the fate of horizons under quantum corrections

EA, J. Anero & E. Velasco, 2207.12721 [hep-th]

GR is on-shell finite to one loop ('t Hooft and Veltman)

$$L_{\infty}^{(1)} = \frac{1}{n-4}\frac{1}{(4\pi)^2}\int d^n x\sqrt{|g|}\left(\frac{1}{60}R^2 + \frac{7}{10}R_{\mu\nu}^2\right)$$

The first nonvanishing counterterm appears at two loop order(Goroff and Sagnotti)

$$L_{\infty}^{(2)} = \frac{1}{n-4}\frac{209}{2880}\frac{1}{(4\pi)^4}\int d^n x\sqrt{|g|}\,W_{\mu\nu\alpha\beta}W^{\alpha\beta\rho\sigma}W_{\rho\sigma}{}^{\mu$$

We are led to consider the lagrangian

$$L = \sqrt{|g|}\left\{ -\Lambda - \gamma R + \omega\kappa^2\, W_{\mu\nu\alpha\beta} W^{\alpha\beta\rho\sigma} W_{\rho\sigma}{}^{\mu\nu} \right\}$$

The definition of the Weyl tensor depends on the dimension

$$W_{\alpha\beta\gamma\delta} \equiv R_{\alpha\beta\gamma\delta} - \frac{1}{n-2}\left\{ g_{\alpha\gamma}R_{\beta\delta} - g_{\alpha\delta}R_{\beta\gamma} - g_{\beta\gamma}R_{\delta\alpha} + g_{\beta\delta}R_{\gamma\alpha} \right\} +$$

$$+ \frac{1}{(n-1)(n-2)}\, R\left\{ g_{\alpha\gamma}g_{\delta\beta} - g_{\alpha\delta}g_{\gamma\beta} \right\}$$

W^3 would have been a conformal theory in 6 dimensions

provided the six-dimensional definition were taken

weyl squared

$$g_{\mu\nu} \to \Omega(x)^2 g_{\mu\nu}$$

$$W^{\mu}{}_{\nu\rho\sigma} \to W^{\mu}{}_{\nu\rho\sigma}$$

$$S_{Weyl^2} = \int d^4x \sqrt{|g|}\, W_{\mu\nu\alpha\beta} W^{\mu\nu\alpha\beta}$$

This theory is conformal in four dimensions

The EM are the vanishing of the Bach tensor

Bach tensor

$$B_{\alpha\beta} \equiv \nabla^{\lambda} C_{\lambda\alpha\beta} + K^{\mu\nu} W_{\mu\alpha\beta\nu}$$

Cotton tensor

$$C_{\alpha\rho\sigma} \equiv \nabla_{\sigma} K_{\alpha\rho} - \nabla_{\rho} K_{\alpha\sigma}$$

schouten' tensor

$$K_{\mu\nu} \equiv \frac{1}{n-2}\left(R_{\mu\nu} - \frac{1}{2(n-1)} R g_{\mu\nu}\right)$$

$$ds^2 = A(r,t)B^2(r,t)dt^2 - \frac{1}{A(r,t)}dr^2 - r^2 d\Omega_2^2,$$

symmetric criticality **(Weyl,palais)**

$$\boxed{S_{Weyl^2} = \frac{1}{3}\int dtdrr^2 B\left[\frac{F}{r^2 A^3 B^3}\right]^2}$$

$$\begin{aligned} F &= -2r^2 B\dot{A}^2 + r^2 A\left(-\dot{A}\dot{B} + B\ddot{A}\right) + A^3 B^2 \Big[B\left(-2 + 2A - 2rA' + r^2 A''\right) + \\ &+ r\left(3rA'B' - 2A(B' - rB'')\right)\Big] \qquad (17) \end{aligned}$$

This remarkabls structure is shared by all powers of weyl

weyl cube

$$S_{Weyl^3} = \int d^4x \sqrt{|g|}\, W_{\mu\nu\alpha\beta} W^{\alpha\beta\rho\sigma} W_{\rho\sigma}{}^{\mu\nu}$$

$$\boxed{S_{Weyl^3} \qquad = \frac{1}{18}\int dtdrr^2 B\Big[\frac{F}{r^2 A^3 B^3}\Big]^3}$$

The EM in tensor form

$$V_{\mu\nu} \equiv \frac{\delta S^{(W^3)}}{\delta g^{\mu\nu}} = \tfrac{1}{2} W^3 g_{\mu\nu} + \tfrac{1}{4}\left(R g_{\mu\nu} - R_{\mu\nu}\right) W^2 - \tfrac{1}{2}\left(\nabla_\mu \nabla_\nu - \Box g_{\mu\nu}\right) W^2 -$$
$$-3 R^{\alpha\beta} (W^2)_{\mu\alpha\nu\beta} - 3 W_\mu{}^{\alpha\beta\lambda} (W^2)_{\nu\alpha\beta\lambda} - 6 \nabla^\alpha \nabla^\beta (W^2)_{\mu\alpha\nu\beta} \qquad (23$$

- $V_{\mu\nu} = 0 \Rightarrow F = 0$

  Trivially, we have that F is related to the trace of the EM in vacuum as follows,

$$V_{\mu\nu} := \frac{1}{\sqrt{|g|}} \frac{\delta S_{Weyl^3}}{\delta g^{\mu\nu}}, \tag{7}$$

$$V := V_{\mu\nu}\, g^{\mu\nu} = \frac{n-2}{6} W^3. \tag{8}$$

It is then clear that, provided the metric is non-singular, we have[1];

$$V_{\mu\nu} = 0 \Rightarrow V = V_{\mu\nu}\, g^{\mu\nu} = \frac{n-2}{6} W^3 = 0 \xRightarrow{n=4} F = 0. \tag{9}$$

Then when the metric is non degenerate,

$$F \neq 0 \Rightarrow V_{\mu\nu} \neq 0, \tag{10}$$

That is, $F = 0$ is a necessary condition for $(A(r), B(r))$ to constitute a metric that solves the EM.

- $F = 0 \Rightarrow V_{\mu\nu} = 0$ For this we will use the convenient tool that is the *Principle of Symmetric Criticallity*[2] [26, 11, 31].

  From the Euler-Lagrange equations of motion which,

$$\frac{\partial \mathcal{L}}{\partial A(r)} - \frac{\partial}{\partial r}\frac{\partial \mathcal{L}}{\partial A'(r)} + \frac{\partial^2}{\partial r^2}\frac{\partial \mathcal{L}}{\partial A''(r)} = \mathcal{O}\left[\mathcal{L}\right] = 0, \tag{12}$$

$$\frac{\partial \mathcal{L}}{\partial B(r)} - \frac{\partial}{\partial r}\frac{\partial \mathcal{L}}{\partial B'(r)} + \frac{\partial^2}{\partial r^2}\frac{\partial \mathcal{L}}{\partial B''(r)} = \tilde{\mathcal{O}}\left[\mathcal{L}\right] = 0. \tag{13}$$

  Where $\tilde{\mathcal{O}}, \mathcal{O}$ are the corresponding differential operators. Which obviously satisfy the EM above for functions $A(r)$ and $B(r)$ such that,

$$F = 0, \tag{14}$$

  meaning that it is Actually a sufficient condition foe $\{A(r),\, B(r)\}$ to solve the EM.

This concludes the proof that the condition

$$F = 0, \tag{15}$$

is indeed a necessary and sufficient condition for $\{A(r),\, B(r)\}$ to be such that they correspond for the gauge choice,

$$ds^2 = B(r)dt^2 - A(r)dr^2 - r^2 d\Omega_2^2, \tag{16}$$

a solution to the EM.

$$ds^2 = A(r)B(r)^2 dt^2 - \frac{dr^2}{A(r)} - r^2 d\Omega^2$$

Every solution of F=0 is also a solution of the full EM

$$F \equiv B(-2 + 2A - 2rA' + r^2 A'') + r\left(3rA'B' - 2A(B' - rB'')\right) = 0$$

Given any trial value for B, we get an ODE for A

For example, B constant leads to

$$A = 1 + C_1 r + C_2 r^2$$

This includes (a)dS

$$ds^2 = \left(1 - \frac{r^2}{r_\Lambda^2}\right) dt^2 - \frac{1}{1 - \frac{r^2}{r_\Lambda^2}} dr^2 - r^2 d\Omega^2$$

$$ds^2 = \left(1 + \frac{r^2}{r_\Lambda^2}\right) dt^2 - \frac{1}{1 + \frac{r^2}{r_\Lambda^2}} dr^2 - r^2 d\Omega^2$$

$$A(r) = 1 + C_1 r + C_2 r^2$$

$$B(r) = C_2 + 2C_1 \frac{2 + C_1 r}{(C_1^2 - 4C_2)\sqrt{1 + C_1 r + C_2 r^2}}$$

$$ds^2 = B(r)dt^2 - A(r)dr^2 - r^2 d\Omega_2^2$$

$$\boxed{B = 1 + e^{-C_1 r}}$$

$$\boxed{A(r) = \frac{(2 + 2e^{C_1 r} + C_1 r)^2}{(1 + e^{C_1 r})(4 + e^{C_1 r}(4 + C_2 r^2))}}$$

Lots of exact solutions…

It so happens that Schwarzschild (S(a)dS) is NOT a solution on Weyl cube theories.

This fact explains the tension that this dimension six operator induces in Schwarzschild.

Frobenius

$$ds^2 = B(r)dt^2 - A(r)dr^2 - r^2 d\Omega_2^2$$

$$\boxed{\begin{aligned} A(r) &= a_s r^s + a_{s+1} r^{s+1} + \ldots \\ B(r) &= b_t (r^t + b_{t+1} r^{t+1} + \ldots \end{aligned}}$$

families characterized by the exponents (s,t)

The (2,2) solution

$$A(r) = a_2 r^2 + a_3 r^3 + a_4 r^4 + \mathcal{O}(r^4)$$

$$\frac{B(r)}{b_2} = r^2 + b_3 r^3 + \frac{1}{4}\left(\frac{a_3 b_3}{a_2} + 4a_2 + b_3^2\right) r^4 + b_5 r^5 + \mathcal{O}(r^6),$$

horizonless, singular solution

It has been proposed by Stelle & Holdom as a candidate for the endpoint of gravitational collapse

structural stability

$$ds^2 = e^{2\nu} dt^2 - e^{2\psi} \left(d\phi - q_2 dx^2 - q_3 dx^3 - \omega dt \right)^2 - e^{2\mu_1} dx_2^2 - e^{2\mu_3} dx_3^2$$

axial/polar perturbations

$$S = \int d^4x \sqrt{|g|} \left\{ -\Lambda - \gamma R + \omega \kappa^2 W^3 \right\}$$

The A and B functions are the consequence of the weyl cube perturbation

$$ds^2 = \left[1 - \frac{r_s}{r} - \left(\frac{r}{r_\Lambda} \right)^2 + B(r) \right] dt^2 - \frac{1}{\left[1 - \frac{r_s}{r} - \left(\frac{r}{r_\Lambda} \right)^2 + A(r) \right]} dr^2 - r^2 d\Omega^2 \tag{80}$$

$$a_1 = \frac{16r_s^3}{\gamma r^7} - \frac{18r_s^2}{\gamma r^6} + \frac{4r_s^2\Lambda}{\gamma r^4} + \frac{C_1}{r}$$

$$b_1 = \frac{4r_s^3}{\gamma r^7} - \frac{6r_s^2}{\gamma r^6} + \frac{2r_s^2\Lambda}{\gamma r^4} + \frac{C_1 + 6C_2\gamma(r_s - r)}{r} + C_2\Lambda r^2$$

asymptotic behavior is changed

horizons are displaced

$$R_* := \kappa^2\omega\left(\frac{2}{\gamma R_s^3} - c_1\right) + R_s$$

Stelle & Holdom discovered some evidence of the existence of a horizonless, singular solution in theories with higher dimensional operators.

although we have no analytical grasp of the (2,2) solution we have some indications (based on Frobeniu 'analysis) that it survives this deformation.

This solution has been proposed as al alternative endpoint for gravitational collapse.

Its importance cannot be overestimated.

$$ds^2 = \left(r^2 + \ldots\right) dt^2 - \left(r^2 + \ldots\right) \, dr^2 - r^2 d\Omega^2$$

at any rate, weyl cube perturbations profoundly alter the asymptotic behavior of SadS backup

$$
\begin{aligned}
H_{tt} = \frac{1}{144r^6A^7B^5}\Big\{ & 132r^4\omega B^4A'^4\left(rB'-2B\right)^2 - 48A^6B^6\left(3\gamma r^4-8\omega\right) - \\
& - 8A^7B^6\left(9\Lambda r^6-18\gamma r^4+4\omega\right) + 3r^2A^2B^2\mathcal{F}_\omega\left(A,B,A',B',A'',B'',A^{(3)},B^{(3)}\right) + \\
& + A^4\mathcal{G}_\omega\left(A,B,A',B',A'',B'',A^{(3)},B^{(3)},B^{(4)}\right) - \\
& - 3rA^3B\mathcal{H}_\omega\left(A,B,A',B',A'',B'',A^{(3)},B^{(3)},B^{(4)}\right) + \\
& + r^3\omega AB^3A'^2\left(2B-rB'\right)\Big[2rB\left(63rA''B'+A'\left(213rB''-97B'\right)\right) - 265r^2A'B'^2 \\
& + 4B^2\left(149A'-63rA''\right)\Big] + 6A^5B^2\mathcal{K}_\omega\left(A,B,A',B',A'',B'',A^{(3)},B^{(3)},B^{(4)}\right)\Big\},
\end{aligned} \tag{1}
$$

$$
\begin{aligned}
\mathcal{F}_\omega\left(A, B, A', B', \cdots\right) := {} & 105 r^4 \omega A'^2 B'^4 - 2 r^3 \omega B A' B'^2 \left(27 r A'' B' + A'\left(130 r B'' + 23 B'\right)\right) - \\
& + 8 r B^3 \Bigg[14 \omega r A'^3 B' - 2 r^2 \omega A''^2 B' + \omega A'^2 \left(r\left(31 B'' - 14 r B^{(3)}\right) - 14 B'\right) + \\
& + r \omega A' \left(\left(32 A'' - 2 r A^{(3)}\right) B' - 23 r A'' B''\right)\Bigg] + 4 r^2 \omega B^2 \Bigg[r^2 A''^2 B'^2 + \\
& + A'^2 \left(28 r^2 B''^2 - 60 B'^2 + r B'\left(14 r B^{(3)} + 43 B''\right)\right) + \\
& + r A' B' \left(23 r A'' B'' + \left(r A^{(3)} + 13 A''\right) B'\right)\Bigg] - \\
& - 16 B^4 \left(-r^2 \omega A''^2 + 14 \omega r A'^3 - 20 \omega A'^2 + r \omega A'\left(18 A'' - r A^{(3)}\right)\right),
\end{aligned}
\tag{2}
$$

$$
\begin{aligned}
\mathcal{K}_\omega\left(A, B, A', B', \cdots\right) := 8B^4\Big(A'\left(3\gamma r^5 + 2r\omega\right) - 14\omega\Big) + \\
+ 49\omega r^4 B'^4 - 16\omega r^4 B^3 B^{(4)} - 4r^3 B B'^2\left(29\omega r B'' - 5\omega B'\right) + \\
+ 4r^2 B^2\left[9\omega r^2 B''^2 + 3\omega B'^2 + 2rB'\Big(6\omega r B^{(3)} - 3\omega B''\Big)\right],
\end{aligned}
$$

$$
\begin{aligned}
\mathcal{G}_\omega\left(A,B,A',B',\cdots\right) := & -4r^3B^3\Big[\omega B'^3\left(40-87rA'\right)+26r^3\omega B''^3+\\
&+6r^2\omega B'B''\left(14rB^{(3)}+B''\right)+6r\omega B'^2\left(r\left(rB^{(4)}-6B^{(3)}\right)-36B''\right)\Big]+\\
&+24r^2B^4\Big[-rB'\left(\omega B''\left(27rA'-8\right)+2r\omega\left(rB^{(4)}+8B^{(3)}\right)\right)-\\
&-2\omega B'^2\left(3r^2A''-3rA'+2\right)+2r^2\omega\Big((rB^{(3)})^2-5B''^2+\\
&+r\left(rB^{(4)}+2B^{(3)}\right)B''\Big)\Big]+48r^2B^5\Big[A'\Big(\omega B'+r\Big(6\omega rB^{(3)}-\omega B''\Big)\Big)+\\
&+r\left(2\omega r\left(2A''B''+B^{(4)}\right)+B'\left(\omega rA^{(3)}-\omega A''\right)\right)\Big]-\\
&-32B^6\Big[3\alpha r^5A^{(3)}+18\beta r^5A^{(3)}+\\
&+3r^3\omega A^{(3)}-9r^2\omega A''+24r\omega A'-10\omega\Big]+121r^6\omega B'^6-\\
&-6r^5\omega BB'^4\left(85rB''-29B'\right)+\\
&+84r^4\omega B^2B'^2\left(7r^2B''^2-4B'^2+2rB'\left(rB^{(3)}-2B''\right)\right),
\end{aligned}
\tag{4}
$$

$$
\begin{aligned}
\mathcal{H}_\omega\left(A, B, A', B', \cdots\right) := {} & 28r^4\omega B B'^3\left(rA''B' - A'\left(B' - 10rB''\right)\right) - 85r^5\omega A'B'^5 - \\
& - 4r^3\omega B^2 B'\Bigg[A'\left(55r^2B''^2 - 46B'^2 + 2rB'\left(10rB^{(3)} + B''\right)\right) + \\
& + rB'\left(18rA''B'' + \left(rA^{(3)} + A''\right)B'\right)\Bigg] + \\
& + 8rB^4\Bigg[\omega rA'^2\left(19rB'' - 5B'\right) - 2r\omega\Bigg(2\left(A'' - rA^{(3)}\right)B' + \\
& + r\left(rA^{(3)}B'' + A''\left(2rB^{(3)} - 5B''\right)\right)\Bigg) + A'\Bigg(B'\left(13\omega r^2A'' + 4\omega\right) - \\
& - 2r\omega\left(2B'' + r\left(rB^{(4)} - 8B^{(3)}\right)\right)\Bigg)\Bigg] + 2r^2B^3\Bigg[-57\omega rA'^2B'^2 + 4\omega A'\Bigg(13B'^2 + \\
& + 2r^2B''\left(6rB^{(3)} + 5B''\right) + B'\left(r^3B^{(4)} - 46rB''\right)\Bigg) + \\
& + 4r\omega\Bigg(4r^2A''B''^2 - 10A''B'^2 + rB'\Big(rA^{(3)}B'' + \\
& + A''\left(2rB^{(3)} + 5B''\right)\Big)\Bigg)\Bigg] + 8B^5\Bigg[29r\omega A'^2 - 4r\omega\left(rA^{(3)} - 3A''\right) - \\
& - 2\omega A'\left(13r^2A'' + 14\right)\Bigg].
\end{aligned}
\tag{5}
$$

$$
\begin{aligned}
H_{rr} = \frac{1}{144 r^6 A^5 B^6} \Big\{ & 8\left(9\Lambda r^6 - 18\gamma r^4 + 4\omega\right) A^6 B^6 + 11\omega (rA'\left(rB' - 2B\right) B)^3 + \\
& +48(A\,B)^5 \Big(\left(3\gamma r^4 - 4\omega\right) B + r\left(3\gamma r^4 + 2\omega\right) B'\Big) + 3(rB)^2 \omega A A' \left(rB' - 2B\right)^2 \Big[7r^2 A' B'^2 + \\
& -2r\Big(rB'A'' + A'\left(6rB'' - 4B'\right)\Big) B - 4\left(5A' - rA''\right) B^2\Big] + 6A^4 B^6 \Big[48\omega B^2 - \\
& -16\omega B\bigg(r^3 B^{(3)} + 2rB'\bigg) - 4r^2\bigg(\omega B'^2 - 4\omega r\Big(B'' + rB^{(3)}\Big) B' + r^2 \omega B''^2\bigg) - \\
& -12\omega r^3 B^{-2} B'^2 \left(B' + rB''\right) B + 7\omega r^4 B^{-2} B'^4\bigg] + 3rA^2 B\bigg[8\omega B^5 \left(7rA'^2 - 8A' + 4rA''\right) + \\
& +2r^2\bigg(7\omega r A'^2 B'^2 + 4\omega A' B^3 \Big(3B'^2 + r\Big(B'' - 2rB^{(3)}\Big) B' - 2r^2 B''^2\Big) + \\
& +8r\omega A''\left(B' - rB''\right) B'\bigg) - 8rB^4\Big[7\omega r B' A'^2 - 2\omega\Big(4B' + r\Big(rB^{(3)} - 3B''\Big)\Big) A' - \\
& -2r\omega A''\left(rB'' - 3B'\right)\Big] + 4r^3 \omega B' B^2 \bigg(A'\Big(2r^2 B''^2 - 5B'^2 + r\Big(11B'' + rB^{(3)}\Big) B'\Big) + \\
& +rB'A''\left(B' + rB''\right)\bigg) - 2r^4 \omega B'^3 \Big(rB'A'' + A'B\left(7B' + 9rB''\right)\Big) B + 7r^5 \omega A' B'^5\bigg] + \\
& +A^3\bigg(11\omega B'^6 r^6 - 42\omega B B'^4 B'' r^6 + 12\omega B^2 B'^2 \Big(-8B'^2 + r\Big(5B'' + rB^{(3)}\Big) B' + 3r^2 B''^2\Big) r^4 - \\
& +4(rB)^3 \Big(\omega\left(16 + 9rA'\right) B'^3 + 48r\omega B'' B'^2 - 6r^2 \omega B''\Big(5B'' + rB^{(3)}\Big) B' + 2r^3 \omega B''^3\Big) + \\
& +24B^4\bigg(2\omega B''\Big(B'' + rB^{(3)}\Big) r^2 - 2rB'\Big(\omega\left(5 + rA'\right) B'' + 2r\omega B^{(3)}\Big) - \\
& -B'^2\left(\omega A'' r^2 + 2\omega A' r - 2\omega\right)\bigg) r^2 + 48rB^5\bigg(2\omega A' B'' + B^{(3)} r^2 + \\
& +B'\Big(-3r\omega A' + 2\omega\left(A'' r^2 + 1\right)\Big)\bigg) - 32\omega B^6 \left(3A'' r^2 - 6A' r + 4\right)\bigg).
\end{aligned} \tag{6}
$$

B Horizons of SdS and SadS.

Let us give here for the benefit of the reader, an intuitive analysis of the horizon system in de Sitter spaces.

- **Horizons of SdS.**

  Consider the function

$$f(r) = 1 - \frac{r_s}{r} - \frac{r^2}{r_\lambda^2}. \tag{7}$$

  The function goes to $-\infty$ for $r \to 0$, and again to $-\infty$ for $r \to \infty$. In order to examine whether the function becomes positive in some interval, we need to examine whether it has got a maximum value. Their extrema are located at

$$\bar{r}^3 = \frac{r_s r_\lambda^2}{2}. \tag{8}$$

  Let us now examine whether its value at the extrema is positive, in which case there will be two real points at which $f(r) = 0$.

$$f(\bar{r}) = 1 - 3\epsilon^{2/3}, \tag{9}$$

  where

$$\epsilon \equiv \frac{r_s}{2r_\lambda}. \tag{10}$$

  Then we can assert that

  –
$$1 > 3\epsilon^{2/3} \quad \therefore \text{There are two horizons.} \tag{11}$$

  –
$$1 = 3\epsilon^{2/3} \quad \therefore \text{There is only one horizon.} \tag{12}$$

  –
$$1 < 3\epsilon^{2/3} \quad \therefore \text{There is no horizon at all.} \tag{13}$$

- **Horizons of SadS.**

  In this case the adequate function to consider is

$$f(r) = 1 - \frac{r_s}{r} + \frac{r^2}{r_\lambda^2}. \tag{14}$$

  This function evolves from $-\infty$ for $r \to 0$, to $+\infty$ for $r \to \infty$. It has no extrema. Bolzano's theorem then implies the existence of just one horizon.