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Quantization of braided noncommutative field theories

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based on:

MDC, G. Giotopoulos, V. Radovanović, R. J. Szabo, *Braided L_∞-Algebras, Braided Field Theory and Noncommutative Gravity*, arXiv:2103.08939.

MDC, N. Konjik, V. Radovanović, R. J. Szabo, M. Toman, *L_∞-algebra of braided electrodynamics*, arXiv: 2204.06448.

MDC, N. Konjik, V. Radovanović, R. J. Szabo, M. Toman, in preparation.

Motivation

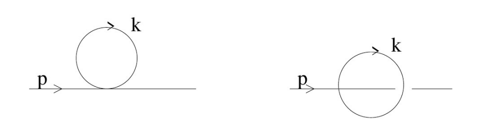
Divergences in QFT, Early Universe, singularities of BHs $\Rightarrow$ QG $\Rightarrow$ Quantum space-time.

One possibility: Noncommutative (NC) and/or nonassociative (NA) space-time.

Original motivation: Heisenberg, regularization of divergent electron self-energy. Nowdays we now that quantization of NC field theories introduses new divergences: UV/IR mixing.

Scalar $\phi^4_\star$ field theory and the Moyal-Weyl $\star$-product (motivated by string theory...)

$$S_\star(\phi) = \int \mathrm{d}^4x \Big(\frac{1}{2}\phi\,(-\Box - m^2)\,\phi + \frac{\lambda}{4!}\phi\star\phi\star\phi\star\phi \Big).$$



Planar diagrams: usual (quadratic divergent) UV behaviour, no improvement from NC deformation.

$$\Pi_1(p) \sim \int \frac{\mathrm{d}^4k}{(2\pi)^4} \frac{1}{(p^2-m^2)^2(k^2-m^2)}.$$

Non-planar diagrams: (consequence of NC deformation) introduce UV/IR mixing.

$$\Pi_2(p) \sim \int \frac{\mathrm{d}^4 k}{(2\pi)^4} \frac{e^{(p\wedge k)}}{(p^2 - m^2)^2(k^2 - m^2)},$$

with $p \wedge k = i\theta^{\mu\nu} p_\mu k_\nu$. UV convergent due to the oscilating factor $e^{(p\wedge k)}$. However, for $\theta \to 0$ or a very small external momentum $p \to 0$ the quadratic UV divergence appears again! Nonrenormalizable theory [(Minwalla, Van Raamsdonk, Seiberg '99)], see also [Bahns et al. '03].

Modification of the action by an oscilator term, renormalizable Grosse-Wulkenhaar model [Grosse, Wulkenhaar '04; Rivasseau et al. '05].

$$S_\star(\phi) = \int \mathrm{d}^4 x \Big(\frac{1}{2} \phi \big(-(\Box + \frac{1}{2}\omega^2 \tilde{x}^2) - m^2 \big) \phi + \frac{\lambda}{4!} \phi \star \phi \star \phi \star \phi \Big),$$

where $\tilde{x}^\mu = 2(\theta^{-1} x)^\mu$.

Gauge theories: no renormalizeble model has been constructed so far [Blaschke '16].

Our approach is based on:

Deformation

Drinfeld twist formalsim: a well defined way to deform a (Hopf) algebra of classical symmetries to a twisted (noncommutative, defomed) Hopf algebra. Module algebras (differential forms, tensors...) are consistently deformed into $\star$-module algebras: noncommutative differential geometry [Aschieri et al. '05...'18].

Construction of NC field theories

L_∞ algebra: Any classical (gauge) field theory described by the corresponding L_∞ algebra [Hohm, Zwiebach '17; Jurco et. al '19]. NC field theories can be encoded in a braided L_∞ algebra[MDC, Giotopoulos, Radovanovic, Szabo '21; Giotopoulos, Szabo '22].

Quantization

BV formalism, homological perturbation theory: algebraic techniques for quantization, can be generalized to NC (braided) field theories [Nguyen, Schenkel, Szabo '21].

Overview

Motivation

Tools

- Deformation by a twist
- L_∞-algebra
- Braided BV and homological perturbation theory

Examples of braided QFT

- Braided $\phi^4_\star$ theory
- Braided electrodynamics

Outlook

NC geometry via the twist deformation

Start from a symmetry algebra g and its universal covering algebra Ug. Then define a twist operator $\mathcal{F}$ as:

-an invertible element of $Ug \otimes Ug$
-fulfills the 2-cocycle condition (ensures the associativity of the $\star$-product).

$$\mathcal{F} \otimes 1(\Delta \otimes \mathrm{id})\mathcal{F} = 1 \otimes \mathcal{F}(\mathrm{id} \otimes \Delta)\mathcal{F}.$$

-additionaly: $\mathcal{F} = 1 \otimes 1 + \mathcal{O}(h)$; h-deformation parameter.

Braiding (noncommutativity): controled by the R-matrix $\mathcal{R} = \mathcal{F}^{-2} = \mathrm{R}^k \otimes \mathrm{R}_k$; triangular $\mathcal{R}_{21} = \mathcal{R}^{-1} = \mathrm{R}_k \otimes \mathrm{R}^k$.

$$\begin{array}{rcl}
\text{Symmetry Hopf algebra } Ug & \xrightarrow{\mathcal{F}} & \text{Twisted symmetry Hopf algebra } Ug^{\mathcal{F}} \\
\text{Module algebra } \mathcal{A} & \xrightarrow{\mathcal{F}} & \star \text{ module algebra } \mathcal{A}_\star \\
a, b \in \mathcal{A},\ a \cdot b \in \mathcal{A} & \xrightarrow{\mathcal{F}} & a \star b = \cdot \circ \mathcal{F}^{-1}(a \otimes b) = \mathrm{R}_k(b) \star \mathrm{R}^k(a).
\end{array}$$

Well known example: Moyal-Weyl twist $\mathcal{F} = e^{-\frac{i}{2}\theta^{\rho\sigma}\partial_\rho \otimes \partial_\sigma}$

$$\begin{aligned}
f \star g\,(x) &= \cdot \circ \mathcal{F}^{-1}(f \otimes g) \\
&= f \cdot g + \frac{i}{2}\theta^{\rho\sigma}(\partial_\rho f) \cdot (\partial_\sigma g) + \mathcal{O}(\theta^2) = \mathrm{R}_k g \star \mathrm{R}^k f \neq g \star f.
\end{aligned}$$

Associative, noncommutative: $\mathcal{R}^{-1} = \mathrm{R}_k \otimes \mathrm{R}^k$ encodes the noncommutativity.

L_∞ algebra and gauge field theory

L_∞-algebra (strong homotopy algebra): generalization of a Lie algebra with higher order brackets.

-Higher spin gauge theories with field-dependent gauge parameters [Berends, Burgers, van Dam '85]

$$(\delta_\alpha \delta_\beta - \delta_\beta \delta_\alpha)\Phi = \delta_{C(\alpha,\beta,\Phi)}\Phi.$$

-Generalized gauge symmetries of closed string field theory involve higher brackets [Zwiebach '15].

-Any classical field theory with generalized gauge symmetries is determined by an L_∞-algebra, due to duality with BV-BRST [Hohm, Zwiebach 17; Jurčo, Raspollini, Sämann, Wolf 18].

-NC gauge field theories in the L_∞ setting discussed in [Blumenhagen et al.'18; Kupriyanov '19].

-L_∞-algebras of ECP gravity, classical and noncommutative [MDC, Giotopoulos, Radovanović, Szabo '20, '21].

L_∞-algebra: $\mathbb{Z}$-graded vector space $V = \bigoplus_{k\in\mathbb{Z}} V_k$ with graded antisymmetric multilinear maps, n-brackets

$$\ell_n : \bigotimes^n V \longrightarrow V \ , \quad v_1 \otimes \cdots \otimes v_n \longmapsto \ell_n(v_1, \dots, v_n)$$

$$\ell_n(\dots, v, v', \dots) = -(-1)^{|v|\,|v'|}\,\ell_n(\dots, v', v, \dots) \ ,$$

where $|v|$ is a degree of $v \in V$.

n-brackets fulfil homotopy relations:

$$n=1: \quad \ell_1(\ell_1(v)) = 0, \quad (V, \ell_1) \text{ is a cochain complex} \ ,$$

$$n=2: \quad \ell_1(\ell_2(v_1, v_2)) = \ell_2(\ell_1(v_1), v_2) + (-1)^{|v_1|}\,\ell_2(v_1, \ell_1(v_2)) \quad \ell_1 \text{ is a derivation of } \ell_2 \ ,$$

$$\begin{aligned} n=3: \quad \ell_1(\ell_3(v_1, v_2, v_3)) = & -\ell_3(\ell_1(v_1), v_2, v_3) - (-1)^{|v_1|}\,\ell_3(v_1, \ell_1(v_2), v_3), \quad \text{Jacobi up to homotopy} \\ & - (-1)^{|v_1|+|v_2|}\,\ell_3(v_1, v_2, \ell_1(v_3)) \\ & - \ell_2(\ell_2(v_1, v_2), v_3) - (-1)^{(|v_1|+|v_2|)\,|v_3|}\,\ell_2(\ell_2(v_3, v_1), v_2) \\ & - (-1)^{(|v_2|+|v_3|)\,|v_1|}\,\ell_2(\ell_2(v_2, v_3), v_1) \end{aligned}$$

$\dots$

Cyclic L_∞-algebra: graded symmetric non-degenerated bilinear pairing $\langle -, - \rangle : V \otimes V \to \mathbb{R}$

$$\langle v_0, \ell_n(v_1, v_2, \dots, v_n)\rangle = (-1)^{n+(|v_0|+|v_n|)\,n+|v_n|\,\sum_{i=0}^{n-1}\,|v_i|}\,\langle v_n, \ell_n(v_0, v_1, \dots, v_{n-1})\rangle, \quad n \geq 1.$$

How do we use this in (gauge) field theories?

Start with $V = V_0 \oplus V_1 \oplus V_2 \oplus V_3$. Then

-gauge parameters $\rho \in V_0$,
-(gauge) fields $A \in V_1$,
-equations of motion $F_A \in V_2$,
-II Noether identites (Bianchi identites) $\mathrm{d}_A F_A \in V_3$.

Gauge transformations: $\delta_\rho A = \ell_1(\rho) + \ell_2(\rho, A) - \frac{1}{2}\ell_3(\rho, A, A) + \ldots$

EoM: $F_A = \ell_1(A) - \frac{1}{2}\ell_2(A, A) - \frac{1}{3!}\ell_3(A, A, A) + \ldots$

Action: $S(A) = \frac{1}{2}\langle A, \ell_1(A)\rangle - \frac{1}{3!}\langle A, \ell_2(A, A)\rangle + \ldots$

Noether identities: $\mathrm{d}_A F_A = \ell_1(F_A) + \ell_2(F_A, A) + \ldots$

Using the cyclicity of the pairing $\langle\ ,\ \rangle$, the variational principle is easily implemented

$$\delta S(A) = \langle \delta A\,,\ F_A\rangle\ .$$

Example: $3D$ non-Abelian Chern-Simons theory

We define: $\rho = \rho^a T^a \in V_0$, $A = A^a T^a \in V_1$, $F_A \in V_2$ and $\mathrm{d}_A F_A \in V_3$

The non-vanishing ℓ_n brackets are given by:

1-bracket ℓ_1

$$\ell_1(\rho) = \mathrm{d}\rho \ \in \ V_1 \ , \ \ell_1(A) = \mathrm{d}A \ \in \ V_2 \ , \ \ell_1(F_A) = \mathrm{d}F_A \ \in \ V_3 .$$

2-bracket ℓ_2

$$\begin{aligned} &\ell_2(\rho_1, \rho_2) = i[\rho_1, \rho_2], \quad \ell_2(\rho, A) = i[\rho, A], \quad \ell_2(\rho, F_A) = i[\rho, F_A] \\ &\ell_2(A_1, A_2) = i[A_1, A_2], \quad \ell_2(A, F_A) = i[A, F_A]. \end{aligned}$$

These reproduce:

$$\begin{aligned} \delta_\rho A &= \ell_1(\rho) + \ell_2(\rho, A) = \mathrm{d}\rho + i[\rho, A], \\ [\delta_{\rho_1}, \delta_{\rho_2}] &= \delta_{-\ell_2(\rho_1,\rho_2)} = \delta_{-i[\rho_1,\rho_2]} \ , \\ F_A &= \ell_1(A) - \frac{1}{2}\,\ell_2(A, A) = \mathrm{d}A - \frac{i}{2}\,[A, A], \\ \delta_\rho F_A &= \ell_2(\rho, F_A) = i[\rho, F_A], \\ \mathrm{d}_A F_A &= \ell_1(F_A) - \ell_2(A, F_A) = \mathrm{d}F_A - \frac{i}{2}\,[A, F_A], \\ S &= \frac{1}{2}\langle A, \ell_1(A)\rangle - \frac{1}{3!}\langle A, \ell_2(A, A)\rangle = \frac{1}{2}\int_M \mathrm{Tr}\Big(A \wedge \mathrm{d}A - \frac{i}{3}\,A \wedge [A, A]\Big). \end{aligned}$$

How do we deform this?

Braided L_∞-algebra

Generalization of a quantum Lie algebra [Woronowicz '89; Majid '94].

Rigorously: A braided L_∞-algebra is an L_∞-algebra $(V, \{\ell_n\})$ in the symmetric monoidal category ${}_{\mathcal{F}}\mathcal{M}^\sharp$. What does it means, how does it work?

- $\mathbb{Z}$-graded real vector space $V = \bigoplus_{k\in\mathbb{Z}} V_k$. Usually we work with

$$V = V_0 \oplus V_1 \oplus V_2 \oplus V_3.$$

- maps/brackets: $\ell_n^\star : \bigotimes^n V \to V$

$$\ell_n^\star(v_1 \otimes \cdots \otimes v_n) = \ell_n(v_1 \otimes_\star \cdots \otimes_\star v_n),$$

with $v \otimes_\star v' := \mathcal{F}^{-1}(v \otimes v') = \bar{\mathrm{f}}^\alpha(v) \otimes \bar{\mathrm{f}}_\alpha(v')$ for $v, v' \in V$. The brackets are graided and braied symmetric!

$$\ell_n^\star(\dots, v, v', \dots) = -(-1)^{|v|\,|v'|}\,\ell_n^\star(\dots, \mathrm{R}_k(v'), \mathrm{R}^k(v), \dots)\ .$$

For example: $3D$ CS gauge theory $\ell_2(\rho, A) = i[\rho, A]$ is deformed to

$$\begin{aligned}\ell_2^\star(\rho, A) =& i[\bar{\mathrm{f}}^k(\rho), \bar{\mathrm{f}}_k(A)] = i[\rho, A]_\star = -i[\mathrm{R}_k(A), \mathrm{R}^k(\rho)]_\star \\ =& i\rho^a \star A^b[T^a, T^b].\end{aligned}$$

The braided commutator closes in the corresponding Lie algebra!

- braided homotopy relations:

$$\ell_1^\star(\ell_1^\star(v)) = 0 \ ,$$

$$\ell_1^\star(\ell_2^\star(v_1, v_2)) = \ell_2^\star(\ell_1^\star(v_1), v_2) + (-1)^{|v_1|}\, \ell_2^\star(v_1, \ell_1^\star(v_2)) \ ,$$

$$\begin{aligned}
&\ell_2^\star(\ell_2^\star(v_1, v_2), v_3) - (-1)^{|v_2|\,|v_3|}\, \ell_2^\star(\ell_2^\star(v_1, \mathrm{R}_k(v_3)), \mathrm{R}^k(v_2)) \\
&\quad + (-1)^{(|v_2|+|v_3|)\,|v_1|}\, \ell_2^\star(\ell_2^\star(\mathrm{R}_k(v_2), \mathrm{R}_j(v_3)), \mathrm{R}^j\mathrm{R}^k(v_1)) \\
&= -\ell_3^\star(\ell_1^\star(v_1), v_2, v_3) - (-1)^{|v_1|}\, \ell_3^\star(v_1, \ell_1^\star(v_2), v_3) \\
&\quad - (-1)^{|v_1|+|v_2|}\, \ell_3^\star(v_1, v_2, \ell_1^\star(v_3)) - \ell_1^\star(\ell_3^\star(v_1, v_2, v_3)) \ , \\
&\cdots
\end{aligned}$$

- To have a well defined variational principle, we demand strict cyclicity:

$$\begin{aligned}
&\langle v_2, v_1\rangle_\star = \langle\ ,\ \rangle \circ \mathcal{F}^{-1}(v_2 \otimes v_1) = \langle \mathrm{R}_k(v_1), \mathrm{R}^k(v_2)\rangle_\star = \langle v_1, v_2\rangle_\star, \\
&\langle v_0, \ell_n^\star(v_1, v_2, \dots, v_n)\rangle_\star = \langle v_n, \ell_n^\star(v_0, v_1, \dots, v_{n-1})\rangle_\star.
\end{aligned}$$

  Twist operator fulfilling this is a compatible Drinfel'd twists. It define a strictly cyclic braided L_∞-algebra.

Braided gauge theory via braided L_∞-algebra

Just like in the classical (commutative) case, a braided L_∞-algebra defines a braided field theory.

Braided gauge transformations

$$\delta^\star_\rho A = \ell^\star_1(\rho) + +\ell^\star_2(\rho, A) - \frac{1}{2}\ell^\star_3(\rho, A, A) + \dots \ .$$

Braided equations of motion

$$\begin{aligned} F^\star_A &= \ell^\star_1(A) - \frac{1}{2}\ell^\star_2(A, A) - \frac{1}{6}\ell^\star_3(A, A, A) + \dots \ = 0, \\ \text{Braided 3D CS:} \quad F^\star_A &= \ell^\star_1(A) - \tfrac{1}{2}\,\ell^\star_2(A, A) = \mathrm{d}A - \frac{i}{2}\,[A, A]_\star = 0 \ . \end{aligned}$$

Braided Noether identity does not follow from the variation of an action. Instead it is obtained as a combination of homotopy relations

$$\mathrm{d}^\star_A F^\star_A = \ell^\star_1(F^\star_A) - \tfrac{1}{2}\left(\ell^\star_2(A, F^\star_A) - \ell^\star_2(F^\star_A, A)\right) + \tfrac{1}{4}\,\ell^\star_2\left(\mathrm{R}_k(A), \ell^\star_2(\mathrm{R}^k(A), A)\right) + \dots = 0$$

Braided 3D CS:

$$\mathrm{d}^\star_A F^\star_A = \mathrm{d}F^\star_A - \frac{i}{2}\,[A, F^\star_A]_\star + \frac{i}{2}\,[F^\star_A, A]_\star + \tfrac{1}{4}\,[\mathrm{R}_k(A), [\mathrm{R}^k(A), A]_\star]_\star = 0.$$

Braided gauge invariant action

$$S(A) = \sum_{n=1}^{\infty} \frac{1}{(n+1)!} (-1)^{\frac{1}{2}\,n\,(n-1)} \langle A, \ell_n^\star(A,\ldots,A)\rangle\ ,$$

Braided 3D CS:
$$S_\star(A) = \frac{1}{2}\,\big\langle A, \ell_1^\star(A)\big\rangle_\star - \frac{1}{6}\,\big\langle A, \ell_2^\star(A,A)\big\rangle_\star$$
$$= \frac{1}{2}\int_M \mathrm{Tr}\Big(A\wedge_\star \mathrm{d}A - \frac{i}{3}\,A\wedge_\star [A,A]_\star\Big)\ .$$

It is braided gauge invariant $\delta_\rho^\star S_\star(A) = 0$.

Comments on the braided 3D CD theory

-"naive" deformation of the classical theory

-braided II Noether identity: new term (inhomogeneous in EoM), vanishes in the commutative limit. Important, introduce interdependence of EoM, consequence of braided gauge symmetry.

-braided gauge transformations have braided Leibniz rule:

$$\delta_\rho^\star(\phi_1\star\phi_2) = \delta_\rho^\star\phi_1\star\phi_2 + \mathrm{R}_k\phi_1\star\delta^\star_{\mathrm{R}^k(\rho)}\phi_2.$$

Braided BV formalism

Developed in [Nguyen, Schenkel, Szabo '21], following [Costello, Gwilliam '16)] and [Jurco et al. '19].

- Start from the (braided, cyclic) L_∞ algebra that defines the theory $(V, \ell_n^\star, \langle\,,\,\rangle_\star)$.
- Introduce the braided symmetric algebra $\mathrm{Sym}_{\mathcal{R}} V$ and extend the L_∞ structure to it:

$$v_1 \odot_\star (v_2) = (-1)^{|v_1||v_2|} \mathrm{R}_k(v_2) \odot_\star \mathrm{R}^k(v_1)$$

and

$$\begin{aligned} \boldsymbol{\ell}_1^\star(a_1 \otimes v_1) &= a_1 \otimes \ell_1^\star(v_1) \ , \\ \boldsymbol{\ell}_2^\star(a_1 \otimes v_1, a_2 \otimes v_2) &= (a_1 \odot_\star \mathrm{R}_k(a_2)) \otimes \ell_2^\star(\mathrm{R}^k(v_1), v_2) \ , \\ &\cdots \\ \langle\!\langle a_1 \otimes v_1, a_2 \otimes v_2 \rangle\!\rangle_\star &= (a_1 \odot_\star \mathrm{R}_k(a_2)) \, \langle \mathrm{R}^k(v_1), v_2 \rangle_\star \ , \end{aligned}$$

for $a_1, a_2 \in \mathrm{Sym}_{\mathcal{R}} V[2]$, $v_1, v_2 \in V$.

Compare with extending the Lie algebra $[T^a, T^b] = if^{abc} T^c$ to the algebra of differential forms with $\wedge$ product

$$[A_1, A_2] = [A_1^a T^a, A_2^b T^b] = A_1^a \wedge A_2^b [T^a, T^b].$$

- The contracted coordinate functions $\boldsymbol{\xi} \in (\mathrm{Sym}_{\mathcal{R}} V[2]) \otimes V$ are construced using the basis in V, τ_k, and the corresponding dual (via pairing) basis in $V[3]$, τ^k and $\langle \tau_k, \tau^j \rangle = \delta_k^j$

$$\boldsymbol{\xi} = \sum_k \tau_k \otimes \tau^k .$$

- The braided BV action $\mathcal{S}^\star_{\mathrm{BV}} \in \mathrm{Sym}_{\mathcal{R}} V[2]$ is defined as

$$\begin{aligned}\mathcal{S}^\star_{\mathrm{BV}} &= \frac{1}{2} \langle\!\langle \boldsymbol{\xi} \,, \boldsymbol{\ell}_1^\star(\boldsymbol{\xi}) \rangle\!\rangle_\star - \frac{1}{3!} \langle\!\langle \boldsymbol{\xi} \,, \boldsymbol{\ell}_2^\star(\boldsymbol{\xi}, \boldsymbol{\xi}) \rangle\!\rangle_\star - \frac{1}{4!} \langle\!\langle \boldsymbol{\xi} \,, \boldsymbol{\ell}_3^\star(\boldsymbol{\xi}, \boldsymbol{\xi}, \boldsymbol{\xi}) \rangle\!\rangle_\star + \dots \\ &= \mathcal{S}^\star_{(0)} + \mathcal{S}^\star_{\mathrm{int}} .\end{aligned}$$

  $S^\star_{\mathrm{BV}}$ satisies the classical master equation

$$\{ S^\star_{\mathrm{BV}}, S^\star_{\mathrm{BV}} \}_\star = 0$$

  with $\{\phi_1, \phi_2\}_\star = \langle \phi_1, \phi_2 \rangle_\star$ for $\phi_{1,2} \in V[2]$ and extended (braided, gradied) to the full $\mathrm{Sym}_{\mathcal{R}} V[2]$.

- The operator $Q = \ell_1^\star + \{ S^\star_{\mathrm{int}}, \ \}_\star$ satisfies $Q^2 = 0$ and

$$Q\{\phi_1, \, \phi_2\}_\star = \{Q\phi_1, \, \phi_2\}_\star + (-1)^{|a_1|} \{\phi_1, Q\phi_2\}_\star .$$

- The algebra of classical observables: $(\mathrm{Sym}_{\mathcal{R}} V[2], Q, \{ \ , \ \}_\star)$.

- The (braided) algebra of quantum observables: $(\mathrm{Sym}_{\mathcal{R}} V[2]), Q_{\mathrm{BV}}, \{\ ,\ \}_\star)$ with
$$Q_{\mathrm{BV}} = \ell_1^\star + \{S_{\mathrm{int}}^\star, \ \}_\star + i\hbar\Delta_{\mathrm{BV}}.$$
The braided BV Laplacian Δ_{BV}
$$\Delta_{\mathrm{BV}}(1) = 0, \quad \Delta_{\mathrm{BV}}(\phi_1) = 0, \quad \Delta_{\mathrm{BV}}(\phi_1 \odot_\star \phi_2) = \{\phi_1, \phi_2\}_\star ,$$
$$\Delta_{\mathrm{BV}}(\phi_1 \odot_\star \cdots \odot_\star \phi_n) = \sum_{a<b} \pm \langle \phi_a, \mathrm{R}_{k_{a+1}} \cdots \mathrm{R}_{k_{b-1}}(\phi_b)\rangle_\star \ \phi_1 \odot_\star \cdots \odot_\star \phi_{a-1}$$
$$\odot_\star \mathrm{R}^{k_{a+1}}(\phi_{a+1}) \odot_\star \cdots \odot_\star \mathrm{R}^{k_{b-1}}(\phi_{b-1}) \odot_\star \phi_{b+1} \odot_\star \cdots \odot_\star \phi_n \ .$$
$$\ell_1^\star \Delta_{\mathrm{BV}} + \Delta_{\mathrm{BV}} \ell_1^\star = 0, \quad \Delta_{\mathrm{BV}}^2 = 0, \quad \Delta_{\mathrm{BV}}(\mathcal{S}_{\mathrm{int}}^\star) = 0.$$
These properties enable $Q_{\mathrm{BV}}^2 = 0$!
- The braided BV laplacian Δ_{BV} encodes the braided Wick theorem and the interaction action $\mathcal{S}_{\mathrm{int}}^\star$ encodes interaction (vertices).

Braided homological perturbation theory

How do we calculate corelation functions? We use the (braided) homological perturbation lemma.

- On V_∞ algebra $V[2]$: propagators h define (braided) strong deformation retracts:

$$\begin{tikzcd} (V_\bullet[2], \ell_1^*) \arrow[loop above, "h"] \arrow[r, shift left, "p"] & (H^\bullet(V_\bullet[2]), 0) \arrow[l, shift left, "i"] \end{tikzcd}$$

- This can be extended to the space of observables $h \to H$:

$$\begin{tikzcd} (\mathrm{Sym}_R V_\bullet[2], \ell_1^{*\,ext.}) \arrow[loop above, "H"] \arrow[r, shift left, "P"] & (\mathrm{Sym}_R H^\bullet(V[2]), 0) \arrow[l, shift left, "J"] \end{tikzcd}$$

- A preturbation δ defines a new (braided) strong deformation retract

$$\begin{tikzcd} (\mathrm{Sym}_R V[2], \underbrace{\ell_1^{ext.} + \delta}_{Q_{BV}}) \arrow[loop above, "\tilde{H}"] \arrow[r, shift left, "\tilde{P}"] & (\mathrm{Sym}_R H^\bullet(V[2]), \tilde{\delta}_1) \arrow[l, shift left, "\tilde{J}"] \end{tikzcd}$$

Braided homological perturbation lema defines the perturbed projection map $\tilde{P} = \mathsf{P} + \mathsf{P}_{\delta}$ with

$$\mathsf{P}_{\delta} = \mathsf{P}\left(\mathrm{id}_{\mathrm{Sym}_{\mathcal{R}} V[1]} - \boldsymbol{\delta}\,\mathsf{H}\right)^{-1} \boldsymbol{\delta}\,\mathsf{H}\ .$$

In the classical case (no NC deformation) gives the path integral [Doubek, Jurčo, Pulmann '17].

The new projection P_{δ} gives correlation functions for the braided QFT:

$$\begin{aligned} G_n^{\star}(x_1, \dots, x_n) &= \langle 0|\mathrm{T}[\phi(x_1) \star \cdots \star \phi(x_n)]|0\rangle_{\star} := \mathsf{P}_{\delta}(\delta_{x_1} \odot_{\star} \cdots \odot_{\star} \delta_{x_n}) \\ &= \sum_{p=1}^{\infty} \mathsf{P}\big((\boldsymbol{\delta}\,\mathsf{H})^p(\delta_{x_1} \odot_{\star} \cdots \odot_{\star} \delta_{x_n})\big)\ , \end{aligned}$$

where $\delta_{x_a}(x) := \delta(x - x_a)$ are Dirac distributions supported at the insertion points x_a of the physical field $\phi \in V^1$.

Braided $\phi^4_\star$ theory

For simplicity: 4D Minkowski space-time, Moyal-Weyl twist and a real massive scalar field ϕ with ϕ^4 interaction.

Classical theory is given by the graided vector space $V = V_1 \oplus V_2$ with $V_1 = V_2 = \Omega^0(\mathbb{R}^{1,3})$ and the brackets

$$\ell_1(\phi) = -(\Box + m^2)\phi, \quad \ell_3(\phi_1, \phi_2, \phi_3) = -\lambda\phi_1\phi_2\phi_3.$$

The cyclic pairing

$$\langle \phi, \phi^+ \rangle = \int \mathrm{d}^4x \ \phi \ \phi^+ \ ,$$

for $\phi \in V^1$ and $\phi^+ \in V^2$ then defines the usual action

$$S(\phi) = \frac{1}{2}\langle \phi, \ell_1(\phi), \phi \rangle - \frac{1}{24}\langle \phi, \ell_3(\phi, \phi, \phi)\rangle = \int \mathrm{d}^4x \ \Big(\frac{1}{2}\phi\,\big(-\Box - m^2\big)\,\phi - \frac{\lambda}{24}\phi^4\Big).$$

Braided NC scalar field theory: the same vector space V with

$$\ell_1^\star(\phi) = -\,(\Box + m^2)\phi, \quad \ell_3^\star(\phi_1, \phi_2, \phi_3) = \lambda\phi_1 \star \phi_2 \star \phi_3$$

$$\begin{aligned} S_\star(\phi) =& \frac{1}{2}\,\langle \phi, \ell_1(\phi)\rangle_\star - \frac{1}{24}\,\langle \phi, \ell_3^\star(\phi, \phi, \phi)\rangle_\star =: S_0(\phi) + S_{\rm int}(\phi) \\ =& \int \mathrm{d}^4x\Big(\frac{1}{2}\,\phi\,\big(-\Box - m^2\big)\,\phi + \frac{\lambda}{4!}\,\phi\star\phi\star\phi\star\phi\Big). \end{aligned}$$

The same as the usual $\phi^4_\star$ theory!

P and H maps:

$$\mathrm{P}(1) = 1 \quad \text{and} \quad \mathrm{P}(\varphi_1 \odot_\star \cdots \odot_\star \varphi_n) = 0, \quad \mathrm{H}(1) = 0 \ ,$$

$$\mathrm{H}(\varphi_1 \odot_\star \cdots \odot_\star \varphi_n) = \frac{1}{n} \sum_{a=1}^{n} \pm \, \varphi_1 \odot_\star \cdots \odot_\star \varphi_{a-1} \odot_\star \mathrm{h}(\varphi_a) \odot_\star \varphi_{a+1} \odot_\star \cdots \odot_\star \varphi_n \ ,$$

for all $\varphi_a = \phi_a + \phi_a^+ \in V[1]$ and $\mathrm{h}(\phi^+)(x) = -\frac{1}{\Box + m^2}\, \phi^+(x)$.

Quantization, free theory: perturbation $\boldsymbol{\delta} = i\,\hbar\,\Delta_{\mathrm{BV}}$

$$G_n^\star(x_1, \dots, x_n)^{(0)} = \langle 0 | \mathrm{T}[\phi(x_1) \star \cdots \star \phi(x_n)] | 0 \rangle_\star := \mathrm{P}_{\boldsymbol{\delta}}(\delta_{x_1} \odot_\star \cdots \odot_\star \delta_{x_n})$$
$$= \sum_{p=1}^{\infty} \mathrm{P}\big((i\hbar\Delta_{\mathrm{BV}}\, \mathrm{H})^p(\delta_{x_1} \odot_\star \cdots \odot_\star \delta_{x_n})\big) \ ,$$

2-point function: free propagator

$$G_2^\star(x_1, x_2)^{(0)} = i\,\hbar\,\Delta_{\mathrm{BV}}\, \mathrm{H}(\delta_{x_1} \odot_\star \delta_{x_2}) = -i\hbar G(x_1 - x_2) = -i\,\hbar \int_k \frac{e^{-i\,k\cdot(x_1 - x_2)}}{k^2 - m^2}$$
$$= \underbracket{\phi_1\, \phi_2}.$$

4-point function: braided Wick theorem

$$G_4^\star(x_1, x_2, x_3, x_4)^{(0)} = (i\,\hbar\,\Delta_{\mathrm{BV}}\, \mathrm{H})^2\, (\delta_{x_1} \odot_\star \delta_{x_2} \odot_\star \delta_{x_3} \odot_\star \delta_{x_4})$$
$$= \underbracket{\phi_1\, \phi_2}\, \underbracket{\phi_3\, \phi_4} + \phi_1\, \mathrm{R}_\alpha(\phi_3)\, \mathrm{R}^\alpha(\phi_2)\, \phi_4 + \underbracket{\phi_1\, \phi_4}\, \underbracket{\phi_2\, \phi_3} \ .$$

Quantization, interacting theory: perturbation $\delta = i\,\hbar\,\Delta_{\rm BV} + \{\mathcal{S}^\star_{\rm int}, -\}_\star$ with

$$\mathcal{S}^\star_{\rm int} = -\tfrac{1}{24}\,\langle\!\langle \boldsymbol{\xi}\,,\,\ell^\star_3(\boldsymbol{\xi},\boldsymbol{\xi},\boldsymbol{\xi})\rangle\!\rangle_\star$$

and

$$\boldsymbol{\xi} = \int_k \big(\mathsf{e}_k\otimes\mathsf{e}^k + \mathsf{e}^k\otimes\mathsf{e}_k\big)\,, \qquad \mathsf{e}_k = e^{-ikx},\ \mathsf{e}^k = e^{ikx}$$

contracted coordinate functions $\boldsymbol{\xi} \in (\mathrm{Sym}_{\mathcal{R}} L[2])\otimes L$. The explicit form of $\mathcal{S}^\star_{\rm int}$ is

$$\mathcal{S}^\star_{\rm int} = \int_{k_1,\dots,k_4} V(k_1,k_2,k_3,k_4)\ \mathsf{e}^{k_1}\odot_\star\mathsf{e}^{k_2}\odot_\star\mathsf{e}^{k_3}\odot_\star\mathsf{e}^{k_4}$$

with

$$V(k_1,k_2,k_3,k_4) = \frac{\lambda}{4!}\ e^{\frac{i}{2}\sum\limits_{a<b} k_a\cdot\theta\,k_b}\ (2\pi)^4\,\delta(k_1+k_2+k_3+k_4)\,.$$

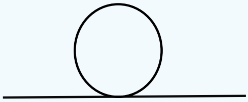
The interacting n-point function is defined as

$$\begin{aligned} G^\star_n(x_1,\dots,x_n)^{\rm int} &= \langle 0|\mathrm{T}[\phi(x_1)\star\cdots\star\phi(x_n)]|0\rangle^{\rm int} \\ &= \sum_{p=1}^{\infty} \mathsf{P}\big((i\,\hbar\,\Delta_{\rm BV}\,\mathsf{H} + \{\mathcal{S}_{\rm int},-\}_\star\,\mathsf{H})^p(\delta_{x_1}\odot_\star\cdots\odot_\star\delta_{x_n})\big)\ . \end{aligned}$$

2-point function at 1-loop:

$$
\begin{aligned}
G_2^\star(x_1,x_2)^{(1)} &= (i\,\hbar\,\Delta_{\mathrm{BV}}\,\mathsf{H})^2\,\{\mathcal{S}_{\mathrm{int}},\mathsf{H}(\delta_{x_1}\odot_\star\delta_{x_2})\}_\star \\
&= \dots \\
&= \frac{\hbar^2\,\lambda}{2}\int_{k_1,k_2}\frac{e^{-i\,k_1\cdot(x_1-x_2)}}{\big(k_1^2-m^2\big)^2\,\big(k_2^2-m^2\big)}\ .
\end{aligned}
$$

is the same as in the commutative case!



No nonplanar diagrams and no UV/IR mixing at 1-loop. Consistent with [Oeckel '00], discussed in [Balachandran et al. '06; Bu et al. '06; Fiore, Wess '07].

Braided 4D electrodynamics

4D Minkowski space-time, Moyal-Weyl twist, massive spinor field ψ, $U(1)$ gauge field A_μ. An example of L_∞ algebra with gauge and matter fields. More examples discussed in [Gomes et al. '20].

The braided L_∞ algebra of spinor electrodynamics:

$$\mathcal{A} = \begin{pmatrix} \bar\psi \\ \psi \\ A_\mu \end{pmatrix}, \quad F_{\mathcal{A}} = \begin{pmatrix} F_{\bar\psi} \\ F_\psi \\ (F_A)_\mu \end{pmatrix},$$

$$\ell_1^\star(\rho) = \begin{pmatrix} 0 \\ 0 \\ \frac{1}{e}\partial_\mu\rho \end{pmatrix}, \quad \ell_2^\star(\rho, \mathcal{A}) = \begin{pmatrix} -i\mathrm{R}_k(\bar\psi)\star\mathrm{R}^k(\rho) \\ i\rho\star\psi \\ i[\rho, A]_\star = 0 \end{pmatrix},$$

$$\ell_1^\star(F_{\mathcal{A}}^\star) = \partial_\mu(F_A^\star)^\mu, \quad \ell_2^\star(\mathcal{A}, F_{\mathcal{A}}^\star) = -ie\big(\bar\psi\star F_{\bar\psi} - \mathrm{R}_k(F_\psi)\star\mathrm{R}^k(\psi)\big),$$

$$\ell_1^\star(\mathcal{A}) = \begin{pmatrix} i\gamma^\mu\partial_\mu\psi \\ -i\gamma^\mu\partial_\mu\bar\psi \\ -\partial_\mu\partial_\nu A^\nu + \partial_\nu\partial^\nu A_\mu \end{pmatrix}, \quad \ell_2^\star(\mathcal{A}_1, \mathcal{A}_2) = -\frac{e}{2}\begin{pmatrix} \gamma^\mu A_{1\,\mu}\star\psi_2 + \mathrm{R}_k\gamma^\mu A_{2\,\mu}\star\mathrm{R}^k\psi_1 \\ \bar\psi_1\star\gamma^\mu A_{2\,\mu} + \mathrm{R}_k\bar\psi_2\star\gamma^\mu\mathrm{R}^k A_{1\,\mu} \\ \bar\psi_1\gamma^\mu\star\psi_2 + \mathrm{R}_j\bar\psi_2\gamma^\mu\star\mathrm{R}^j\psi_1 \end{pmatrix}.$$

Braided action

$$S = \int \mathrm{d}^4x\,\Big\{-\frac{1}{4}F^{\mu\nu}\star F_{\mu\nu} + \bar\psi\star i\gamma^\mu\partial_\mu\psi + \frac{e}{2}\Big(\bar\psi\star A_\mu\gamma^\mu\star\psi + \bar\psi\star\mathrm{R}_k(A_\mu)\gamma^\mu\star\mathrm{R}^k(\psi)\Big)\Big\}.$$

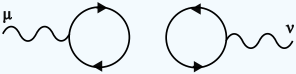
Comments:

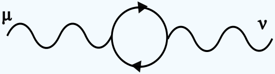
-braided NC electrodynamics remains abelian: no photon self-interactions.

-the photon-fermion verteks is different compared to the $\star$-electrodynamics.

Quantization: homological perturbation theory. Preeliminary results: 1-loop photon self energy

$$
\begin{aligned}
G^{\star}_{A_\mu,A_\nu}(x_1,x_2)^{(1)} &= \langle 0|\mathrm{T}[A_\mu(x_1)\star A_\nu(x_2)]|0\rangle^{(1)}_{\star} \\
&= (i\,\hbar\,\Delta_{\mathrm{BV}}\,\mathsf{H})^2\left\{\mathcal{S}_{\mathrm{int}},\mathsf{H}\left\{\mathcal{S}_{\mathrm{int}},\mathsf{H}\big(\delta^{A_\mu}_{x_1}\odot_\star\delta^{A_\nu}_{x_2}\big)\right\}_\star\right\}_\star \\
&\quad + i\,\hbar\,\Delta_{\mathrm{BV}}\,\mathsf{H}\left\{\mathcal{S}_{\mathrm{int}},\mathsf{H}\,(i\,\hbar\,\Delta_{\mathrm{BV}}\,\mathsf{H})\big\{\mathcal{S}_{\mathrm{int}},\mathsf{H}\big(\delta^{A_\mu}_{x_1}\odot_\star\delta^{A_\nu}_{x_2}\big)\big\}_\star\right\}_\star \\
&=: \mathcal{G}^1_{\mu\nu}(x_1,x_2)+\mathcal{G}^2_{\mu\nu}(x_1,x_2)\ .
\end{aligned}
$$





$$
\frac{i}{\hbar}\,\Pi^{\mu\nu}_{\star 2}(p) = -q^2\int\frac{\mathrm{d}^4k}{(2\pi)^4}\quad\frac{\cos^2\left(\frac{i}{2}\theta^{\mu\nu}p_\mu k_\nu\right)}{\left((p-k)^2-m^2\right)\left(k^2-m^2\right)}\,\mathrm{Tr}\big((\not p-\not k-m)\,\gamma^\mu\,(\not k+m)\,\gamma^\nu\big)\ .
$$

Unlike in the $\star$ electrodynamics:

-fermion bubble gives a nontrivial NC contribution.

-no non-planar diagrams, but UV/IR mixing present.

Outlook

- We deformed the L_∞-algebra to a braided L_∞-algebra (mathematically well defined in a proper category).

  -well defined way to construct a braided L_∞-algebra starting from the classical one.

  -enables constructions of new NC field theories (unexpected deformations, different from the "naive" expectations).

- Quantization

  -no UV/IR mixing (no non-planar diagram) in $\phi^4_\star$ braided QFT

  -no non-planar diagrams in braided QED, but UV/IR mixing seems to be present at 1-loop.

- Future work

  -better understanding of braided symmetries and classical braided field theories, new solutions of the classical equations (in gravity)

  -better understanding of braided QFT: relations between non-planar diagrams, UV/IR mixing, (braided) gauge symmetry