NONLOCAL 4D Einstein-Gauss-Bonnet Gravity

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work in coll. with M M Maglio, M. Creti', S. Lionetti, R. Tommasi

OUTLINE

Topological terms are present in chiral and conformal anomaly actions. They are not related to renormalization but are associated with finite Ward identities.

Their role has been recognized in lower dimensional physics, including condensed matter theory (Topological matter and Quantum spin Hall effect)

FOR EXAMPLE

In d=2 The Einstein Hilbert term is topological, and a form of dilaton gravity
appears using the same procedure that we are going to discuss in d=4.
This procedure is tightly connected with DimReg

The analysis can be extended to any even dimensions.

I am going to emphasize the relation between 4D EGB theories and the conformal anomaly action.
Such 4DEGB theories are local. We are going to derive a nonlocal version
of such theories, in close analogy with the structure of the conformal anomaly action

Conformal field theory in momentum space and anomaly actions in gravity: The analysis of three- and four-point function

#13

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The conformal anomaly action to fourth order (4T) in $d = 4$ in momentum space

#6

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Einstein Gauss-Bonnet theories as ordinary, Wess-Zumino conformal anomaly actions

Claudio Corianò (Salento U. and INFN, Lecce), Matteo Maria Maglio (GGI, Florence and U. Heidelberg, ITP) (Jan 19, 2022)

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Topological Corrections and Conformal Backreaction in the Einstein Gauss-Bonnet/Weyl Theories of Gravity at D=4

Claudio Corianò (INFN, Lecce and Salento U.), Matteo Maria Maglio (GGI, Florence and U. Heidelberg, ITP), Dimosthenis Theofilopoulos (INFN, Lecce and Salento U.) (Mar 8, 2022)

e-Print: 2203.04213 [hep-th]

**The search for corrections to general relativity (GR) and to its Einstein-Hilbert (EH) action by higher derivative terms,
is characterized by a large number of both older and of more recent proposals.**

Their goal is to address unsolved issues, such as the nature of dark energy and the mechanism of inflation of the early universe, in a more satisfactory way.

QUANTUM EFFECTS back-react on the classical metric.

We integrate out matter generating an effective action that can be studied in detail using standard quantum field theory methods, being
the gravitational action classical

important connection QFT-CMT

Topological insulators and topological semimetals are both new classes of quantum materials, which are characterized by surface states induced by the topology of the bulk band structure. Topological Dirac or Weyl semimetals show linear dispersion around nodes, termed the Dirac or Weyl points, as the three-dimensional analog of graphene.

REVIEWS

Nature, 2021

Check for updates

Experimental signatures of the chiral anomaly in Dirac–Weyl semimetals

N. P. Ong ✉ and Sihang Liang

A **Weyl semimetal** is a solid state crystal whose low energy excitations are Weyl fermions that carry electrical charge even at room temperatures.

Chiral Anomaly in interacting Condensed Matter Systems

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(Dated: February 9, 2021)

The chiral anomaly is a fundamental quantum mechanical phenomenon which is of great importance to both particle physics and condensed matter physics alike. In the context of QED it manifests as the breaking of chiral symmetry in the presence of electromagnetic fields. It is also known that anomalous chiral symmetry breaking can occur through interactions alone, as is the case for interacting one dimensional systems. In this paper we investigate the interplay between these two modes of anomalous chiral symmetry breaking in the context of interacting Weyl semimetals.

typical axion Lagrangian ofQED

Axion E&M

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \frac{1}{2} \partial_\mu \varphi \partial^\mu \varphi + \frac{1}{4} \widetilde{g}\, \varphi\, F_{\mu\nu} \widetilde{F}^{\mu\nu},$$

$$\Box\varphi - \frac{\widetilde{g}}{4} F^{\mu\nu} \widetilde{F}_{\mu\nu} = 0.$$

Modified Maxwell's equations

$$\Box\varphi = -\widetilde{g}\, \boldsymbol{E} \cdot \boldsymbol{B}.$$

$$\boldsymbol{B} = 0,$$

$$\begin{cases} \frac{\partial \boldsymbol{B}}{\partial t} + \nabla \times \boldsymbol{E} = 0, \\ \Box\varphi = -\widetilde{g}\, \boldsymbol{E} \cdot \boldsymbol{B}, \\ \nabla \cdot \boldsymbol{E} = \widetilde{g}\, \nabla\varphi \cdot \boldsymbol{B}, \\ \nabla \times \boldsymbol{B} - \frac{\partial \boldsymbol{E}}{\partial t} = -\widetilde{g} \boldsymbol{B} \frac{\partial \varphi}{\partial t} + \widetilde{g} \boldsymbol{E} \times \nabla\varphi. \end{cases}$$

$$\Box(\boldsymbol{E} - \frac{1}{2}\widetilde{g}\varphi\boldsymbol{B}) = -\frac{1}{2}\widetilde{g}\varphi\,\Box\boldsymbol{B},$$
$$\Box(\boldsymbol{B} + \frac{1}{2}\widetilde{g}\varphi\boldsymbol{E}) = \frac{1}{2}\widetilde{g}\varphi\,\Box\boldsymbol{E}.$$

$$\boldsymbol{D} \equiv \boldsymbol{E} - \frac{1}{2}\widetilde{g}\varphi\boldsymbol{B}$$

$$\boldsymbol{H} \equiv \boldsymbol{B} + \frac{1}{2}\widetilde{g}\varphi\boldsymbol{E}.$$

$$\theta = \frac{1}{2}\widetilde{g}\Delta\varphi,$$

Rotation of the polarization plane

in the eikonal limit (neglect second derivatives on the axion field

$$\Delta\boldsymbol{E} \equiv \boldsymbol{E}(L) - \boldsymbol{E}(0) = \frac{1}{2}\widetilde{g}\Delta\varphi\boldsymbol{H}(0).$$

$$\Delta\varphi \equiv \varphi(L) - \varphi(0).$$

This is not the only parameterization. A second one is the **longitudinal/transverse (LT) decomposition**

$$W^{\lambda\mu\nu} = \frac{1}{8\pi^2}\left[W^{L\,\lambda\mu\nu} - W^{T\,\lambda\mu\nu}\right],$$

$$W^{L\,\lambda\mu\nu} = w_L\, k^\lambda \varepsilon[\mu,\nu,k_1,k_2]$$

De Rafael et al

developed in the study of g-2 of the muon

It corrects an erro r in the book by Kerson Huang on particle theory

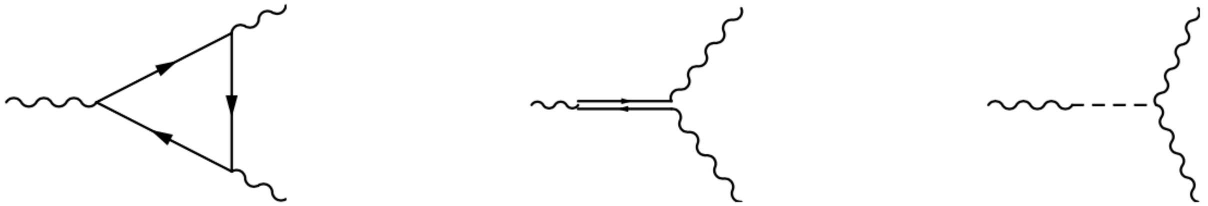
Only the L part contributes to the Ward Identity

$$\begin{aligned} W^{T}{}_{\lambda\mu\nu}(k_1,k_2) \;&=\; w_T^{(+)}\left(k^2,k_1^2,k_2^2\right)\, t^{(+)}_{\lambda\mu\nu}(k_1,k_2) + w_T^{(-)}\left(k^2,k_1^2,k_2^2\right)\, t^{(-)}_{\lambda\mu\nu}(k_1,k_2) \\ &\quad + \widetilde{w}_T^{(-)}\left(k^2,k_1^2,k_2^2\right)\, \widetilde{t}^{(-)}_{\lambda\mu\nu}(k_1,k_2), \end{aligned}$$

$$\begin{aligned} t^{(+)}_{\lambda\mu\nu}(k_1,k_2) \;&=\; k_{1\nu}\,\varepsilon[\mu,\lambda,k_1,k_2] - k_{2\mu}\,\varepsilon[\nu,\lambda,k_1,k_2] - (k_1\cdot k_2)\,\varepsilon[\mu,\nu,\lambda,(k_1-k_2)] \\ &\quad + \frac{k_1^2+k_2^2-k^2}{k^2}\, k_\lambda\, \varepsilon[\mu,\nu,k_1,k_2]\;, \\ t^{(-)}_{\lambda\mu\nu}(k_1,k_2) \;&=\; \left[(k_1-k_2)_\lambda - \frac{k_1^2-k_2^2}{k^2}\, k_\lambda\right]\varepsilon[\mu,\nu,k_1,k_2] \\ \widetilde{t}^{(-)}_{\lambda\mu\nu}(k_1,k_2) \;&=\; k_{1\nu}\,\varepsilon[\mu,\lambda,k_1,k_2] + k_{2\mu}\,\varepsilon[\nu,\lambda,k_1,k_2] - (k_1\cdot k_2)\,\varepsilon[\mu,\nu,\lambda,k]. \end{aligned}$$

Tensor structures involved
In the LT parameterization

$$
\begin{aligned}
w_L(s_1, s_2, s) &= -\frac{4i}{s} \\
w_T^{(+)}(s_1, s_2, s) &= i\frac{s}{\sigma} + \frac{i}{2\sigma^2}\Big[(s_{12}+s_2)(3s_1^2 + s_1(6s_{12}+s_2) + 2s_{12}^2)\log\frac{s_1}{s} \\
&+ (s_{12}+s_1)(3s_2^2 + s_2(6s_{12}+s_1) + 2s_{12}^2)\log\frac{s_2}{s} \\
&+ s(2s_{12}(s_1+s_2) + s_1 s_2(s_1+s_2+6s_{12}))\Phi(s_1, s_2)\Big]
\end{aligned}
$$



(a) (b) (c)

The triangle diagram in the fermion case (a), the collinear fermion configuration responsible for the anomaly (b) and a diagrammatic representation of the exchange via an intermediate state (dashed line) (c).

The signature of the chiral anomaly is in the the generation of 1 pole in the axial vector channel

The nonlocal action related to the topological contribution

$$\mathcal{S}_{eff} = -\frac{e^2}{16\pi^2}\int d^4x \int d^4y\, [\epsilon^{\mu\nu\rho\sigma} F_{\mu\nu} F_{\rho\sigma}]_x \Box^{-1}_{xy}\, [\partial^\lambda B_\lambda]_y,$$

in local form

$$\mathcal{S}_{eff}[\eta,\chi;A,B] = \int d^4x\, \Big\{(\partial^\mu \eta)\,(\partial_\mu \chi) - \chi\, \partial^\mu B_\mu + \frac{e^2}{8\pi^2}\, \eta\, F_{\mu\nu} \tilde{F}^{\mu\nu}\Big\}$$

Giannotti and Mottola

these two degrees of freedom are entangled. Anomaly produces entanglement

$$\Box\,\eta = -\partial^\lambda B_\lambda\,,$$

$$\Box\,\chi = \frac{e^2}{8\pi^2}\, F_{\mu\nu}\tilde{F}^{\mu\nu} = \frac{e^2}{16\pi^2}\, \epsilon^{\mu\nu\rho\sigma} F_{\mu\nu} F_{\rho\sigma}\,.$$

no optical activity in the eikonal limit

A Nonlocal Effective Action of Axial Electrodynamics for Topological Matter

Claudio Corianò,[1] Mario Cretì,[1] Stefania D'Agostino,[2] Ewelina Hankiewicz,[3] and Matteo Maria Maglio[4]

to appear

Turning to gravity and the conformal anomaly

$$R^{\mu\nu} - \frac{1}{2} g^{\mu\nu} R = \langle T^{\mu\nu} \rangle$$

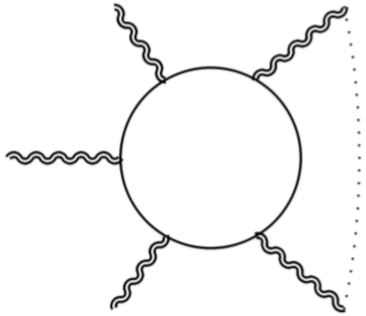
$$\langle T^{\mu\nu}(x_1))\rangle_g = \langle T^{\mu\nu}(x_1))\rangle_\eta + \int \langle T^{\mu\nu}(x_1)) T^{\rho\sigma}(x_2)) \delta g_{\rho\sigma}(x_2)\rangle + \frac{1}{2} \int dx_1 dx_2 \langle T^{\mu\nu}(x_1) T^{\mu_1\nu_1}(x_2) T^{\mu_2\nu_2}(x_2) \delta g_{\rho\sigma}(x_2)\rangle \delta g_{\rho\sigma}(x_3)$$

usually the computations are performed as fluctuatons around flat space

but they can be extended to Weyl flat spaces where

$$g_{\mu\nu} = \bar{g}_{\mu\nu} e^{2\phi}$$

$$\mathcal{S}(g) = \sum_n$$



(n-point)

corrections to gravity induced by quantum corrections, assuming that the matter that is integrated out is conformal

The conformal anomaly is related to a nonzero trace of the stress energy tensor

$$\langle T^{\mu}_{\mu} \rangle = \mathscr{A}(z),$$

$$\mathscr{A}(z) = -\frac{1}{8}\left[2bC^2 + 2b'\left(E - \frac{2}{3}\Box R\right) + 2cF^2\right],$$

$$\begin{aligned} C^2 &= C_{\lambda\mu\nu\rho}C^{\lambda\mu\nu\rho} = R_{\lambda\mu\nu\rho}R^{\lambda\mu\nu\rho} - 2R_{\mu\nu}R^{\mu\nu} + \frac{R^2}{3} \\ E &= {}^*\!R_{\lambda\mu\nu\rho}\,{}^*\!R^{\lambda\mu\nu\rho} = R_{\lambda\mu\nu\rho}R^{\lambda\mu\nu\rho} - 4R_{\mu\nu}R^{\mu\nu} + R^2. \end{aligned}$$

In a flat metric background the expression of such functional reduces to the simple form

$$\mathscr{A}(z) = \sum_i \frac{\beta_i}{2g_i} F_i^{\alpha\beta}(z) F^i_{\alpha\beta}(z),$$

The simplest example is the TJJ vertex, where the nonlocality takes the form of a conformal anomaly pole

We consider the standard QED lagrangian

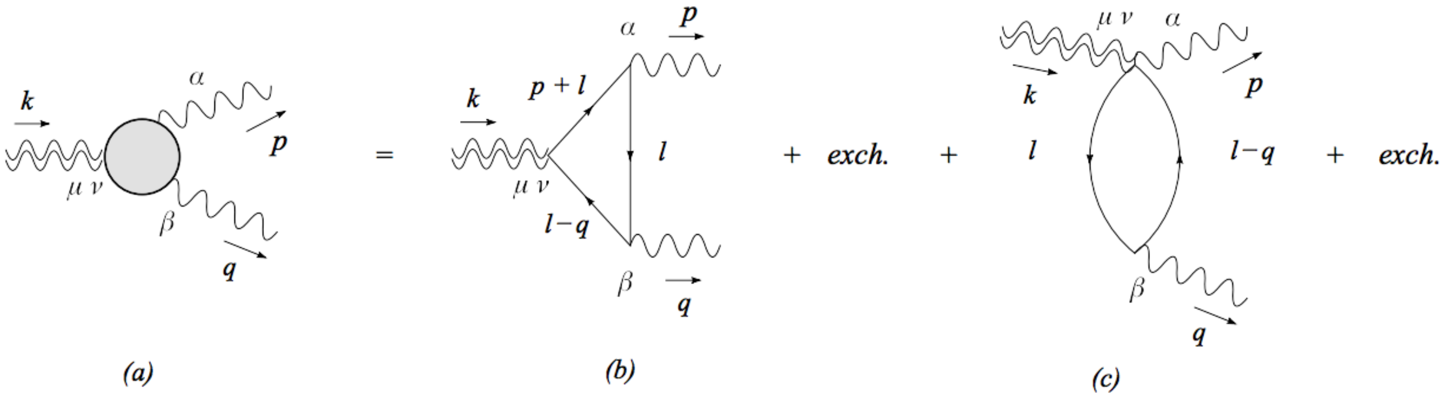
$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + i\,\bar{\psi}\gamma^{\mu}(\partial_{\mu} - i\,e\,A_{\mu})\psi - m\bar{\psi}\psi,$$

$$T^{\mu\nu} \equiv T^{\mu\nu}_{f} + T^{\mu\nu}_{fp} + T^{\mu\nu}_{ph}$$

$$T^{\mu\nu}_{f} = -i\bar{\psi}\gamma^{(\mu}\overleftrightarrow{\partial}^{\nu)}\psi + g^{\mu\nu}(i\bar{\psi}\gamma^{\lambda}\overleftrightarrow{\partial}_{\lambda}\psi - m\bar{\psi}\psi),$$

$$T^{\mu\nu}_{fp} = -\,eJ^{(\mu}A^{\nu)} + eg^{\mu\nu}J^{\lambda}A_{\lambda}\,,$$

$$T^{\mu\nu}_{ph} = F^{\mu\lambda}F^{\nu}{}_{\lambda} - \frac{1}{4}g^{\mu\nu}F^{\lambda\rho}F_{\lambda\rho},$$



$$\langle T^{\mu\nu}_{p}(z)\rangle_{A} \equiv \int D\psi D\bar{\psi}\; T^{\mu\nu}_{p}(z)\; e^{i\int d^4x\mathcal{L}+\int J\cdot A(x)d^4x}$$
$$= \langle T^{\mu\nu}_{p}\, e^{i\int d^4x\, J\cdot A(x)}\rangle$$

M. Maglio, Delle Rose, C.C.

$$
\begin{aligned}
R^\mu &= \bar\lambda^a \bar\sigma^\mu \lambda^a + \frac{1}{3}\left(-\bar\chi_i \bar\sigma^\mu \chi_i + 2i\phi_i^\dagger \mathcal{D}^\mu_{ij}\phi_j - 2i(\mathcal{D}^\mu_{ij}\phi_j)^\dagger \phi_i\right), \\
S^\mu_A &= i(\sigma^{\nu\rho}\sigma^\mu \bar\lambda^a)_A F^a_{\nu\rho} - \sqrt{2}(\sigma_\nu \bar\sigma^\mu \chi_i)_A (\mathcal{D}^\nu_{ij}\phi_j)^\dagger - i\sqrt{2}(\sigma^\mu \bar\chi_i)\mathcal{W}_i^\dagger(\phi^\dagger) \\
&- ig(\phi_i^\dagger T^a_{ij}\phi_j)(\sigma^\mu \bar\lambda^a)_A + S^\mu_{I\,A}, \\
T^{\mu\nu} &= -F^{a\,\mu\rho}F^{a\,\nu}{}_\rho + \frac{i}{4}\left[\bar\lambda^a \bar\sigma^\mu(\delta^{ac}\overrightarrow{\partial^\nu} - g\,t^{abc}A^{b\,\nu})\lambda^c + \bar\lambda^a\bar\sigma^\mu(-\delta^{ac}\overleftarrow{\partial^\nu} - g\,t^{abc}A^{b\,\nu})\lambda^c + (\mu \leftrightarrow \nu)\right] \\
&+ (\mathcal{D}^\mu_{ij}\phi_j)^\dagger(\mathcal{D}^\nu_{ik}\phi_k) + (\mathcal{D}^\nu_{ij}\phi_j)^\dagger(\mathcal{D}^\mu_{ik}\phi_k) + \frac{i}{4}\Big[\bar\chi_i\bar\sigma^\mu(\delta_{ij}\overrightarrow{\partial^\nu} + igT^a_{ij}A^{a\,\nu})\chi_j \\
&+ \bar\chi_i\bar\sigma^\mu(-\delta_{ij}\overleftarrow{\partial^\nu} + igT^a_{ij}A^{a\,\nu})\chi_j + (\mu\leftrightarrow\nu)\Big] - \eta^{\mu\nu}\mathcal{L} + T^{\mu\nu}_I,
\end{aligned}
$$

in susy nonlocalities are unified

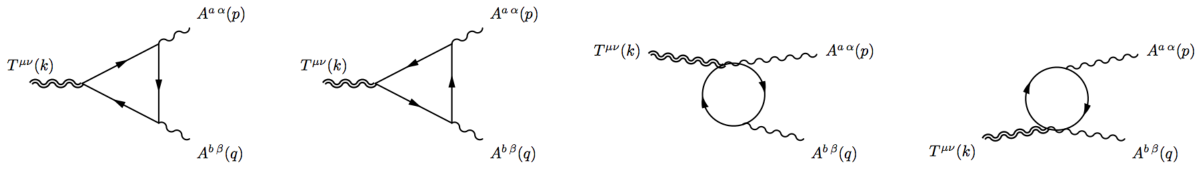
$$
\begin{aligned}
\partial_\mu R^\mu &= \frac{g^2}{16\pi^2}\left(T(A) - \frac{1}{3}T(R)\right)F^{a\,\mu\nu}\tilde F^a_{\mu\nu}, \\
\bar\sigma_\mu S^\mu_A &= -i\frac{3\,g^2}{8\pi^2}\left(T(A) - \frac{1}{3}T(R)\right)\left(\bar\lambda^a\bar\sigma^{\mu\nu}\right)_A F^a_{\mu\nu}, \\
\eta_{\mu\nu}T^{\mu\nu} &= -\frac{3\,g^2}{32\pi^2}\left(T(A) - \frac{1}{3}T(R)\right)F^{a\,\mu\nu}F^a_{\mu\nu}.
\end{aligned}
$$

"Unification of anomalies in N=1
Ferrara Zumino supermultiplet

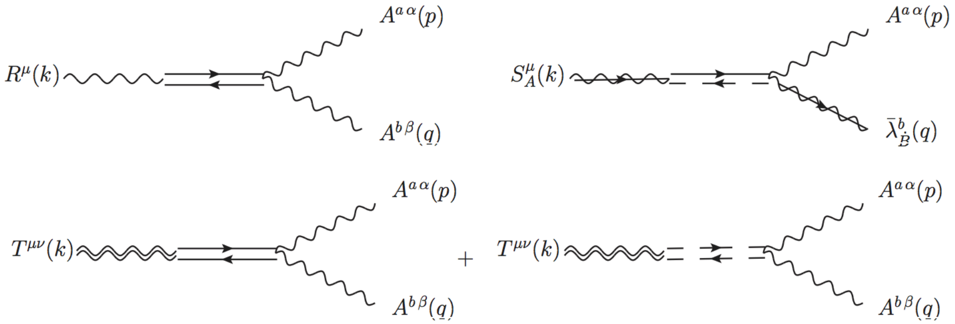
The nonlocal structure of the effective action is established by a direct computation

$$\Gamma^{\mu\alpha\beta}_{(R)}(p,q) = i\frac{g^2\,T(R)}{12\pi^2}\,\Phi_1(k^2,m^2)\,\frac{k^\mu}{k^2}\varepsilon[p,q,\alpha,\beta]\,,$$

$$\Gamma^{\mu\alpha}_{(S)}(p,q) = i\frac{g^2T(R)}{6\pi^2\,k^2}\,\Phi_1(k^2,m^2)\,s_1^{\mu\alpha}$$



(a) (b) (c) (d)



Delle Rose, CC

$$S_{\text{axion}} = -\frac{g^2}{4\pi^2}\left(T(A)-\frac{T(R)}{3}\right)\int d^4z\,d^4x\,\partial^\mu B_\mu(z)\,\frac{1}{\Box_{zx}}\,\frac{1}{4}F_{\alpha\beta}(x)\tilde F^{\alpha\beta}(x)$$

$$S_{\text{dilatino}} = \frac{g^2}{2\pi^2}\left(T(A)-\frac{T(R)}{3}\right)\int d^4z\,d^4x\left[\partial_\nu\Psi_\mu(z)\sigma^{\mu\nu}\sigma^\rho\frac{\overleftarrow{\partial}_\rho}{\Box_{zx}}\,\bar\sigma^{\alpha\beta}\bar\lambda(x)\frac{1}{2}F_{\alpha\beta}(x)+h.c.\right]$$

$$S_{\text{dilaton}} = -\frac{g^2}{8\pi^2}\left(T(A)-\frac{T(R)}{3}\right)\int d^4z\,d^4x\,\left(\Box h(z)-\partial^\mu\partial^\nu h_{\mu\nu}(z)\right)\,\frac{1}{\Box_{zx}}\,\frac{1}{4}F_{\alpha\beta}(x)F^{\alpha\beta}(x).$$

Is the effective action for gravity, obtained by
integrating out matter
local or nonlocal?

We integrate out a conformal sector, estrapolating from Sakharov's idea of induced gravity and see the effect on gravity

4D EGB theory proceeds quite closely as in the derivation of the anomaly actions,
but in a purely classical context. In other words we borrow a Dimensionaal Regularization procedure, trying to evade Lovelock't theorem

We want to derive second order equations of motion by a singular limit on the GB term

The integratin of quantum matter, in general, induce an effective action which includes several quadratic corrections

Sakharov's induced gravity idea

Assume you are given a Lorentzian manifold

Make no assumptions about the dynamics of this geometry;
leave it free to flap in the breeze.

The geometry is considered as a classical background.

Consider one-loop quantum field theory on this manifold.

$$\int \mathrm{d}^4x\sqrt{-g}\,\{c_0 + c_1\, R(g) + c_2\,(\text{“}R^2\text{”})\}\,.$$

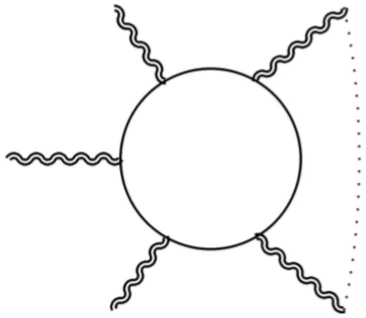
Compare this with the standard Lagrangian for Einstein gravity

$$\int \mathrm{d}^4x\sqrt{-g}\left\{-\Lambda - \frac{R(g)}{16\pi\, G} + K\,(\text{“}R^2\text{”}) + \mathcal{L}_{\text{matter}}\right\}.$$

the one loop effective action automatically contains terms proportional to the cosmological constant,
the Einstein–Hilbert action, plus “curvature-squared” terms.

Start by considering the one-loop contribution
to the effective action for a scalar field

$$\mathcal{S}(g) = \sum_n$$



(n-point)

$$S_0(g,\chi) \;=\; \frac{1}{2}\int d^dx\,\sqrt{-g}\,\left[g^{\mu\nu}\nabla_\mu\chi\nabla_\nu\chi - c_0\,R\,\chi^2\right],$$

$$\mathcal{S}_g = -\frac{1}{2}\ln\det(\Delta_g + m^2 + \xi R) = -\frac{1}{2}\mathrm{Tr}\ln(\Delta_g + m^2 + \xi R).$$

$$\ln(b/a) = \int_0^\infty \frac{\mathrm{d}x}{x}\left[e^{-ax} - e^{-bx}\right].$$

$$\mathcal{S}_g = \mathcal{S}_{g_0} + \frac{1}{2}\mathrm{Tr}\int_{\kappa^{-2}}^\infty \frac{\mathrm{d}s}{s}\left[\exp(-s[\Delta_g + m^2 + \xi R]) - \exp(-s[\Delta_{g_0} + m^2 + \xi R_0])\right].$$

introduced a cutoff to regulate the result use the heat kernel expansion as s → 0

$$\exp(-s[\Delta_g + m^2 + \xi R]) = \frac{\sqrt{-g}}{(4\pi\, s)^2}\left[a_0(g) + a_1(g)s + a_2(g)s^2 + \cdots\right].$$

Now define Tr to include a trace over both spacetime and any internal indices

The coefficients are denoted as the Seeley–DeWitt

$$\begin{aligned} \mathcal{S}_g &= \mathcal{S}_{g_0} + \frac{1}{32\pi^2}\mathrm{Tr}\Bigg\{[a_0(g) - a_0(g_0)]\,\frac{\kappa^4}{2} + [a_1(g) - a_1(g_0)]\,\kappa^2 \\ &\quad + [a_2(g) - a_2(g_0)]\,\ln(\kappa^2/m^2)\Bigg\} + \mathrm{UV\ finite.} \end{aligned}$$

$$a_0(g) = 1.$$

$$a_1(g) = k_1 \; R(g) - m^2.$$

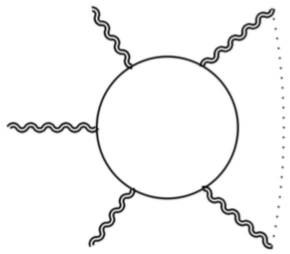
$$\begin{aligned} a_2(g) &= k_2 \; C_{abcd}C^{abcd} + k_3 \; R_{ab}R^{ab} + k_4 \; R^2 + k_5 \; \nabla^2 R \\ & \quad -m^2 \; k_1 \; R(g) + \frac{1}{2}m^4. \end{aligned}$$

We can extract the coefficients in the one-loop effective action.
Define the gravitational couplings as follows

$$\int \mathrm{d}^4x\sqrt{-g}\left\{-\Lambda - \frac{R(g)}{16\pi \, G} + K_2 \; C_{abcd} \; C^{abcd} + K_4 \; R^2 + \mathcal{L}_{\mathrm{matter}}\right\}.$$

we can investigate the structure of the effective action in the conformal case. S0 is conformally invariant.
We integrate out conformal matter

$$\mathcal{S}(g) = \sum_n$$



$$\mathcal{Z}_B(g) = \mathcal{N} \int D\chi e^{-S_0(g,\chi)},$$

$$e^{-\mathcal{S}_B(g)} = \mathcal{Z}_B(g) \leftrightarrow \mathcal{S}_B(g) = -\log \mathcal{Z}_B(g).$$

$$S_0(g,\chi) \quad = \quad \frac{1}{2}\int d^dx\,\sqrt{-g}\left[g^{\mu\nu}\nabla_\mu\chi\nabla_\nu\chi - c_0\,R\,\chi^2\right],$$

The computation of this action can be performed exactly in momentum space. The theory is conformal at one loop, modulo the appearance of a trace anomaly after renormalization

$$\mathcal{S}(g)_B \equiv \mathcal{S}(\bar{g})_B + \sum_{n=1}^{\infty} \frac{1}{2^n n!} \int d^dx_1 \ldots d^dx_n \sqrt{g_1} \ldots \sqrt{g_n}\, \langle T^{\mu_1\nu_1} \ldots\, T^{\mu_n\nu_n} \rangle_{\bar{g}B} \delta g_{\mu_1\nu_1}(x_1) \ldots \delta g_{\mu_n\nu_n}(x_n),$$

if we use dimensional regularization then Weyl invariance is broken

$$g_{\mu\nu} \to e^{2\sigma(x)} g_{\mu\nu}, \qquad \delta_\sigma g_{\mu\nu} = 2\sigma g_{\mu\nu}$$

$$\delta_\sigma \mathcal{S} = \frac{1}{(4\pi)^2} \int d^4x \sqrt{g} \left(c_1 R_{\mu\nu\rho\sigma} R^{\mu\nu\rho\sigma} + c_2 R_{\mu\nu} R^{\mu\nu} + c_3 R^2 + c_4 \Box R \right),$$

The nonzero variation, induced by the renormalization of the loops, to all orders in the external gravitons, has to satisfy a WEss Zumino consistency condiiton

$$[\delta_{\sigma_1}, \delta_{\sigma_2}]\, \mathcal{S}_R = 0,$$

This is a finite Ward identity, inducing a topological term

the renormalized effective action has a Weyl variation of the form

$$\delta_\sigma \mathcal{S}_R = \frac{1}{(4\pi)^2} \int d^4x \sqrt{g} \delta\sigma(x) \left(aE + bC^2 + c\Box R \right).$$

renormalized effective action

$$\mathcal{S}_R(g,d) = \mathcal{S}_B(g,d) + b'\frac{1}{\epsilon}V_E(g,d) + b\frac{1}{\epsilon}V_{C^2}(g,d).$$

$$(C^{(4)})^2 \equiv R_{\mu\nu\alpha\beta}R^{\mu\nu\alpha\beta} - 2R_{\mu\nu}R^{\mu\nu} + \frac{1}{3}R^2.$$

$$V_{C^2}(g,d) \equiv \mu^{\varepsilon}\int d^dx\,\sqrt{-g}\,C^2,$$

$$V_E(g,d) \equiv \mu^{\varepsilon}\int d^dx\,\sqrt{-g}\,E,$$

$$E = R^2 - 4R^{\mu\nu}R_{\mu\nu} + R^{\mu\nu\rho\sigma}R_{\mu\nu\rho\sigma}.$$

This is the point at which topological corrections appear. We identify it in the Euler Poincare' density, E

Specific linear combinations of higher derivative invariants in the action may be deprived of double poles in their propagators. In general, an expansion of the Riemann tensor in the fluctuations hμν around a flat Minkoswki vacuum,

$$R_{\mu\nu\rho\sigma} = R^{(1)}_{\mu\nu\rho\sigma} + R^{(2)}_{\mu\nu\rho\sigma} + \ldots ,$$

shows immediately that at quadratic level the theory, is plagued by propagating double poles, corresponding to the kinetic operator, as one can easily derive from the action

there are double poles, in general, but not if you take a and b as in the Euler density E

$$\mathcal{S}_2^{(2)} = \frac{1}{4}\int d^d x \sqrt{g}\left((a+4)h^{\mu\nu}\Box^2 h_{\mu\nu} + (b-1)h\Box^2 h\right),$$

$$V_E^{\mu\nu} = 4R_{\mu\alpha\beta\sigma}R_\nu{}^{\alpha\beta\sigma} - 8R_{\mu\alpha\nu\beta}R^{\alpha\beta} - 8R_{\mu\alpha}R_\nu{}^\alpha + 4RR_{\mu\nu} - g_{\mu\nu}E,$$

$$V_E^{\mu\nu} = \frac{\delta}{\delta g_{\mu\nu}}V_E,$$

Do we evade Lovelock't theorem ?

The theorem: at d=4 only the Einstein-Hilbert action
+ cosmological constant
generates equation of motion of the second order

Horndeski theories: generate equations of motion of the second order but with the inclusion of scalar fields

$$\mathcal{S}_{EH} = \int d^d x \sqrt{g}(M_P^2 R + \Lambda)$$

$$\delta_g(\sqrt{g}E) = \sqrt{g}\,\nabla_\sigma \delta X^\sigma,$$

$$\delta X^\sigma = \varepsilon^{\mu\nu\alpha\beta}\varepsilon^{\sigma\lambda\gamma\tau}\delta_g \Gamma^\eta_{\nu\lambda} g_{\mu\eta} R_{\alpha\beta\gamma\tau},$$

$$\mathcal{S}_{EGB} = S_{EH} + \alpha V_E,$$

$$\frac{1}{\kappa}\left(R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R + \Lambda_0 g_{\mu\nu}\right) + \alpha(V_E(d))_{\mu\nu} = 0$$

$$\mathcal{S}_{EGB} = \mathcal{S}_{EH} + \mathcal{S}_{GB}(d) \qquad \mathcal{S}_{GB}(d) = \frac{\alpha}{\epsilon} V_E(d)$$

In d=4 the density is topological

We need it in order to satisfy the Wess Zumino consistency condition

but it is not essential for renormalization

V_E shares properties similar to those of the EH action at $d = 2$,

$$\mathcal{S}_{EH}(d) \equiv \int d^d x \sqrt{g} R,$$

due to the topological nature of both functionals.

If we move away from d=4, obviously, things change

4d EInstein Gauss Bonnet Theory

We learn that it is possible to modify classical GR just by the inclusion of of a GB term, in which we perform a singular renormalization of the coupling

$$\sqrt{g}E = \sqrt{\bar{g}}e^{(d-4)\phi}\left\{\bar{E} + (d-3)\bar{\nabla}_\mu \bar{J}^\mu(\bar{g},\phi) + (d-3)(d-4)\bar{K}(\bar{g},\phi)\right\}, \qquad g_{\mu\nu} = e^{2\phi(x)}\bar{g}_{\mu\nu}$$

$$\bar{J}^\mu(\bar{g},\phi) = 8\bar{R}^{\mu\nu}\bar{\nabla}_\nu\phi - 4\bar{R}\bar{\nabla}^\mu\phi + 4(d-2)(\bar{\nabla}^\mu\phi\bar{\Box}\phi - \bar{\nabla}^\mu\bar{\nabla}^\nu\phi\bar{\nabla}_\nu\phi + \bar{\nabla}^\mu\phi\bar{\nabla}_\lambda\phi\bar{\nabla}^\lambda\phi),$$

$$\bar{K}(\bar{g},\phi) = 4\bar{R}^{\mu\nu}\bar{\nabla}_\mu\phi\bar{\nabla}_\nu\phi - 2\bar{R}\bar{\nabla}_\lambda\phi\bar{\nabla}^\lambda\phi + 4(d-2)\bar{\Box}\phi\bar{\nabla}_\lambda\phi\bar{\nabla}^\lambda\phi + (d-1)(d-2)(\bar{\nabla}_\lambda\phi\bar{\nabla}^\lambda\phi)^2.$$

$$\mathcal{S}_E^{WZ} = \lim_{\epsilon\to 0}\frac{1}{\epsilon}\left(V_E(\bar{g}e^{2\phi},d) - V_E(\bar{g},d)\right),$$

$$\mathcal{S}_E^{WZ} = \lim_{\epsilon \to 0} \frac{1}{\epsilon} \left(V_E(\bar{g} e^{2\phi}, d) - V_E(\bar{g}, d) \right),$$

Wess Zumino efffective action

$$\mathcal{S}_{C^2}^{WZ} = \lim_{\epsilon \to 0} \frac{1}{\epsilon} \left(V_{C^2}(\bar{g} e^{2\phi}, d) - V_{C^2}(\bar{g}, d) \right),$$

a similar Weyl gauging procedure is applied to the EH term

$$\tilde{S}_{EGBW_1} = \frac{1}{16\pi G} \int d^4x \sqrt{g}\; e^{-2\phi} \left([R + 6\nabla_\lambda \phi \nabla^\lambda \phi] - 2e^{-2\phi} \Lambda \right) + \mathcal{S}_f(4)$$
$$+ \int d^4x \sqrt{g} \left[-\phi(b'E + bC^2) - b' \left(4G^{\mu\nu} (\nabla_\mu \phi \nabla_\nu \phi) + 2(\nabla_\lambda \phi \nabla^\lambda \phi)^2 - 4\bar{\Box} \phi \nabla_\lambda \phi \nabla^\lambda \phi \right) \right].$$

EGB can be obtained from the expression above by removing the Weyl tensor terms and the finite, Weyl invariant contributions

In general, one obtains a dilaton gravity action.

Anomaly actions are nonlocal, and with the inclusion of a dilaton is essentially isolating an extra degree of freedom.

The locality of the action si related to the fact that we have performed a specific choice for the extension of the topological contribution.

A finite renormalization of the topological term will allow us to remove the dilaton and will generate a nonlocal 4DEGB theory

4D EGB and the anomaly action have important points in common.

In both cases we end up with quartic theories, in which the dilaton is an asymptotic field.

Eliminating the dilaton takes us to a different effective action

$$\sqrt{g}\left(E - \frac{2}{3}\,\Box R\right) = \sqrt{\bar{g}}\left(\bar{E} - \frac{2}{3}\,\bar{\Box}\bar{R} + 4\bar{\Delta}_4\phi\right),$$

in d=4 (Riegert)

$$\Delta_4 = \nabla^2 + 2\,R^{\mu\nu}\nabla_\mu\nabla_\nu - \frac{2}{3}\,R\Box + \frac{1}{3}\,(\nabla^\mu R)\nabla_\mu\,.$$

$$\sqrt{-g}\,\Delta_4\chi = \sqrt{-\bar{g}}\,\bar{\Delta}_4\chi,$$

WEYL GAUGING OF THE CURRENTS

$$J(x) = \bar{J}(x) + 4\sqrt{g}\Delta_4\phi(x), \qquad \bar{J}(x) \equiv \sqrt{\bar{g}}\left(\bar{E} - \frac{2}{3}\bar{\Box}\bar{R}\right), \qquad J(x) \equiv \sqrt{g}\left(E - \frac{2}{3}\Box R\right)$$

$$(\sqrt{-g}\,\Delta_4)_x D_4(x,y) = \delta^4(x,y).$$

$$\phi(x) = \frac{1}{4}\int d^4y\, D_4(x,y)(J(y) - \bar{J}(y)).$$

USE THE GREEN FUNCTION D_4
TO ELIMINATE THE DILATON

$$\mathcal{S}_{WZ} = \mathcal{S}_{anom}(g) - \mathcal{S}_{anom}(\bar{g}),$$

$$\mathcal{S}_{anom}(g) = \frac{1}{8}\int d^4x d^4y J(x) D_4(x,y) J(y),$$

$$\mathcal{S}_{\rm anom}(g) = \frac{1}{8}\int d^4x\sqrt{-g_x}\,\left(E-\frac{2}{3}\Box R\right)_x\int d^4x'\sqrt{-g_{x'}}\,D_4(x,x')\left[\frac{b'}{2}\,\left(E-\frac{2}{3}\Box R\right)+b\,C^2\right]_{x'}.$$

RIEGERT'S ACTION

NONLOCAL 4D EGB

$$\tilde{V}_E = \int d^d x \sqrt{g} \left(E_4 + \epsilon \frac{R^2}{2(d-1)^2} \right).$$

DR extension of the GB term

$$\delta_\phi(\sqrt{g} E_{ext}) = \delta\phi\epsilon \left(\sqrt{g} E_{ext} - \frac{2}{d-1} \sqrt{g} \Box R \right),$$

$$\delta_\phi \int d^d x \sqrt{g} E_{ext} = \epsilon \sqrt{g} (E_{ext} - \frac{2}{d-1} \Box R).$$

$$\tilde{V}_E = \int d^d x \sqrt{g} E_{ext},$$

$$\mathcal{S}_{GB}^{(WZ)} = \frac{\alpha}{\epsilon} \left(\tilde{V}_E(\bar{g}_{\mu\nu} e^{2\phi}, d) - \tilde{V}_E(\bar{g}_{\mu\nu}, d \right).$$

$$\mathcal{S}_{GB}^{(WZ)} = \alpha \int d^4x\, \sqrt{-\bar{g}}\, \left\{ \left(\overline{E} - \frac{2}{3}\, \bar{\Box}\, \overline{R} \right) \phi + 2\, \phi \bar{\Delta}_4 \phi \right\},$$

$$\begin{aligned} \mathcal{S}_{GB}^{(WZ)} \quad &= \quad \frac{\alpha}{8} \int d^4x\, \sqrt{-g}\, \int d^4x'\, \sqrt{-g'}\, \left(E_4 - \frac{2}{3}\, \Box\, R \right)_x \\ &\qquad \times D_4(x, x')\, \left(E - \frac{2}{3}\, \Box\, R \right)_{x'}, \end{aligned}$$

$$\begin{aligned} \mathcal{S}_{\rm anom}(g, \phi) \equiv -\frac{1}{2} \int d^4x\, \sqrt{-g}\, &\Big[(\Box \phi)^2 - 2 \big(R^{\mu\nu} - \tfrac{1}{3} R g^{\mu\nu} \big) (\nabla_\mu \phi)(\nabla_\nu \phi) \Big] \\ &+ \frac{1}{2} \int d^4x\, \sqrt{-g}\, \Big[\big(E - \tfrac{2}{3}\, \Box\, R \big) \Big]\, \phi, \end{aligned}$$

$$
\begin{aligned}
g_{\mu\nu} &= g^{(0)}_{\mu\nu} + g^{(1)}_{\mu\nu} + g^{(2)}_{\mu\nu} + \cdots \equiv \eta_{\mu\nu} + h_{\mu\nu} + h^{(2)}_{\mu\nu} + \dots \\
\phi &= \phi^{(0)} + \phi^{(1)} + \phi^{(2)} + \dots
\end{aligned}
$$

$$
\boxed{
\begin{aligned}
\overline{\Box}^2\phi^{(0)} &= 0 \\
(\sqrt{-g}\Delta_4)^{(1)}\phi^{(0)} + \overline{\Box}^2\phi^{(1)} &= \left[\sqrt{-g}\left(\frac{E}{2} - \frac{\Box R}{3}\right)\right]^{(1)} = -\frac{1}{3}\,\overline{\Box}R^{(1)} \\
(\sqrt{-g}\Delta_4)^{(2)}\phi^{(0)} + (\sqrt{-g}\Delta_4)^{(1)}\phi^{(1)} + \overline{\Box}^2\phi^{(2)} &= \left[\sqrt{-g}\left(\frac{E}{2} - \frac{\Box R}{3}\right)\right]^{(2)} \\
&= \frac{1}{2}E^{(2)} - \frac{1}{3}\,[\sqrt{-g}\,\Box R]^{(2)},
\end{aligned}
}
$$

$$
\phi^{(1)} = -\frac{1}{3\overline{\Box}}\,R^{(1)} \qquad \phi^{(2)} = \frac{1}{\overline{\Box}^2}\left\{(\sqrt{-g}\Delta_4)^{(1)}\frac{1}{3\overline{\Box}}\,R^{(1)} + \frac{1}{2}E^{(2)} - \frac{1}{3}\,[\sqrt{-g}\,\Box R]^{(2)}\right\}.
$$

$$\mathcal{S}^{(3)}_{\rm anom} = -\frac{1}{2}\int d^4x\, \Big\{2\,\phi^{(1)}\,\overline{\Box}^{\,2}\phi^{(2)} + \phi^{(1)}\big(\sqrt{-g}\Delta_4\big)^{(1)}\,\phi^{(1)}\Big\}$$

$$+\frac{1}{2}\int d^4x\left\{\left(-\frac{2}{3}\,\overline{\Box}\,R^{(1)}\right)\phi^{(2)} + \left(E^{(2)} - \frac{2}{3}\,\sqrt{-g}\,\Box\,R\right)^{(2)}\phi^{(1)}\right\}.$$

$$\left(\sqrt{g}\,\Box^2\right)^{(1)} \equiv \delta\left(\sqrt{g}\,\Box^2\right) = \left(\frac{(\sqrt{g}\,\Box)^2}{\sqrt{g}}\right)^{(1)} \qquad \delta\left(\frac{1}{\sqrt{g}}\right)(\sqrt{g}\,\Box)^2 = -\delta(\sqrt{g})\,\Box^2,$$

$$\left(\sqrt{g}\,\Box^2\right)^{(1)} = -\delta(\sqrt{g})\,\Box^2 + \frac{1}{\sqrt{g}}\delta(\sqrt{g}\,\Box)(\sqrt{g}\,\Box) + \frac{1}{\sqrt{g}}(\sqrt{g}\,\Box)\,(\delta(\sqrt{g}\,\Box)),$$

The nonlocal GB theory

$$\mathcal{S}^{(2)}_{\rm anom} = -\frac{1}{2}\int d^4x\,\phi^{(1)}\,\overline{\square}^2\phi^{(1)} + \frac{1}{2}\int d^4x\,\left(-\frac{2}{3}\,\overline{\square}\,R^{(1)}\right)\phi^{(1)} = \frac{1}{18}\int d^4x\,\left(R^{(1)}\right)^2,$$

TT

$$\mathcal{S}^{(3)}_{\rm anom} = \frac{1}{9}\int d^4x\int d^4x'\int d^4x''\Big\{(\partial_\mu R^{(1)})_x\left(\frac{1}{\overline{\square}}\right)_{xx'}\left(R^{(1)\mu\nu} - \frac{1}{3}\eta^{\mu\nu}R^{(1)}\right)_{x'}\left(\frac{1}{\overline{\square}}\right)_{x'x''}(\partial_\nu R^{(1)})_x$$
$$-\frac{1}{6}\int d^4x\int d^4x'\left(E^{(2)}\right)_x\left(\frac{1}{\overline{\square}}\right)_{xx'}R^{(1)}_{x'} + \frac{1}{18}\int d^4x\,R^{(1)}\left(2\,R^{(2)} + (\sqrt{-g})^{(1)}R^{(1)}\right),$$

TTT

This theory defines the anomaly contribution to the original Einstein equation, but it will miss some Weyl invariant terms, which will come only from the computation of the graviton vertices

What about the Weyl invariant terms ?
They can be computed exactly and correspond to the finite parte of the TT, TTT etc correlators.

They satisfy anomalous conformal Ward identities and can be expressed in terms of a free dield theory realization. The same result can be used for ANY CFT.
The reason being that the scaling dimension of T is d, and the genral solution of the CWIs depend only on 3 parameters in d=4

ALl the form factors of the TTT are determined uniquely only modulo 3 constants

The general 3-graviton vertex (TTT) of conformal field theories in momentum space in $d = 4$

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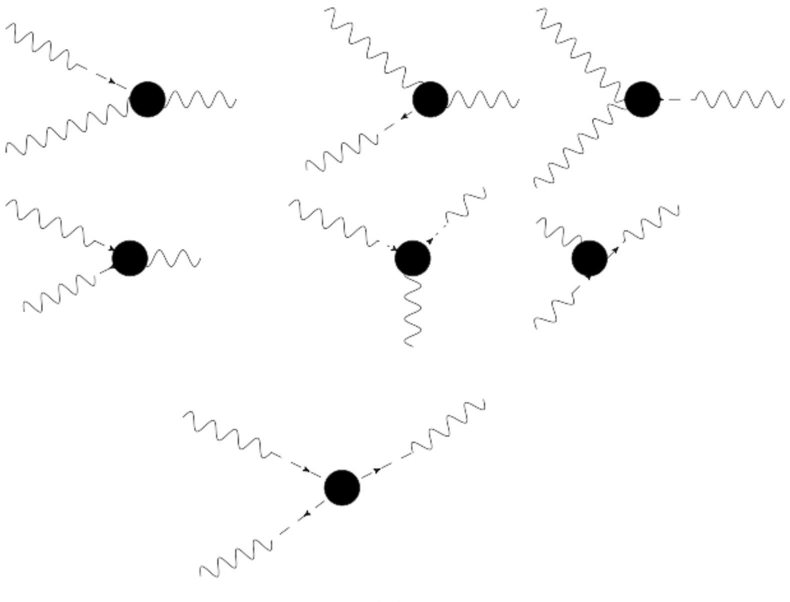
$$
\begin{aligned}
A_1^{Ren} = {} & \pi^2(4n_F - 2n_G - n_S)\Big\{ \frac{1}{45\sigma^5}\bigg[s^9 - 13s^8(s_1+s_2) + 2s^7\left(25s_1^2 + 77s_1s_2 + 25s_2^2\right) - 2s^6(s_1+s_2)\left(41s_1^2 - 9s_1s_2 + 41s_2^2\right) \\
& + s^5\left(44s_1^4 - 922s_1^3s_2 + 5088s_1^2s_2^2 - 922s_1s_2^3 + 44s_2^4\right) + 2s^4(s_1+s_2)\left(22s_1^4 + 823s_1^3s_2 - 3360s_1^2s_2^2 + 823s_1s_2^3 + 22s_2^4\right) \\
& - 2s^3\left(41s_1^6 + 461s_1^5s_2 + 2537s_1^4s_2^2 - 8598s_1^3s_2^3 + 2537s_1^2s_2^4 + 461s_1s_2^5 + 41s_2^6\right) + 2s^2(s_1-s_2)^2(s_1+s_2)\big(25s_1^4 - 7s_1^3s_2 \\
& + 2562s_1^2s_2^2 - 7s_1s_2^3 + 25s_2^4\big) - s(s_1-s_2)^4\left(13s_1^4 - 102s_1^3s_2 - 422s_1^2s_2^2 - 102s_1s_2^3 + 13s_2^4\right) + (s_1-s_2)^6(s_1+s_2)\left(s_1^2 - 8s_1s_2 + s_2^2\right)\bigg] \\
& - \frac{4s^2}{15\sigma^6}\bigg[s_2^5\left(-35s^4 - 2469s^2s_1^2 + 2428ss_1^3 + 726s_1^4\right) + s_2^6(35s^3 + 135s^2s_1 + 448ss_1^2 - 1052s_1^3) + s_2^2(s-s_1)^3\big(s^4 - 24s^3s_1 - 675s^2s_1^2 \\
& - 1348ss_1^3 - 300s_1^4\big) + 3s_2^4(7s^5 - 45s^4s_1 + 581s^3s_1^2 + 925s^2s_1^3 - 1850ss_1^4 + 242s_1^5) - s_2^3(s-s_1)\big(7s^5 - 101s^4s_1 - 705s^3s_1^2 \\
& + 4151s^2s_1^3 + 1376ss_1^4 - 1052s_1^5\big) - 27s_1^2s_2(s-s_1)^5(s+s_1) + s_1^2(s-s_1)^7 + s_2^8(7s + 27s_1) - 3s_2^7(s-2s_1)(7s+50s_1) - s_2^9\bigg]\bar{B}_0(s) \\
& - \frac{4s_1^2}{15\sigma^6}\bigg[-s^9 + s^8(7s_1 + 27s_2) - 3s^7(s_1 - 2s_2)(7s_1 + 50s_2) + s^6\left(35s_1^3 + 135s_1^2s_2 + 448s_1s_2^2 - 1052s_2^3\right) + s^5\big(-35s_1^4 \\
& - 2469s_1^2s_2^2 + 2428s_1s_2^3 + 726s_2^4\big) + 3s^4\left(7s_1^5 - 45s_1^4s_2 + 581s_1^3s_2^2 + 925s_1^2s_2^3 - 1850s_1s_2^4 + 242s_2^5\right) - s^3(s_1-s_2)\big(7s_1^5 - 101s_1^4s_2 \\
& - 705s_1^3s_2^2 + 4151s_1^2s_2^3 + 1376s_1s_2^4 - 1052s_2^5\big) + s^2(s_1-s_2)^3\left(s_1^4 - 24s_1^3s_2 - 675s_1^2s_2^2 - 1348s_1s_2^3 - 300s_2^4\right) \\
& - 27ss_2^2(s_1-s_2)^5(s_1+s_2) + s_2^2(s_1-s_2)^7\bigg]\bar{B}_0(s_1) - \frac{4s_2^2}{15\sigma^6}\bigg[-s^9 + s^8(27s_1 + 7s_2) + 3s^7(2s_1 - s_2)(50s_1 + 7s_2) + s^6\big(-1052s_1^3 \\
& + 448s_1^2s_2 + 135s_1s_2^2 + 35s_2^3\big) + s^5\left(726s_1^4 + 2428s_1^3s_2 - 2469s_1^2s_2^2 - 35s_2^4\right) + 3s^4\big(242s_1^5 - 1850s_1^4s_2 + 925s_1^3s_2^2 + 581s_1^2s_2^3 \\
& - 45s_1s_2^4 + 7s_2^5\big) - s^3(s_1-s_2)\left(1052s_1^5 - 1376s_1^4s_2 - 4151s_1^3s_2^2 + 705s_1^2s_2^3 + 101s_1s_2^4 - 7s_2^5\right) + s^2(s_1-s_2)^3\big(300s_1^4 + 1348s_1^3s_2 \\
& + 675s_1^2s_2^2 + 24s_1s_2^3 - s_2^4\big) + 27ss_1^2(s_1-s_2)^5(s_1+s_2) - s_1^2(s_1-s_2)^7\bigg]\bar{B}_0(s_2) + \frac{16s^2s_1^2s_2^2}{\sigma^6}\bigg[3s^6 - 4s^5(s_1+s_2) \\
& + s^4\left(-11s_1^2 + 40s_1s_2 - 11s_2^2\right) + 12s^3(2s_1 - s_2)(s_1+s_2)(s_1 - 2s_2) - s^2\left(11s_1^4 + 36s_1^3s_2 - 108s_1^2s_2^2 + 36s_1s_2^3 + 11s_2^4\right) \\
& - 4s(s_1-s_2)^2(s_1+s_2)\left(s_1^2 - 9s_1s_2 + s_2^2\right) + (s_1-s_2)^4\left(3s_1^2 + 8s_1s_2 + 3s_2^2\right)\bigg]C_0(s, s_1, s_2)
\end{aligned}
\tag{G.1}
$$



(a)

finite parts

TT andTTT

anomaly parts are in agreement with the expansion of the nonlocal action

The anomaly part of the TTTT has also been computed

The conformal anomaly action to fourth order (4T) in $d = 4$ in momentum space #4

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The anomaly parts can be derived directly from the Conformal Ward identities, not just from the nonlocal action

$$\mathcal{S}_A = \int d^4x_1 d^4x_2 \langle T\cdot h(x_1) T\cdot h(x_2)\rangle + \int d^4x_1 d^4x_2 d^4x_3 \langle T\cdot h(x_1) T\cdot h(x_2) T\cdot h(x_3)\rangle_{pole}$$
$$+ \int d^4x_1 d^4x_2 d^4x_3 d^4x_4 \left(\langle T\cdot h(x_1) T\cdot h(x_2) T\cdot h(x_3) T\cdot h(x_4)\rangle_{pole} + \right.$$
$$\left. + \langle T\cdot h(x_1) T\cdot h(x_2) T\cdot h(x_3) T\cdot h(x_4)\rangle_{0T}\right),$$

$$\begin{aligned}
&\langle T^{\mu_1\nu_1}(p_1) T^{\mu_2\nu_2}(p_2) T^{\mu_3\nu_3}(p_3) T^{\mu_4\nu_4}(\bar{p}_4)\rangle_{poles} = \\
&= \frac{\pi^{\mu_1\nu_1}(p_1)}{3} \langle T(p_1) T^{\mu_2\nu_2}(p_2) T^{\mu_3\nu_3}(p_3) T^{\mu_4\nu_4}(\bar{p}_4)\rangle_{anomaly} + (perm.) \\
&- \frac{\pi^{\mu_1\nu_1}(p_1)}{3}\frac{\pi^{\mu_2\nu_2}(p_2)}{3} \langle T(p_1) T(p_2) T^{\mu_3\nu_3}(p_3) T^{\mu_4\nu_4}(\bar{p}_4)\rangle_{anomaly} + (perm.) \\
&+ \frac{\pi^{\mu_1\nu_1}(p_1)}{3}\frac{\pi^{\mu_2\nu_2}(p_2)}{3}\frac{\pi^{\mu_3\nu_3}(p_2)}{3} \langle T(p_1) T(p_2) T(p_3) T^{\mu_4\nu_4}(\bar{p}_4)\rangle_{anomaly} + (perm.) \\
&- \frac{\pi^{\mu_1\nu_1}(p_1)}{3}\frac{\pi^{\mu_2\nu_2}(p_2)}{3}\frac{\pi^{\mu_3\nu_3}(p_3)}{3}\frac{\pi^{\mu_4\nu_4}(p_4)}{3} \langle T(p_1) T(p_2) T(p_3) T(\bar{p}_4)\rangle_{anomaly}.
\end{aligned}$$

$$
\begin{aligned}
&\langle T^{\mu_1\nu_1}(p_1)T^{\mu_2\nu_2}(p_2)T^{\mu_3\nu_3}(p_3)T^{\mu_4\nu_4}(\bar{p}_4)\rangle_{0-residue} = \\
&= \mathcal{I}^{\mu_1\nu_1}_{\alpha_1}(p_1)\, p_{1\beta_1}\, \langle T^{\alpha_1\beta_1}(p_1)T^{\mu_2\nu_2}(p_2)T^{\mu_3\nu_3}(p_3)T^{\mu_4\nu_4}(\bar{p}_4)\rangle_{anomaly} + (perm.) \\
&- \Bigg\{\bigg[\mathcal{I}^{\mu_2\nu_2}_{\alpha_2}(p_2)\,\mathcal{I}^{\mu_1\nu_1}_{\alpha_1}(p_1)\, p_{2\beta_2}\, p_{1\beta_1}\, \langle T^{\alpha_1\beta_1}(p_1)T^{\alpha_2\beta_2}(p_2)T^{\mu_3\nu_3}(p_3)T^{\mu_4\nu_4}(p_4)\rangle_{anom} \\
&\qquad + \mathcal{I}^{\mu_1\nu_1}_{\alpha_1}(p_1)\,\frac{\pi^{\mu_2\nu_2}(p_2)}{3}\, p_{1\beta_1}\, \langle T^{\alpha_1\beta_1}(p_1)T(p_2)T^{\mu_3\nu_3}(p_3)T^{\mu_4\nu_4}(\bar{p}_4)\rangle_{anomaly} \\
&\qquad + \frac{\pi^{\mu_1\nu_1}(p_1)}{3}\,\mathcal{I}^{\mu_2\nu_2}_{\alpha_2}(p_2)\, p_{2\beta_2}\, \langle T(p_1)T^{\alpha_2\beta_2}(p_2)T^{\mu_3\nu_3}(p_3)T^{\mu_4\nu_4}(\bar{p}_4)\rangle_{anomaly}\bigg] + (perm.)\Bigg\} \\
&+ \Bigg\{\bigg[\mathcal{I}^{\mu_1\nu_1}_{\alpha_1}(p_1)\,\mathcal{I}^{\mu_2\nu_2}_{\alpha_2}(p_2)\,\frac{\pi^{\mu_3\nu_3}(p_3)}{3}\,\langle T^{\alpha_1\beta_1}(p_1)T^{\alpha_2\beta_2}(p_2)T(p_3)T^{\mu_4\nu_4}(\bar{p}_4)\rangle_{anomaly} + (13) + (23)\bigg] + (perm.) \\
&+ \bigg[\mathcal{I}^{\mu_1\nu_1}_{\alpha_1}(p_1)\frac{\pi^{\mu_2\nu_2}(p_2)}{3}\,\frac{\pi^{\mu_3\nu_3}(p_3)}{3}\, p_{1\beta_1}\,\langle T^{\alpha_1\beta_1}(p_1)T(p_2)T(p_3)T^{\mu_4\nu_4}(\bar{p}_4)\rangle_{anomaly} + (12) + (13)\bigg] + (perm.)\Bigg\} \\
&\quad - \Bigg\{\mathcal{I}^{\mu_1\nu_1}_{\alpha_1}(p_1)\,\mathcal{I}^{\mu_2\nu_2}_{\alpha_2}(p_2)\,\frac{\pi^{\mu_3\nu_3}(p_3)}{3}\,\frac{\pi^{\mu_4\nu_4}(p_4)}{3}\, p_{1\beta_1}\, p_{2\beta_2}\,\langle T^{\alpha_1\beta_1}(p_1)T^{\alpha_2\beta_2}(p_2)T(p_3)T(p_4)\rangle_{anomaly} \\
&\qquad\quad + (13) + (23) + (14) + (24) + (13)(24)\Bigg\} \\
&\quad - \Bigg\{\mathcal{I}^{\mu_1\nu_1}_{\alpha_1}(p_1)\,\frac{\pi^{\mu_2\nu_2}(p_2)}{3}\,\frac{\pi^{\mu_3\nu_3}(p_3)}{3}\,\frac{\pi^{\mu_4\nu_4}(p_4)}{3} p_{1\beta_1}\,\langle T^{\alpha_1\beta_1}(p_1)T(p_2)T(p_3)T(\bar{p}_4)\rangle_{anomaly} \\
&\qquad\quad + (12) + (13) + (14)\Bigg\}.
\end{aligned} \tag{9.65}
$$

this is anew component that appears at the TTTT level from the anomalous CWI's

(Maglio, Theofilopoulos, CC)

It needs to be checked if the nonlocal anomaly action agrees with this exact derivation.
(work in progress)

Conclusions:
We have full control of the quantum corrections in the conformal sector up to cubic order in the gravitational fluctuations.

They reproduce, in their anomaly parts, the nonlocal anomaly action.
TT, TTT are known and written in a simplified form, extract from free field theory by matching with the result from the anomalous Conformal Ward identities

$$R^{\mu\nu} - \frac{1}{2} g^{\mu\nu} R = \langle T^{\mu\nu} \rangle$$

One can compute, for instance the impact of these corrections in the production of GW in the early universe.

THANK YOU