Cobordism, K-theory and tadpoles

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Introduction

Motivation

- Mathematical formulation of quantum gravity?

- Signatures of quantum gravity in low energy EFTs?

These two questions can be addressed together!

No global symmetries in quantum gravity

- Not any EFT is consistent with quantum gravity
  $\Rightarrow$ Swampland Program [Vafa, '05]

- **There are no global symmetries in quantum gravity**
  [Misner, Wheeler '57; Banks, Dixon '88; Kallosh, Linde, Linde, Susskind '95; Harlow, Ooguri '18].
  Among the most solid swampland conjectures.

- Recent proposal [McNamara, Vafa '19]:
  cobordism conjecture, generalising no global symmetries.

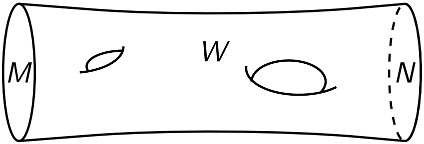
The cobordism conjecture relates the two questions to one another.

Cobordism

Cobordism: definition

Consider n-dim compact manifolds M and N without boundary.
A **bordism** is a $(n+1)$-dim compact manifold W such that

$$\partial W = M \sqcup N$$



Being bordant is an equivalence relation, $[M] \sim [N]$.

Set of equivalence classes is an abelian group, **cobordism group**

$$\Omega_n = \{\text{compact } n\text{-dim manifolds without boundary}\}/\sim$$

Cobordim as generalized homology

- (Co)Homology groups of point carry no information

$$H_n(\mathrm{pt}) = 0 \qquad (\text{if } n > 0)$$

  since every cycle on pt of positive dimension is a boundary.

- Cobordism groups of point do carry information

$$\Omega_n(\mathrm{pt}) \neq 0$$

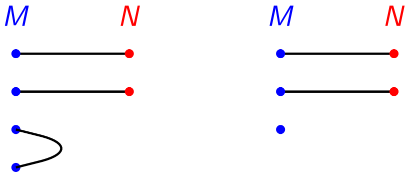
  since **not every compact manifold is a boundary**.

- This information is topological and physical.

A (co)homology theory whose groups of pt are generically non-vanishing is called generalized (co)homology. Cobordism and K-theory are examples.

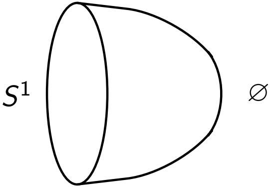
Simple examples

- $\Omega_0(\mathrm{pt}) = \mathbb{Z}_2$.
  $M = \sqcup_m \mathrm{pt}$ and $N = \sqcup_n \mathrm{pt}$ bordant iff $m+n$ is even



bordant — not bordant

- $\Omega_1(\mathrm{pt}) = 0$. Indeed the circle is a boundary. Notice $0 = [\varnothing]$



Spin and Spin^c structures

Manifolds can be endowed with *structure*. This is inherited by the bordism group Ω_n^ξ. We will consider mainly

- **Spin structure**: $w_2(TM) = 0$

n	0	1	2	3	4	5	6	7	8	9	10
Ω_n^{Spin}	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	0	$\mathbb{Z}$	0	0	0	$2\mathbb{Z}$	$2\mathbb{Z}_2$	$3\mathbb{Z}_2$

- **Spin^c structure**: $W_3(TM) = 0$

n	0	1	2	3	4	5	6	7	8	9	10
$\Omega_n^{\mathrm{Spin^c}}$	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$2\mathbb{Z}$	0	$2\mathbb{Z}$	0	$4\mathbb{Z}$	0	$4\mathbb{Z}$

These are examples of stable tangential structures.

Bordism invariants

They are maps

$$\mu_n : \Omega_n^{\xi} \to A,$$

with A an abelian group (e.g. $\mathbb{Z}$).
They take the same value within the whole class $[M]$ (**invariant**).

In some cases, they admit an integral representation in terms of (generalised) cohomology classes

$$\mu = \int_{[M]} \omega \qquad \text{where} \qquad \omega \in H^n(M, A)$$

Example: the bordism invariants of $\Omega_6^{\mathrm{Spin^c}} = \mathbb{Z} \oplus \mathbb{Z}$ are

$$\int_{M_6} \mathrm{td}_6 = \int_{M_6} \frac{c_2 c_1}{24} \qquad \text{and} \qquad \int_{M_6} \frac{c_1^3}{2}$$

Cobordism conjecture

The cobordism conjecture

[McNamara, Vafa '19]

For **any** d-dim EFT, $\exists$ quantum gravity structure QG such that

$$\Omega_n^{\mathrm{QG}}(\mathrm{pt}) = 0, \qquad \forall n \leq d,$$

i.e. the groups contain just the trivial element $0 = [\varnothing]$.

- The QG-structure need not to be unique.
- If $\Omega_n^{\mathrm{QG}}(\mathrm{pt}) \neq 0$, then there is a global $(d-n-1)$-form symmetry in the EFT. Not allowed in quantum gravity.
- If $\Omega_n^{\mathrm{QG}}(\mathrm{pt}) = 0$, all compactifications on any manifold in the unique class $0 = [M_n^{\mathrm{QG}}]$ are bordant. Uniqueness of QG.

How to proceed?

[McNamara, Vafa '19]

QG-structure is not known. Try with educated guess $\widetilde{QG}$.

If $\Omega_n^{\widetilde{QG}} \neq 0$ we have a global symmetry.

- **Breaking**: $\exists$ defect with correct charge such that

$$\Omega_n^{\widetilde{QG}} \to \Omega_n^{\widetilde{QG}+\text{defects}} = 0 \qquad \text{killed}$$

- **Gauging**: The class $0 = [M] \in \Omega_n^{\widetilde{QG}} \neq 0$ gives a consistent EFT. Then, introduce gauge fields to kill the group

$$0 = \Omega_n^{\widetilde{QG}+\text{gauge fields}} \to \Omega_n^{\widetilde{QG}} \qquad \text{co-killed}$$

(In [NC, Andriot, Carqueville '22] we propose to use the Whitehead tower as organizing principle pointing towards QG structure.)

- Cobordism: language to classify compact manifolds (**closed string backgrounds**) without fixing topology

- K-theory: proper language for D-branes (**open strings**) [Witten '98]

Is there an *"open-closed"* correspondence between them?

Cobordism and K-theory

K-theory: intuitive definition

[Witten '98]

- Consider n $D9$-$\overline{D9}$ branes with U(n) bundles E and F

$$(E, F) = E - F$$

- Creation/annihilation of m pairs with *same* bundle H leaves (E,F) invariant

$$(E \oplus H, F \oplus H) \sim (E, F)$$

- The set of equivalence classes is the (reduced) K-theory group

$$K(X) = \{\text{vector bundles over X}\}/\sim$$

D-branes and K-theory

D-branes are classified by K-theory. [Witten '98]

Dp-branes on $\mathbb{R}^{10}$ with $p = 9 - n$ are classified by

- **Type I**: real K-theory $KO^{-n}(\mathrm{pt})$

n	0	1	2	3	4	5	6	7	8	9	10
$KO^{-n}(\mathrm{pt})$	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	0	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$
D-brane	$D9$	$\widehat{D8}$	$\widehat{D7}$	-	$D5$	-	-	-	$D1$	$\widehat{D0}$	$\widehat{D(-1)}$

- **Type II**: complex K-theory $K^{-n}(\mathrm{pt})$

n	0	1	2	3	4	5	6	7	8	9
$K^{-n}(\mathrm{pt})$	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0
D-brane	$D9$	-	$D7/D8$	-	$D5/D6$	-	$D3/D4$	-	$D1/D2$	-

Cobordism vs K-theory

n	Ω_n^{Spin}	KO^{-n}	$\Omega_n^{\text{Spin}^c}$	K^{-n}
0	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}$
1	$\mathbb{Z}_2$	$\mathbb{Z}_2$	0	0
2	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	$\mathbb{Z}$
3	0	0	0	0
4	$\mathbb{Z}$	$\mathbb{Z}$	$2\mathbb{Z}$	$\mathbb{Z}$
5	0	0	0	0
6	0	0	$2\mathbb{Z}$	$\mathbb{Z}$
7	0	0	0	0
8	$2\mathbb{Z}$	$\mathbb{Z}$	$4\mathbb{Z}$	$\mathbb{Z}$

Atiyah-Bott-Shapiro orientation

Relation between cobordism and K-theory dates back to **ABS-orientation** [Atiyah, Bott, Shapiro '64]

$$\begin{aligned} \alpha_n &: \quad \Omega_n^{\mathrm{Spin}}(\mathrm{pt}) \;\to\; \mathrm{KO}^{-\mathrm{n}}(\mathrm{pt}) \\ \alpha_n^c &: \quad \Omega_n^{\mathrm{Spin}^c}(\mathrm{pt}) \to \mathrm{K}^{-\mathrm{n}}(\mathrm{pt}) \end{aligned}$$

explicitly given by the refined A-roof and Todd genus

$$\alpha_n([M]) = \begin{cases} \hat{A}(M) & n = 8k \\ \frac{1}{2}\hat{A}(M) & n = 8k+4 \\ \dim H \mod 2 & n = 8k+1 \\ \dim H^+ \mod 2 & n = 8k+2 \\ 0 & \text{otherwise} \end{cases} \qquad \alpha_n^c([M]) = \mathrm{Td}(M)$$

Starting point to prove theorem by [Hopkins, Hovey '92], see also [Conner, Floyd '66; Landweber '76; Kreck, Stolz '93].

Note: α_n, α_n^c are bordism invariants.

Physical consequences

- Cobordism and K-theory charges are related.
  They must undergo the same fate in quantum gravity
  (see also [Uranga '00; Blumenhagen, Brinkmann, Makridou '19; Damian, Loaiza-Brito '19])

- The **combination** of cobordism and K-theory charges should be either gauged or broken. Schematically

$$\text{cobordism} + \text{K-theory} = 0$$

$$\text{closed strings} + \text{open strings} = 0$$

In the following, I will discuss **gauging**.
For breaking, **see A. Makridou's talk** tomorrow.

Tadpoles from bottom-up

[Blumenhagen, NC '21; Blumenhagen, NC, Kneißl, Makridou '22]

Gauging cobordism

- In [Blumenhagen, NC '21] it is shown that gauging cobordism + K-theory can lead to string theory **tadpoles**.
- Generalisation pt$\to X$ in [Blumenhagen, NC, Kneißl, Makridou '22]. Results interpreted as dimensional reduction of EFT on X.

Tadpole: integrated Bianchi identity

$$0 = \int_M dF_{n-1} = \int_M J_n$$

Total charge on a compact manifold should vanish

Goal: To reconstruct J_n without knowing string theory.

Constructing the current

α_n^c is natural candidate for the current

$$0 = \int_M dF_{n-1} = \alpha_n^c(M) + \dots$$

however there can be additional contributions.

1. Bordism charge might not be detected by α_n^c completely.
   $\Rightarrow$ **Add all bordism invariants** (α_n^c is just one)

   $$0 = \int_M dF_{n-1} = \sum_{i \in \text{inv}} a_i \, \mu_n^i$$

2. There might be defects: branes classified by $K^{-n}(\mathrm{pt})$.
   $\Rightarrow$ **Include defects**.

Thus we get a combination of cobordism and K-theory

$$0 = \int_{[M]} dF_{n-1} = \sum_{i \in \text{inv}} a_i \, \mu_n^i + \sum_{j \in \text{def}} \int_{[M]} Q_j \, \delta^n(\Delta_{10-n,j})$$

Example: gauging $\Omega_6^{\mathrm{Spin}^c}$

We have $\Omega_6^{\mathrm{Spin}^c} = \mathbb{Z} \oplus \mathbb{Z}$ with invariants

$$\mu_6^1 \equiv \alpha_6^c = \int td_6 = \int \frac{1}{24} c_1 c_2, \qquad \mu_6^2 = \int \frac{1}{2} c_1^3$$

- (Magnetic) 5-form global symmetry, gauged by C_4
- $K^{-6}(\mathrm{pt})$ classifies D3-branes

Combining we get

$$\int_B \sum_i Q_i \, \delta^{(6)}(\Delta_{4,i}) = \int_B \left(\tfrac{a_1}{24} \, c_2(B) \, c_1(B) + \tfrac{a_2}{2} \, c_1^3(B) \right) \equiv \frac{\chi(Y)}{24}$$

Matching with known D3-brane tadpole cancellation in F-theory for $a_1 = 12$ and $a_2 = 30$. [Sethi, Vafa, Witten '96]

Notice that c_3 cannot appear since it is **not bordism invariant**.

From groups of pt to groups of X

[Blumenhagen, NC, Kneißl, Makridou '22]

- The above discussion is just for groups of pt.
  It can be generalised pt $\to X$, with X a topological space.

- The groups are enlarged

$$\Omega(X) = \Omega(\mathrm{pt}) \oplus \tilde{\Omega}(X), \qquad K(X) = K(\mathrm{pt}) \oplus \tilde{K}(X),$$

  so potentially more global symmetries.

- What is their interpretation?

- $X = BG$ used for anomalies of G.
  Instead, we take X to be a manifold, such as spheres, tori, CY.

Some results

For $X = \{S^k, T^k, K3, CY_3\}$, we find ($k = dim(X)$)

$$K^{-n}(X) = \bigoplus_{m=0}^{k} b_{k-m}(X) K^{-n-m}(\mathrm{pt})$$

$$\Omega^{\mathrm{Spin}^c}_{n+k}(X) = \bigoplus_{m=0}^{k} b_m(X) \Omega^{\mathrm{Spin}^c}_{n+k-m}(\mathrm{pt})$$

- We show that they reproduce pattern of global symmetries stemming from dimensional reduction on X.
- They classify $(d-1-k-n)$-form charges in $D = d-k$ dimensions, arising from dimensional reduction of $d-1-n$, $d-2-n$, ..., $d-1-k-n$ form charges along the k, $k-1$, ..., 0 cycles X.

Example: $X = CY_3$

$$K^0(CY_3) = K_6(CY_3) = b_6 \underbrace{K^0(\mathrm{pt})}_{\mathbb{Z}} \oplus b_4 \underbrace{K^{-2}(\mathrm{pt})}_{\mathbb{Z}} \oplus b_2 \underbrace{K^{-4}(\mathrm{pt})}_{\mathbb{Z}} \oplus b_0 \underbrace{K^{-6}(\mathrm{pt})}_{\mathbb{Z}}$$

$$\Omega_6^{\mathrm{Spin}^c}(CY_3) = b_6 \underbrace{\Omega_0^{\mathrm{Spin}^c}(\mathrm{pt})}_{\mathbb{Z}} \oplus b_4 \underbrace{\Omega_2^{\mathrm{Spin}^c}(\mathrm{pt})}_{\mathbb{Z}} \oplus b_2 \underbrace{\Omega_4^{\mathrm{Spin}^c}(\mathrm{pt})}_{\mathbb{Z}\oplus\mathbb{Z}} \oplus b_0 \underbrace{\Omega_6^{\mathrm{Spin}^c}(\mathrm{pt})}_{\mathbb{Z}\oplus\mathbb{Z}}$$

- All terms give 3-form symmetries in 4D
- Combining groups of pt with same $(0, 2, 4, 6)$ index, we can construct tadpoles in 4D.
- In fact, they are the dimensional reduction of tadpoles for the 10D (9,7,5,3)-form symmetries.

Interpretation for $-k \leq n < 0$ less clear (no K-theory groups)

In [Blumenhagen, NC, Kneißl, Makridou '22] we propose that

- $\Omega_{\text{EVEN}}(X)$, **gauged**: contribute to tadpoles of $n \geq 0$ groups.
  New contributions?
- $\Omega_{\text{ODD}}(X)$, **broken**

$\Omega_6(X)$	$b_6\Omega_0(\mathrm{pt})$	$b_4\Omega_2(\mathrm{pt})$	$b_2\Omega_4(\mathrm{pt})$	$b_0\Omega_6(\mathrm{pt})$
	C_{10}	C_8	C_6	C_4
	$O9$	$\mathrm{F}(CY_4)_{c_1(M_6)}$	$\mathrm{tr}(R\wedge R)_{\mathrm{D9,O9}}$	$\mathrm{F}(CY_4)_{c_1c_2,c_1^3(M_6)}$
$\Omega_4(X)$	$b_4\Omega_0(\mathrm{pt})$	$b_2\Omega_2(\mathrm{pt})$	$b_0\Omega_4(\mathrm{pt})$	–
	C_8	C_6	C_4	–
	$O7$	$N7_{c_1(M_4)}$	$\mathrm{tr}(R\wedge R)_{\mathrm{D7,O7}}$	–
$\Omega_2(X)$	$b_2\Omega_0(\mathrm{pt})$	$b_0\Omega_2(\mathrm{pt})$	–	–
	C_6	C_4	–	–
	$O5$	$N5_{c_1(M_2)}$	–	–
$\Omega_0(X)$	$b_0\Omega_0(\mathrm{pt})$	–	–	–
	C_4	–	–	–
	$O3$	–	–	–

Conclusion

- The absence of global symmetries seems to be a fact of QG. It holds true also when enlarging notion of symmetry, such as to include cobordism
- Cobordism and K-theory are closed-open string versions of global symmetries
- Their combination must be either broken or gauged
- This statement has predictive power. [Montero, Vafa '20; Dierigl, Heckmann '20; Hamada, Vafa, '21; Blumenhagen, NC, Kneißl, Makridou '22]

Outlook

- Cobordism groups with more structure (gauge fields, compact manifolds, . . .)
  [Blumenhagen, NC, Kneißl, Makridou, '22]

- Clarify origin of tadpoles from bottom-up

- Is cobordism conjecture combined with K-theory enough to reconstruct tadpoles in string theory (String Lamppost Principle)?

- Are there new objects in string theory detected by cobordism? This can happen when breaking but also when gauging.

Thank you!

Extra slides

Example: gauging Ω_1^{Spin}

Torsion charges require care. Consider $\Omega_1^{\mathrm{Spin}} = \mathbb{Z}_2 = KO^{-1}(\mathrm{pt})$ with invariant

$$\mu_1 \equiv \alpha_1$$

and $KO^{-1}(\mathrm{pt})$ classifies $\widehat{D8}$-branes.

We get $\mathbb{Z}_2$-valued charge neutrality condition

$$\int_M \sum_i Q_i \delta^{(1)}(\Delta_{9,i}) = \mathbf{a}\, \alpha_1 \mod 2$$

- **a=even**: RHS decouples. Even number of $\widehat{D8}$-branes needed and $KO^{-1}(\mathrm{pt})$ is gauged. **New defect** needed to break Ω_1^{Spin}.
- **a=odd**: single $\widehat{D8}$-brane on S_p^1 (having $\alpha_1(S_p^1) = 1$) allowed since vanishing total charge, $1 + 1 = 0$ mod 2.
  Unlikely: S_p^1 valid background without $\widehat{D8}$.

Computing groups of X

The groups $\Omega(X)$, $K(X)$ can be computed using the Atiyah-Hirzebruch spectral sequence.
It is a tool to calculate generalised (co)homology theories.

- Start from ordinary (co)homology
- Refine the approximation by means of **differentials**
- Eventually, solve an **extension problem** (extra information needed)

Certain differentials are physically associated to Freed-Witten anomalies. [Diaconescu, Moore, Witten '00; Maldacena, Moore, Seiberg '01]

Interpretation: K-theory

$$K^{-n}(X) = \bigoplus_{m=0}^{k} b_{k-m}(X) K^{-n-m}(\mathrm{pt})$$

- They classify codimension $(n+m)$-branes wrapping $(k-m)$-cycles of X. Consistent with expectation from dimensional reduction.
- By construction, these branes do not suffer from FW anomalies, otherwise they would not survive the spectral sequence.
- All sites populated. Completeness hypothesis.
- Simlar result for KO-theory, for $X = \{S^k, T^k, K3\}$

Interpretation: Cobordism

$$\Omega_{n+k}^{\mathrm{Spin^c}}(X) = \bigoplus_{m=0}^{k} b_m(X)\Omega_{n+k-m}^{\mathrm{Spin^c}}(\mathrm{pt})$$

Each non-vanishing term in the RHS means that the $(n+k)$-manifold M is wrapped around non-trivial m-cycle of X.

Two qualitatively different cases:

- $n \geq 0$: There is associated K-theory group $K_{n+k}(X) = K^{-n}(X)$ with string interpretation.

  Cobordism reproduces expectation from dim. reduction.

- $-k \leq n < 0$: No K-theory analogous in physics.

  Cobordism interpretation more speculative

Breaking cobordism

- Breaking a cobordism symmetry requires the presence of defects to cancel the charge
- This statement has predictive power: already in [McNamara, Vafa '19] new defects are predicted in string theory
- More developments in [Montero, Vafa '20; Dierigl, Heckmann '20; Hamada, Vafa, '21; Debray, Dierigl, Heckmann, Montero '21]

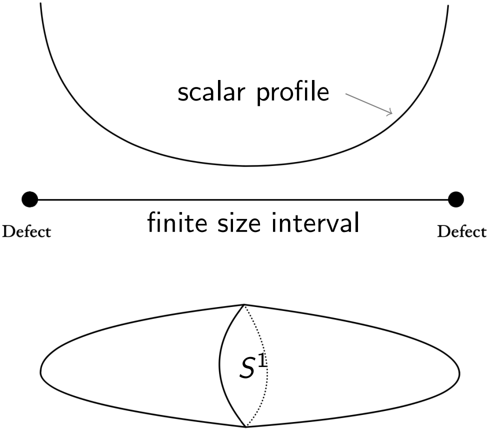
Dynamical cobordism

[Angius, Buratti, Calderon-Infante, Delgado, Huertas, Uranga '21; '21; '22]

- Breaking cobordism can be intertwined with dynamics of scalar fields
- Setups with net amount of energy in the vacuum (NS tadpole), giving rise to a scalar potential driving scalars at **infinite distance in field space**
- If this happens at **finite spacetime distance**, spacetime effectively ends!
- At the **end of the world** we find the defects (branes) predicted by the cobordism conjectures and such that

$$\Delta \sim e^{-\frac{\delta}{2}D}, \qquad |R| \sim e^{\delta D}$$

A picture



Some examples

- 10d massive IIA. ETW defects are O8/D8
- 10d USp(32) theory (namely IIB with $\overline{D9}/O9$).
  [Sugimoto '99; Dudas, Mourad '00]
  ETW defects are 8-branes (see also [Antonelli, Basile '19])
- T-dual of [Sugimoto '99] namely IIA with $\overline{D8}/O8$.
  [Blumenhagen, Font '00] ETW defects are 7-branes.
- More in [Angius, Calderón-Infante, Delgado, Huertas, Uranga '22]

We took [Blumenhagen, Font '00] and looked at solutions of the EOMs with **singularities** at the end of the world.

We interprete these singularities as the **7-branes** expected from the cobordism conjecture.

Backreacted T-dual Sugimoto model

[Blumenhagen, Font '00]

Setup: RR-tadpole-free stack of $\overline{D8}/08$ in type IIA.
Study the backreaction of the stack on spacetime.

$$ds^2 = e^{2\mathcal{A}(r,y)} ds_8^2 + e^{2\mathcal{B}(r,y)}(dr^2 + dy^2)$$

Three solutions were found. All have circle $r \in [-R/2, R/2]$, with $e^{\phi_0} \sim 1/R$, but singularities at $y = \pm\infty$

- Solution I : L_y is infinite - ETW at infinite distance
- Solution II$^-$: L_y is infinite - ETW at infinite distance
- Solution II$^+$: L_y is finite - ETW at finite distance!

According to Dynamical Cobordism, we can now interpret the singularities of Solution II$^+$ as ETW 7-branes (needed to break $\Omega_1^{Spin} = \mathbb{Z}_2$). How do they look like?

Non-isotropic 7-brane solution

[Blumenhagen, NC, Kneißl, Makridou '22]

Ansatz preserving 8D Poincaré invariance but breaking 2D rotational symmetry

$$ds^2 = e^{2\hat{\mathcal{A}}(\rho,\varphi)} ds_8^2 + e^{2\hat{\mathcal{B}}(\rho,\varphi)} (d\rho^2 + \rho^2 d\varphi^2)$$

We found solutions of gravity+dilaton EOMs consistent with the presence of a 2D delta source.
One of these seems to have right properties.

- Same kind of singularity as the backreacted $\overline{D8}/O8$ stack
- Scalings proposed in [Angius, Calderón-Infante, Delgado, Huertas, Uranga '22] satisfied with same δ as the $\overline{D8}/O8$ stack!

Moreover: Coupling to the dilaton $\sim e^{-2\phi}$ in string frame.
New object in string theory?