \epsilon^{\mp}_{\nu} \mathcal{M}^{\pm}_{3}(k)$

And $\mathcal{M}^{\pm}_{3}(k) \propto (M \pm N) e^{\mp k \cdot a}$

as obtained from the classical limit of graviton emission amplitudes.



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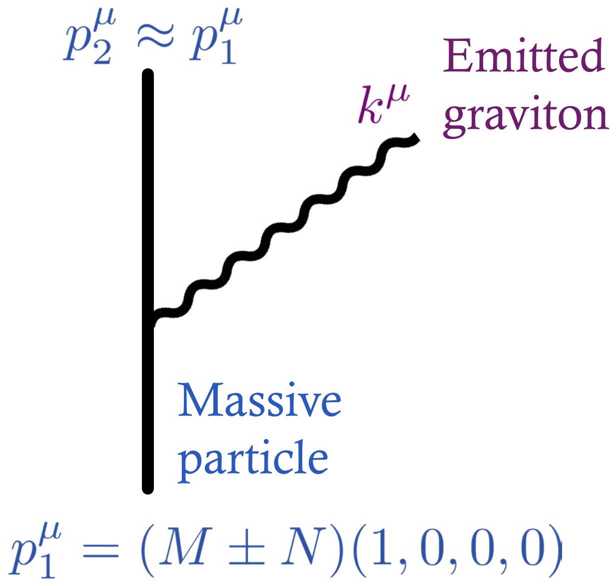

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Part 3: Mapping to the Celestial Torus

WIP w/ Guevara, Himwich, Strominger

We can recast our previous result in a slightly nicer form:

$$h^{+}_{\mu\nu}(x) = G \int_0^\infty \omega d\omega \int_{\mathbb{R}^2} dw d\bar{w} \, \varepsilon^{-}_{\mu\nu}(w,\bar{w}) e^{-\omega \hat{q}(w,\bar{w})\cdot x} \, a(\omega, w, \bar{w})$$

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$\hat{q}^{\mu}(w,\bar{w}) = (w+\bar{w}, 1+w\bar{w}, w-\bar{w}, 1-w\bar{w}),$

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This can be written in a conformal primary basis

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where $\varphi_\Delta(x; z, \bar{z}) = \int_0^\infty d\omega \, \omega^{\Delta-1} \, e^{-\omega \hat{q}(z,\bar{z})\cdot x} = \dfrac{\Gamma(\Delta)}{(q\cdot x)^\Delta}$

Black Hole Construction from the Celestial Torus

Idea:

- Equipped with

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Quantum state on the celestial torus

Thank you!