Fingerprints of the quantum space-time in time-dependent quantum mechanics: An emergent geometric phase

Anwesha Chakraborty
S. N. Bose National Centre for Basic sciences, Kolkata,India

AC, P Nandi, B Chakraborty, *Nucl.Phys.B 975 (2022) 115691, arXiv:2110.04730*

Workshop on Noncommutative and generalized geometry in string theory, gauge theory and related physical models
Corfu Summer School

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Quantum superposition of two different stationary mass distributions
R. Penrose, Gen. Rel. Grav. 8, 5, (1996).

Aim and motivation

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- 1.To provide a consistent formulation of quantum mechanics in noncommutative space-time, where time is also an operator.
- 2.To study the effect of space-time non-commutativity on the dynamics of a time-dependent system.

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So it is interesting to study a time dependent system placed in a non-commutative space-time!
Although the effect of non-commutativity becomes significant at very high energy scale, it is intriguing to speculate that there should be some relics of this effect at low energy level.
Mechanism for the quantum natured gravitons to entangle masses
Sougato Bose, Anupam Mazumdar, Martine Schut, and Marko Toroš, *Phys. Rev. D 105, 106028*

Summary of our work

We have shown that a time dependent displaced harmonic oscillator system when placed in a NC space-time, gives rise to geometrical phase shift, if evolved adiabatically.

C L Mehta,E C G Sudarshan, Phys. Lett. (1996)22,5
P Carruthers, M M Nieto, American J. Phys.33, 537 (1965)

Plan of the talk

- Development of usual quantum mechanics from classical toy model: A brief review.
- Noncommutative symplectic structure of space-time through a classical model
- Quantum mechanics of Non-commutative (NC) space-time
- Displaced harmonic oscillator in NC space-time
- Adiabatic evolution and emergence of geometric phase
- Comments

Time reparametrization invariant form of action

To treat time and space at equal footing:

$\rightarrow$Introduce τ, a monotonically increasing function of t :

$$\dot{t} = \frac{dt}{d\tau} > 0 \;\; and \;\; \dot{x}_i = \left(\frac{dx_i}{d\tau}\right); \quad i = 0, 1, x_0 = t, x_1 = x$$

(**A A Deriglazov, American J. Phys.79, 882 (2011)**)

$$S = \int dt\, L^t = \int d\tau\, L^\tau$$

Example:

$$L^t(x, \dot{x}, t) = \frac{1}{2} m \left(\frac{dx}{dt}\right)^2 - V(x,t) \Rightarrow L^\tau = \frac{1}{2} m \frac{\dot{x}^2}{\dot{t}} - \dot{t} V(x,t) \qquad (1)$$

Canonical Momenta: $p_x = m\frac{\dot{x}}{\dot{t}}$

$$p_t = -\frac{p_x^2}{2m} - V(x,t) = -H \qquad (2)$$

$$\textbf{Constraint:} \qquad \phi = p_t + H \approx 0 \tag{3}$$

$$\textbf{Total Hamiltonian:} \qquad H^\tau_T = H^\tau + \sigma(\tau)\phi = \sigma(\tau)\phi \tag{4}$$

$H^\tau = 0$ is the canonical Hamiltonian corresponding to L^τ.

First Order Form of Lagrangian:

$$L^\tau_f = p_\mu \dot{x}^\mu - \sigma(\tau)(p_t + H) \tag{5}$$

Constraint Analysis $\rightarrow$ Computation of Dirac Bracket:

$$\{x^\mu, x^\nu\}_D = 0 = \{p_\mu, p_\nu\}_D; \quad \{x^\mu, p_\nu\}_D = \delta^\mu{}_\nu \tag{6}$$

The system is still left with a first class constraint $\phi = p_t + H \approx 0$, which is basically the generator of gauge transformation and responsible for the τ evolution in the system.

Quantization:

$$[\hat{t},\hat{x}]=0=[\hat{p}_t,\hat{p}_x],\ [\hat{t},\hat{p}_t]=i=[\hat{x},\hat{p}_x]\ \ (\hbar=1) \tag{7}$$

We introduce simultaneous eigen-state of $\hat{t}$ and $\hat{x}$:

$$\hat{t}\,|t,x\rangle=t\,|t,x\rangle\,,\quad \hat{x}\,|t,x\rangle=x\,|t,x\rangle\,. \tag{8}$$

The representations of the phase space operators in this Hilbert space:

$$\begin{aligned}&\langle t,x|\hat{x}|\psi\rangle=x\psi(t,x);\quad \langle t,x|\hat{t}|\psi\rangle=t\psi(t,x)\\ &\langle\hat{p}_x|\psi\rangle=-i\partial_x\psi(t,x);\quad \langle t,x|\hat{p}_t|\psi\rangle=-i\partial_t\psi(t,x)\end{aligned} \tag{9}$$

$$\textbf{Inner product :-}\ \ <\psi|\phi>_=\int dtdx\psi^\star(t,x)\phi(t,x) \tag{10}$$

and $\psi(t,x)\,=\,\langle\,t,x\,|\,\psi\,\rangle\ \in\ L^2(\mathbb{R}^2)\ [\psi(t,x)\to 0\ for\ t,x\to\pm\infty]$

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However to find a consistent probabilistic interpretation, we identify the physical Hilbert space states by imposing **Schrödinger's Constraint:** $(\hat{p}_t + \hat{H})|\psi; phy\rangle = 0$ Gauge invariance of the physical states !

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$$i\frac{\partial}{\partial t}\psi_{phy}(t, x) = \left[-\frac{1}{2m}\frac{\partial^2}{\partial x^2} + V(x, t)\right]\psi_{phy}(t, x) \tag{11}$$

Now to get a probabilistic interpretation, we derive the continuity equation :

$$\frac{\partial \rho}{\partial t} + \frac{\partial J_x}{\partial x} = 0 \tag{12}$$

with $\rho = \psi^{\star}_{phy}(x,t)\psi_{phy}(x,t)$; $J_x = \frac{1}{2m} Im(\psi^{\star}_{phy}(x,t)\overleftrightarrow{\partial_x}\psi_{phy}(x,t))$.

Perfoming spatial integration,

$$\partial_t \int_{-\infty}^{\infty} \rho dx = -\int_{-\infty}^{\infty} (\partial_x J_x) dx = 0 \tag{13}$$

Only by considering $\psi_{phy}(x,t) \to 0$ as $x \to \pm\infty$, we can show the conservation of total probability.

Induced inner product for the Hilbert space $L^2(\mathbb{R}^1)$

$$\langle \psi; phy|\phi; phy\rangle_t = \int_{-\infty}^{\infty} dx \ \psi^{*}_{phy}(x,t)\phi_{phy}(x,t) < \infty \tag{14}$$

Quantum Space-time: A toy model

$$L_f^{\tau,\theta} = L_f^{\tau} + \frac{\theta}{2}\epsilon^{\mu\nu} p_\mu \dot{p}_\nu, \quad \mu,\nu = 0,1 \tag{15}$$

Dirac Brackets:

$$\{x^\mu, x^\nu\}_D = \theta\epsilon^{\mu\nu}; \quad \{p_\mu, p_\nu\}_D = 0; \quad \{x^\mu, p_\nu\}_D = \delta^\mu{}_\nu \tag{16}$$

Non-commutative Heisenberg Algebra:

$$[\hat{t}, \hat{x}] = i\theta; \quad [\hat{p}_t, \hat{p}_x] = 0, \ [\hat{t}, \hat{p}_t] = i = [\hat{x}, \hat{p}_x] \tag{17}$$

Hilbert Space:

$$\mathcal{H}_c = Span\left\{ |n\rangle = \frac{(b^\dagger)^n}{\sqrt{n!}}|0\rangle; \ b = \frac{\hat{t} + i\hat{x}}{\sqrt{2\theta}}; \ b|0\rangle = 0 \right\} \tag{18}$$

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$$\hat{\mathcal{A}}_\theta = \{|\Psi) = \Psi(\hat{t}, \hat{x}) = \Psi(\hat{b}, \hat{b}^\dagger) = \sum_{m,n} c_{n,m}|m\rangle\langle n|\} \tag{19}$$

(non-commutative associative algebra,not a Hilbert space as yet !)

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$$\mathcal{H}_q = Span\left\{ \psi(\hat{t}, \hat{x}) \equiv |\psi(\hat{t}, \hat{x})); \; ||\psi||_{HS} = \sqrt{tr_c(\psi^\dagger\psi)} < \infty \right\} \subset \hat{\mathcal{A}}_\theta \tag{20}$$

Coherent states

The actions of phase space operators on $|\psi(\hat{t},\hat{x}))$ are given by,

$$\hat{T}|\psi(\hat{t},\hat{x})) = \hat{t}|\psi(\hat{t},\hat{x})), \quad \hat{X}|\psi(\hat{t},\hat{x})) = \hat{x}|\psi(\hat{t},\hat{x})),$$
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Coherent state basis:Maximally localized space time event

$$|z\rangle = e^{-\bar{z}b+zb^{\dagger}}|0\rangle \;\in \mathcal{H}_c \tag{22}$$

$$b|z\rangle = z|z\rangle, \;\; where\;\; b = \frac{\hat{t}+i\hat{x}}{\sqrt{2\theta}}; \;\; z = \frac{t+ix}{\sqrt{2\theta}};$$

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$$\int \frac{d^2z}{\pi}|z,\bar{z}) \star_V (z,\bar{z}| = \int dtdx\,|x,t) \star_V (x,t| = \mathbf{1}_q, \tag{25}$$

where the product $*_V$ is given by,

$$\star_V = e^{\overleftarrow{\partial_z}\overrightarrow{\partial_{\bar{z}}}} = e^{\frac{i\theta}{2}(-i\delta_{ij}+\epsilon_{ij})\overleftarrow{\partial_i}\overrightarrow{\partial_j}}; \quad \epsilon_{01} = 1 \tag{26}$$

Inner Product:$L^2{}_*(\mathbb{R}^2)$

$$(\psi|\phi) = \int dtdx\ \psi^*(x,t) \star_V \phi(x,t) \tag{27}$$

For $\hat{\psi}(\hat{x},\hat{t}) \in \mathcal{H}_q$: $(z,\bar{z}|\hat{\psi})$ is the corresponding symbol.

$$\psi(x,t) = \frac{1}{\sqrt{2\pi\theta}}(z,\bar{z}|\hat{\psi}(\hat{x},\hat{t})) = \frac{1}{\sqrt{2\pi\theta}}\langle z|\hat{\psi}(\hat{x},\hat{t})|z\rangle \tag{28}$$

Isomorphism between operator algebra and symbol algebra

$$(z|\hat{\psi}(\hat{x},\hat{t})\hat{\phi}(\hat{x},\hat{t})) = (z|\hat{\psi}(\hat{x},\hat{t})) \star_V (z|\hat{\phi}(\hat{x},\hat{t})) \tag{29}$$

Physical Hilbert space and Induced Inner-product

$$\text{First class constraint: } (\hat{P}_t + \hat{H})|\hat{\psi}(\hat{x},\hat{t}))_{phy} = 0 \tag{30}$$

Space-time representation of Schrödinger's equation:

$$(z,\bar{z}|(\hat{P}_t + \hat{H})|\hat{\psi}(\hat{x},\hat{t}))_{phy} = 0 \tag{31}$$

$$\partial_t \rho_\theta + \partial_x J^x_\theta = 0 \tag{32}$$

where

$$\begin{aligned} \rho_\theta &= \psi^*_{phy}(x,t) \star_V \psi_{phy}(x,t) > 0, \\ J^x_\theta &= \frac{1}{2im}[\psi^*_{phy} \star_V (\partial_x \psi_{phy}) - (\partial_x \psi^*_{phy}) \star_V \psi_{phy}]. \end{aligned} \tag{33}$$

Inner product in $L^2_*(\mathbb{R}^1)$ (Physical Hilbert Space)

$$\langle \hat{\psi}_{phy} | \hat{\phi}_{phy} \rangle_t = \int_{-\infty}^{\infty} dx\ \psi^*_{phy}(x,t) \star_V \phi_{phy}(x,t) < \infty. \tag{34}$$

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Displaced Harmonic Oscillator in Quantum Space-time

Hamiltonian of a displaced harmonic oscillator is given by-

$$H = \frac{\hat{p}_x^2}{2m} + \frac{1}{2}m\omega^2\hat{x}^2 + f(t)\hat{x} + g(t)\hat{p}_x \tag{35}$$

where $[\hat{x}, \hat{p}_x] = i$.

On quantum space-time $[\hat{T}, \hat{X}] = i\theta$, the Hamiltonian can be-

$$\hat{H} = \frac{\hat{P}_x^2}{2m} + \frac{1}{2}m\omega^2\hat{X}^2 + \frac{1}{2}[f(\hat{T})\hat{X} + \hat{X}f(\hat{T})] + g(\hat{T})\hat{P}_x \tag{36}$$

$$\begin{aligned}
&i\partial_t(z,\bar{z}|\hat{\psi}_{phy}) = (z,\bar{z}|\hat{H}|\hat{\psi}_{phy}) \\
\Rightarrow &i\partial_t\psi_{phy}(x,t) = \\
&\left(x,t\,\middle|\,\frac{P_x^2}{2m} + \frac{1}{2m}m\omega^2X^2 + \frac{1}{2}\{f(T)X + Xf(T)\} + g(T)P_x\right) \star \psi_{phy}(x,t)
\end{aligned} \tag{37}$$

Space-time representation

$$i\partial_t\psi_{phy}(x,t) = \Big[\frac{P_x^2}{2m}+\frac{1}{2}m\omega^2X_\theta^2+\frac{1}{2}\{f(T_\theta)X_\theta+X_\theta f(T_\theta)\}+g(T_\theta)P_x\Big]\psi_{phy}(x,t) \tag{38}$$

$$\begin{aligned}(x,t|\hat{X}\,\Psi(\hat{x},\hat{t})) &= \frac{1}{\sqrt{2\pi\theta}}(z,\bar{z}|\hat{x}\Psi) = \frac{1}{\sqrt{2\pi\theta}}\ \langle z|\hat{x}|z\rangle \star_V (z,\bar{z}|\Psi(\hat{x},\hat{t})) \\ &= X_\theta\Psi(x,t) \end{aligned} \tag{39}$$

$$P_x = -i\partial_x$$

$$X_\theta = x + \frac{\theta}{2}(\partial_x - i\partial_t) = SxS^{-1}$$

$$T_\theta = t + \frac{\theta}{2}(\partial_t + i\partial_x) = S^\dagger t(S^\dagger)^{-1}$$

where $\quad S = e^{\frac{\theta}{4}(\partial_t^2+\partial_x^2)}e^{-\frac{i\theta}{2}\partial_t\partial_x} \rightarrow$ A non-unitary operator

This helps us to introduce the following map:

$$S^{-1} : L^2_*(\mathbb{R}^1) \; \rightarrow \; L^2(\mathbb{R}^1)$$

$$S^{-1}\psi_{phy}(x,t) = \psi_c(x,t)$$

It can be shown, for any generic states,

$$\int_t dx \; \psi_{phy}(x,t)^* \;\; \star \phi_{phy}(x,t) = \int_t dx \; \psi^*_c(x,t)\phi_c(x,t) \tag{40}$$

Note that, the integrands are not however equal!

$$\begin{aligned} &i\partial_t\psi_c(t,x) \\ &= \left[\alpha(t)p_x^2 + \frac{1}{2}m\omega^2x^2 + \gamma(t)(xp_x + p_xx) + f(t)x + g(t)p_x\right]\psi_c(x,t) \end{aligned} \tag{41}$$

The adiabatic and periodic evolution

$$\begin{aligned} H_c &= \alpha(t)p_x^2 + \beta x^2 + \gamma(t)(xp_x + p_x x) + f(t)x + g(t)p_x \\ &= H_{g.h.o} + f(t)x + g(t)p_x \end{aligned} \tag{42}$$

where

$$\begin{aligned} \alpha(t) &= \frac{1}{2m} - \theta\dot{g}(t) \\ \beta &= \frac{1}{2}m\omega^2 \\ \gamma(t) &= -\frac{1}{2}\theta\dot{f}(t) \end{aligned} \tag{43}$$

We consider $\mathbf{R} = (\alpha(t), \beta, \gamma(t))$ to be the time dependent parameter space , which varies periodically with time period of $\mathcal{T}$.
The system Hamiltonian changes *adiabatically* through this parameter so that they make a closed loop Γ in the parameter space, and the Hamiltonian comes back to its initial value.

Diagonalization

$$a = \sqrt{\frac{\beta}{\Omega}}\left[x + \left(\frac{\gamma}{\beta} + i\frac{\Omega}{2\beta}\right)p_x\right];\ \ \Omega(t) = 2\sqrt{\alpha\beta - \gamma^2}$$
$$= A(t)[x + (B(t) + iC(t))p_x] \qquad (44)$$

$$H_c = \Omega(t)(a^\dagger a + \frac{1}{2}) + P(t)a + \bar{P}(t)a^\dagger \qquad (45)$$

where,

$$P(t) = A(t)[C(t)f(t) + i(B(t)f(t) - g(t))]$$

Further, we give a time dependent unitary transformation:

$$H_c \ \rightarrow \ \tilde{H}_c = \mathcal{U}(t)H_c\mathcal{U}^\dagger(t) - i\mathcal{U}(t)\partial_t\mathcal{U}^\dagger(t) \qquad (46)$$

so that $\tilde{H}_c\tilde{\psi}_c = i\partial_t\tilde{\psi}_c$, where $\tilde{\psi}_c = \mathcal{U}(t)\psi_c$

with

$$\mathcal{U}(t) = e^{-(wa - \bar{w}a^\dagger + il)}, \quad w \in \mathbb{C}, \ l \in \mathbb{R} \tag{47}$$

where w and l are given by,

$$\begin{aligned} w &= -ie^{i\int dt\,\Omega(t)} \int dt\,P(t)e^{-i\int dt\,\Omega(t)} \\ l &= \int dt\,\Omega(t)|w|^2 \end{aligned} \tag{48}$$

$$\tilde{H}_c = \Omega(t)(a^\dagger a + \frac{1}{2}) = H_{g.h.o} \tag{49}$$

The Heisenberg equation of motion for the operators a and $a^\dagger$ are given by,

$$\frac{da^\dagger}{dt} = \xi(t)a^\dagger + X(t)a \tag{50}$$

and

$$\frac{da}{dt} = \bar{\xi}(t)a + \bar{X}(t)a^\dagger \tag{51}$$

where X, ξ contains first and higher order time derivative of the parameters $\alpha(t), \gamma(t)$. As they are slowly varying with time, we will drop the second and higher order derivatives.
We finally obtain the leading behaviour for adiabatic transport around a closed loop Γ accomplished in time $\mathcal{T}$, to get

$$a^\dagger(\mathcal{T}) = a^\dagger(0) exp\left[i\int_0^{\mathcal{T}} \Omega \, d\tau + i\int_0^{\mathcal{T}} \left(\frac{1}{\Omega}\right)\frac{d\gamma}{d\tau} d\tau\right] \tag{52}$$

Dependence on non-commutative parameter

In terms of the original parameters,

$$\Phi_B[\Gamma] = -\frac{\theta}{2}\int\int_S \nabla_{\mathbf{R}}\left(\frac{1}{2\sqrt{\frac{1}{2}m\omega^2(\frac{1}{2m}-\theta\dot{g}(t))-\frac{\theta^2}{4}\dot{f}^2(t)}}\right)\times\nabla_{\mathbf{R}}\dot{f}(t) \tag{53}$$

$\Rightarrow$ Whenever a system can be written in terms of Lie group generators, the occurance of geometrical phase is a natural consequence.

B D Roy, G Ghosh, *Phys. Rev. A, Vol.43 (1991) 3217-3220.*

W.-M. Zhang, D. H. Feng, and R. Gilmore, *Rev. Mod. Phys., vol. 62, pp. 867–927, 1990.*

The Hamiltonian of the system can be written in terms of the generator of the su(1,1) algebra elements.

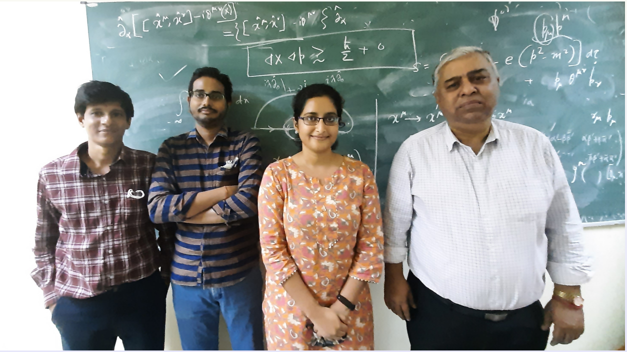
$$\tilde{H}_c = A^\mu(t)K_\mu \tag{54}$$

where $K\mu \equiv (x^2, p_x^2, xp_x + p_x x)$.

Comments

- Forced Harmonic Oscillator placed in Quantum space-time $\Rightarrow$ Dilatation term $\Rightarrow$ gives rise to Berry phase.
- $\theta = 0 \;\; \Rightarrow \text{Berry phase} = \Phi_B = 0$.

- It has been shown that two separated harmonic oscillator states, only interacting via linearalized gravitational field get entangled. Essentially this indicates the nature of quantum nature of gravity, which is seen in the non-relativistic regime.
  Future Work:
  It will be interesting to study the effect of linearized gravity when it interacts with the same system placed in a noncommutative space-time background.



THANK YOU !