Complete complementarity relations in neutrino oscillations

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Summary

1. Entanglement in neutrino oscillations

2. Quantum Correlations & nonlocality in neutrino oscillations

3. Complete Complementarity Relations in neutrino oscillations

4. Quantum Field Theory of neutrino mixing and oscillations

Motivations

- Recent interest in the study of quantumness in neutrino oscillations (entanglement, Leggett-Garg inequalities, quantum coherence, quantum correlations);

- Neutrinos as a resource for quantum information;

- Necessity for a treatment of entanglement in the context of Quantum Field Theory*;

- Neutrino mixing and oscillations in Quantum Field Theory†.

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* M.O.Terra Cunha, J.A.Dunningham and V.Vedral, Proc. Royal Soc. A (2007)
† M.B and G.Vitiello, Ann. Phys. (1995)

Entanglement in neutrino oscillations

Entanglement in neutrino mixing[‡]

– Flavor mixing (neutrinos)

$$|\nu_e\rangle = \quad \cos\theta\,|\nu_1\rangle \,+\, \sin\theta\,|\nu_2\rangle$$

$$|\nu_\mu\rangle = -\sin\theta\,|\nu_1\rangle \,+\, \cos\theta\,|\nu_2\rangle$$

- Correspondence with two-qubit states:

$$|\nu_1\rangle \equiv |1\rangle_1 |0\rangle_2 \equiv |10\rangle, \qquad |\nu_2\rangle \equiv |0\rangle_1 |1\rangle_2 \equiv |01\rangle,$$

$|\rangle_i$ denotes states in the Hilbert space for neutrinos with mass m_i.

$\Rightarrow$ flavor states are entangled superpositions of the mass eigenstates:

$$|\nu_e\rangle = \quad \cos\theta\,|10\rangle \,+\, \sin\theta\,|01\rangle.$$

[‡]M.B., M. Di Mauro and P.Jizba, J. Phys. Conf. Ser. (2007); M.B., F.Dell'Anno, S.De Siena and F.Illuminati, EPL (2009).

Composite structure of Hilbert space for neutrinos

– Necessity of tensor-product structure of Hilbert space for two generations:

Hilbert spaces for fields with different masses are orthogonal to each other[§]

For example, two scalar fields with different masses

$$(\Box + \mu_1^2)\phi_1(x) = 0 \quad , \qquad (\Box + \mu_2^2)\phi_2(x) = 0$$

with boundary conditions $\phi_1(0,\mathbf{x}) = \phi_2(0,\mathbf{x})$ and $\dot{\phi}_1(0,\mathbf{x}) = \dot{\phi}_2(0,\mathbf{x})$

One obtains

$${}_1\langle 0|0\rangle_2 \simeq \exp\left\{-\frac{V}{64\pi^2}\int_0^\infty dk \, \frac{(\mu_1^2-\mu_2^2)^2}{k^2}\right\}$$

which vanishes in the infinite volume limit.

[§] G.Barton, Introduction to Advanced Field Theory, Intersc. Publ. (1963)

Entanglement - mathematical definition

- Given a bipartite system $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B$, a system is entangled, iff

$$\rho_{AB} \neq \sum_k p_k \, \rho_k^{(A)} \otimes \rho_k^{(B)}$$

with $0 \leq p_i \leq 1, \quad \sum_k p_k = 1.$

- For a generic pure state of the form:

$$|\psi\rangle_{AB} = \sum_{ij} c_{ij} |i\rangle_A \otimes |j\rangle_B$$

the condition for entanglement reads

$$|\psi\rangle_{AB} \neq |\phi\rangle_A \otimes |\chi\rangle_B$$

Single-particle entanglement*

– A state like $|\psi\rangle_{A,B} = |0\rangle_A |1\rangle_B + |1\rangle_A |0\rangle_B$ is entangled;

– entanglement among field modes, rather than particles;

– entanglement is a property of composite systems, rather than of many-particle systems;

– entanglement and non-locality are not synonyms;

– single-particle entanglement is as good as two-particle entanglement for applications (quantum cryptography, teleportation, violation of Bell inequalities, etc..).

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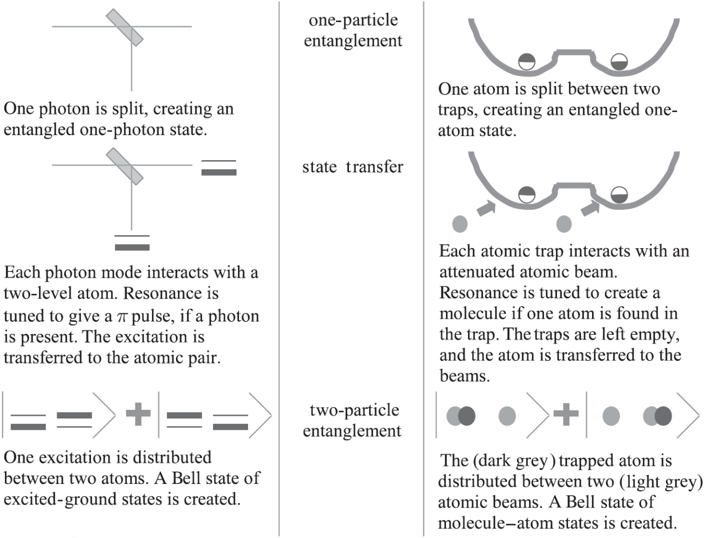
*J.van Enk, Phys. Rev. A (2005), (2006);
M.O.Terra Cunha, J.A.Dunningham and V.Vedral, Proc. Royal Soc. A (2007);
J.A.Dunningham and V.Vedral, Phys. Rev. Lett. (2007).

Protocols for extraction of single-particle entanglement †



† M.O.Terra Cunha, J.A.Dunningham and V.Vedral, Proc. Royal Soc. A (2007)

Multipartite entanglement in neutrino mixing[‡]

– Neutrino mixing (three flavors):

$$|\underline{\nu}_f\rangle \;=\; U(\tilde{\theta},\delta)\,|\underline{\nu}_m\rangle$$

with $|\underline{\nu}_f\rangle = (|\nu_e\rangle, |\nu_\mu\rangle, |\nu_\tau\rangle)^T$ and $|\underline{\nu}_m\rangle = (|\nu_1\rangle, |\nu_2\rangle, |\nu_3\rangle)^T$.

– Mixing matrix (PMNS)

$$U(\tilde{\theta},\delta) \;=\; \begin{pmatrix} c_{12}c_{13} & s_{12}c_{13} & s_{13}e^{-i\delta} \\ -s_{12}c_{23} - c_{12}s_{23}s_{13}e^{i\delta} & c_{12}c_{23} - s_{12}s_{23}s_{13}e^{i\delta} & s_{23}c_{13} \\ s_{12}s_{23} - c_{12}c_{23}s_{13}e^{i\delta} & -c_{12}s_{23} - s_{12}c_{23}s_{13}e^{i\delta} & c_{23}c_{13} \end{pmatrix},$$

where $(\tilde{\theta},\delta) \equiv (\theta_{12}, \theta_{13}, \theta_{23}; \delta)$, $c_{ij} \equiv \cos\theta_{ij}$ and $s_{ij} \equiv \sin\theta_{ij}$.

- Correspondence with three-qubit states:

$$|\nu_1\rangle \equiv |1\rangle_1|0\rangle_2|0\rangle_3 \equiv |100\rangle, \qquad |\nu_2\rangle \equiv |0\rangle_1|1\rangle_2|0\rangle_3 \equiv |010\rangle,$$

$$|\nu_3\rangle \equiv |0\rangle_1|0\rangle_2|1\rangle_3 \equiv |001\rangle$$

[‡] M.B., F.Dell'Anno, S.De Siena, M.Di Mauro and F.Illuminati, PRD (2008).

(Flavor) Entanglement in neutrino oscillations[§]

– Two-flavor neutrino states

$$|\underline{\nu}^{(f)}\rangle = \mathbf{U}(\theta,\delta)\,|\underline{\nu}^{(m)}\rangle$$

where $|\underline{\nu}^{(f)}\rangle = (|\nu_e\rangle, |\nu_\mu\rangle)^T$ and $|\underline{\nu}^{(m)}\rangle = (|\nu_1\rangle, |\nu_2\rangle)^T$ and
$\mathbf{U}(\theta) = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$.

– Flavor states at time t:

$$|\underline{\nu}^{(f)}(t)\rangle = \mathbf{U}(\theta,\delta)\,\mathbf{U}_0(t)\,\mathbf{U}(\theta,\delta)^{-1}\,|\underline{\nu}^{(f)}\rangle \equiv \widetilde{\mathbf{U}}(t)|\underline{\nu}^{(f)}\rangle\,,$$

with $\mathbf{U}_0(t) = \begin{pmatrix} e^{-iE_1 t} & 0 \\ 0 & e^{-iE_2 t} \end{pmatrix}$.

[§] M.B., F.Dell'Anno, S.De Siena and F.Illuminati, EPL (2009).

– Transition probability for $\nu_\alpha \to \nu_\beta$

$$P_{\nu_\alpha \to \nu_\beta}(t) \,=\, |\langle \nu_\beta | \nu_\alpha(t) \rangle|^2 \,=\, |\widetilde{\mathbf{U}}_{\alpha\beta}(t)|^2 \,.$$

• We now take the flavor states at initial time as our qubits:

$$|\nu_e\rangle \equiv |1\rangle_e |0\rangle_\mu \equiv |10\rangle_f, \quad |\nu_\mu\rangle \equiv |0\rangle_e |1\rangle_\mu \equiv |01\rangle_f,$$

• Starting from $|10\rangle_f$ or $|01\rangle_f$, time evolution generates the (entangled) Bell-like states:

$$|\nu_\alpha(t)\rangle \,=\, \widetilde{\mathbf{U}}_{\alpha e}(t)|1\rangle_e|0\rangle_\mu + \widetilde{\mathbf{U}}_{\alpha\mu}(t)|0\rangle_e|1\rangle_\mu, \quad \alpha = e, \mu.$$

Entanglement measure

- Let $\rho = |\psi\rangle\langle\psi|$ be the density operator for a pure state $|\psi\rangle$

Bipartition of the N-partite system $S = \{S_1, S_2, \ldots, S_N\}$ in two subsystems:

$$S_{A_n} = \{S_{i_1}, S_{i_2}, \ldots, S_{i_n}\}, \quad 1 \leq i_1 < i_2 < \ldots < i_n \leq N; (1 \leq n < N)$$

and

$$S_{B_{N-n}} = \{S_{j_1}, S_{j_2}, \ldots, S_{j_{N-n}}\} \quad , 1 \leq j_1 < j_2 < \ldots < j_{N-n} \leq N; i_q \neq j_p$$

- Reduced density matrix of S_{A_n} after tracing over $S_{B_{N-n}}$:

$$\rho_{A_n} \equiv \rho_{i_1, i_2, \ldots, i_n} = Tr_{B_{N-n}}[\rho] = Tr_{j_1, j_2, \ldots, j_{N-n}}[\rho]$$

- Linear entropy associated to such a bipartition:

$$S_L^{(A_n;B_{N-n})}(\rho) \,=\, \frac{d}{d-1}(1 - Tr_{A_n}[\rho^2_{A_n}])\,,$$

d is the Hilbert-space dimension:
$d = \min\{\dim S_{A_n}\,, \dim S_{B_{N-n}}\} = \min\{2^n, 2^{N-n}\}$.

- Average linear entropy (global entanglement):

$$\langle S_L^{(n:N-n)}(\rho)\rangle \,=\, \begin{pmatrix} N \\ n \end{pmatrix}^{-1} \sum_{A_n} S_L^{(A_n;B_{N-n})}(\rho)\,,$$

sum over all the possible bi-partitions of the system in two subsystems, respectively with n and $N-n$ elements ($1 \leq n < N$).

Entanglement measures

It is necessary to distinguish the various entanglement measures for pure and mixed states (which may contain classical correlations).

- **Measures for pure states:**
  - von Neumann entropy
  - Geometric Entanglement
- **Measures for mixed states:**
  - Entanglement of Formation and Concurrence
  - Logarithmic negativity
  - Relative Entropy of Entanglement

Entanglement in neutrino oscillations: two-flavors

Consider the density matrix for the electron neutrino state $\rho^{(e)} = |\nu_e(t)\rangle\langle\nu_e(t)|$, and trace over mode $\mu \;\Rightarrow\; \rho_e^{(e)}$.

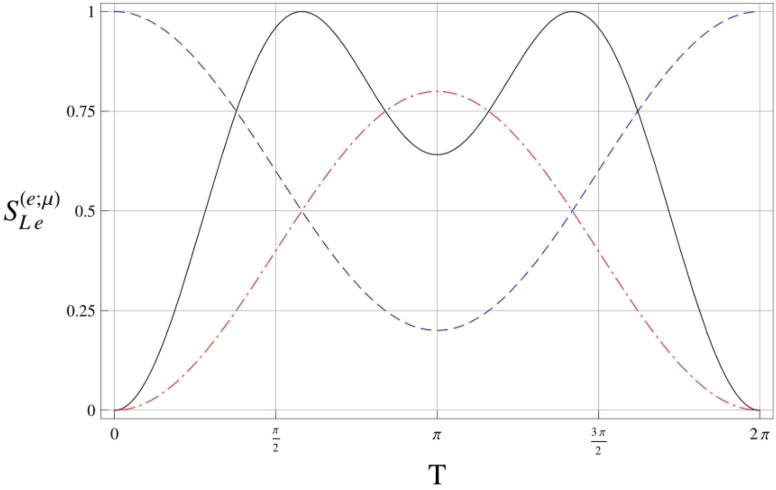
- The associated linear entropy is :

$$S_L^{(e;\mu)}(\rho^{(e)}) \,=\, 4\,|\widetilde{\mathbf{U}}_{e\mu}(t)|^2\,|\widetilde{\mathbf{U}}_{ee}(t)|^2 \,=\, 4\,P_{\nu_e\rightarrow\nu_e}(t)\,P_{\nu_e\rightarrow\nu_\mu}(t)$$

The linear entropy for the state $\rho^{(\alpha)}$ is:

$$\begin{aligned} S_{L\alpha}^{(e;\mu)} = S_{L\alpha}^{(\mu;e)} = \langle S_{L\alpha}^{(1:1)}\rangle &= 4\,|\widetilde{\mathbf{U}}_{\alpha\mu}(t)|^2\,|\widetilde{\mathbf{U}}_{\alpha e}(t)|^2 \\ &= 4|\widetilde{\mathbf{U}}_{\alpha e}(t)|^2\,(1-|\widetilde{\mathbf{U}}_{\alpha e}(t)|^2) \\ &= 4|\widetilde{\mathbf{U}}_{\alpha\mu}(t)|^2\,(1-|\widetilde{\mathbf{U}}_{\alpha\mu}(t)|^2)\,. \end{aligned}$$

- Linear entropy given by product of transition probabilities !



Linear entropy $S_{Le}^{(e;\mu)}$ (full) as a function of the scaled time $T = \frac{2Et}{\Delta m_{12}^2}$, with $\sin^2\theta = 0.314$. Transition probabilities $P_{\nu_e \to \nu_e}$ (dashed) and $P_{\nu_e \to \nu_\mu}$ (dot-dashed) are reported for comparison.

Other results

- Generalization to three flavors. Extension to wave packets (decoherence);*
- Flavor entanglement in Quantum Field Theory.†
- ν–oscillations as a resource for quantum information - Experimental scheme for the transfer of the flavor entanglement of a neutrino beam into a single-particle system with *spatially separated modes*.

*M.B., F.Dell'Anno, S.De Siena and F.Illuminati, EPL (2015).

†M.B., F.Dell'Anno, S.De Siena and F.Illuminati, EPL (2014).

Quantum Correlations & nonlocality in neutrino oscillations

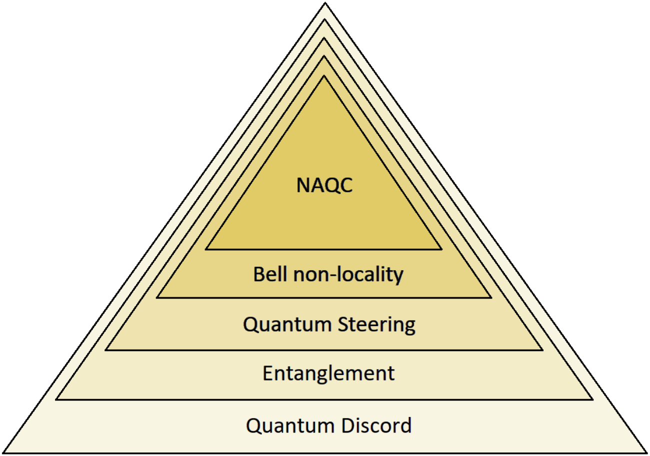
Quantum Correlations

Quantum systems exhibit properties that are beyond our understanding of reality. They show correlations that have no classical counterpart.

Entanglement is the most known of these correlations. But the terminology *quantum correlations* refers to a broader concept:

- **Quantum correlations related to entanglement:**
  - Bell non-locality
  - Entanglement
  - Quantum steering
- **Quantum correlations beyond entanglement:**
  - Quantum discord

Quantum Correlations[‡]



Hierarchy of quantum correlations

[‡] A. Streltsov, G. Adesso, M. B. Plenio. Rev. Mod. Phys. (2017)

Quantum correlations in neutrino oscillations

- Recently, quantum correlations have been investigated in the context of high-energy particle physics;

Focus on neutrinos and mesons, which are candidates for quantum information applications beyond photons.

Quantum correlations in neutrino oscillations (partial list):

A.K. Alok et al., *Quantum correlations in terms of neutrino oscillation probabilities* Nuc. Phys. B (2016)

J.A. Formaggio et al., *Violation of the Leggett- Garg Inequality in Neutrino Oscillation* Phys. Rev. Lett. (2016).

X.-S. Song et al. *Quantifying quantum coherence in experimentally observed neutrino oscillations* Phys. Rev. A (2018)

J.Naikoo et al. *Leggett-Garg inequality in the context of three flavor neutrino oscillation* Phys. Rev. D (2019)

K. Dixit et al., *Study of coherence and mixedness in meson and neutrino systems* Eur. Phys. J. C (2019)

F. Ming et al. *Quantification of quantumness in neutrino oscillations* Eur. Phys. J. C (2020)

L.-J. Li et al. *Characterizing entanglement and measurement's uncertainty in neutrino oscillations* Eur. Phys. J. C (2021)

P.Kurashvili et al *Coherence and mixedness of neutrino oscillations in a magnetic field* Eur. Phys. J. C (2021)

S.Shafaq and P.Mehta *Enhanced violation of Leggett–Garg inequality in three flavour neutrino oscillations via non-standard interactions* J. Phys. G (2021)

K.Dixit, A.K.Alok *New physics effects on quantum coherence in neutrino oscillations* Eur. Phys. J. P (2021)

A.K.Jha, S.Mukherjee, B.A.Bambah *Tri-partite entanglement in neutrino oscillations* Mod. Phys. Lett. A (2021)

A.K.Jha, A. Chatla *Quantum studies of neutrinos on IBMQ processors* Eur. Phys. J. S. T.(2022)

B. Yadav, T.Sarkar, K.Dixit, A.K.Alok *Can NSI affect non-local correlations in neutrino oscillations?* Eur. Phys. J. C (2022)

Z. Askaripour Ravari et al. *Quantum coherence in neutrino oscillation in matter* Eur. Phys. J. P (2022)

Y.W.Li et al. *Genuine tripartite entanglement in three-flavor neutrino oscillations* arXiv preprint arXiv:2205.11058, 2022

A.K.Jha, A.Chatla, B.A.Bambah *Neutrinos as Qubits and Qutrits* arXiv:2203.13485 (2022)

Non-local Advantage of Quantum Coherence[†]

- A state is said to be coherent provided that there are non-zero non-diagonal elements in its matrix representation.

Coherence is quantified by means of the l_1-norm of coherence:[*]

$$C_{l_1}(\rho) = \sum_{i \neq j} |\rho_{i,j}|$$

If the qubit is prepared in either spin up or spin down state along z-direction, then the qubit is incoherent in z-basis ($C^z_{l_1} = 0$) and fully coherent in x- and y-basis ($C^{x(y)}_{l_1} = 1$).

The upper bound beyond which the effects of non-locality emerge is given by:

$$\sum_{i=x,y,z} C_i^{l_1}(\rho) \leq C_{max}.$$

[*] T.Baumgratz, M.Cramer and M.B.Plenio, Phys. Rev. Lett. (2014).

[†] D. Mondal, T. Pramanik, A.K. Pati, Phys. Rev. A (2017).

Non-local Advantage of Quantum Coherence

Consider a bipartite system made of two spatially separated subsystems. Alice performs a measurement Π_i^b on the eigenbasis of σ_i on A and obtains the outcome $b = \{0, 1\}$ with probability $p_{\Pi_i^b} = \mathrm{Tr}[(\Pi_i^b \otimes \mathbf{1})\rho_{AB}]$.

The measured state for the two-qubit state can be obtained as $\rho_{AB|\Pi_i^b} = (\Pi_i^b \otimes \mathbf{1})\rho_{AB}(\Pi_i^b \otimes \mathbf{1})/p_{\Pi_i^b}$ and the conditional state for qubit B is $\rho_{B|\Pi_i^b} = \mathrm{Tr}_A(\rho_{AB|\Pi_i^b})$.

Then Alice tells Bob her measurement choice and Bob has to measure the coherence of qubit B at random in the eigenbases of the other two Pauli matrices σ_j and σ_k.

If the above condition for locality is violated then we cannot have a single-system description of the coherence of subsytem B.

The criterion for achieving a NAQC of qubit B can be written as:

$$N_{l_1}(\rho_{AB}) = \frac{1}{2}\sum_{i,j,b} p(\rho_{B|\Pi^b_{j\neq i}}) C^{\sigma_i}_{l_1}(\rho_{B|\Pi^b_{j\neq i}}) > \sqrt{6}.$$

Quantification of quantumness in neutrino oscillations

- Recently[‡], quantumness in neutrino oscillations has been quantified through correlation measures such as Non-local Advantage of Quantum Coherence (NAQC), quantum steering and Bell non-locality.

– The criterion for NAQC is:

$$N^{l_1}(\rho_{AB}) = \frac{1}{2}\sum_{i,j,b} p(\rho^{b}_{\Pi_{j\neq i}})C^{\sigma_i}_{l_1}(\rho_{B|\Pi_{j\neq i}}) > \sqrt{6}.$$

– Bell non-locality (violation of CHSH inequality):

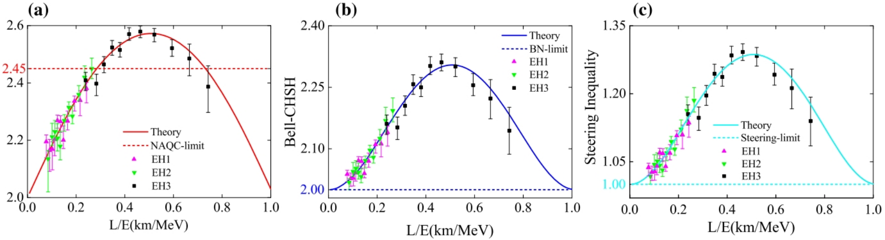
$$B(\rho_{AB}) = |\langle B_{CHSH}\rangle| \leq 2.$$

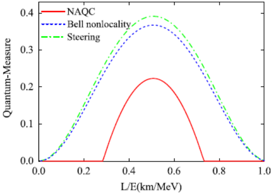
– Quantum steering:

$$F_n(\rho_{AB}, \varsigma) = \frac{1}{\sqrt{n}}\left|\sum_{i=1}^{n} Tr(\rho_{AB} A_i \otimes B_i)\right| \leq 1.$$

[‡] F. Ming, X-K. Song, D. Wang, Eur. Phys. J. C (2020)

Quantumness in neutrino oscillations (Daya Bay) §

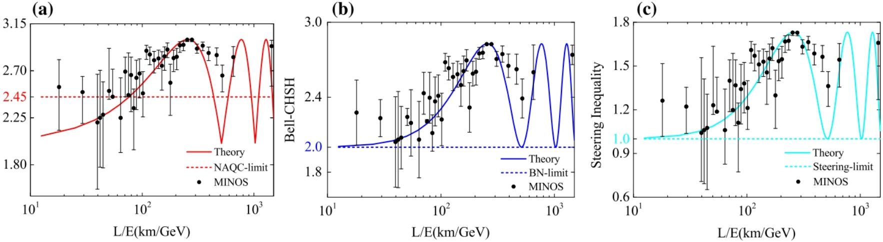


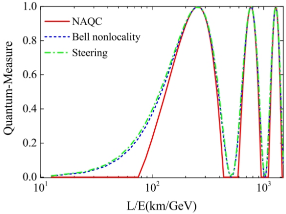


- Daya Bay: $\sin^2 2\theta_{13} = 0.084$ and $\Delta m^2_{ee} = 2.42 \times 10^{-3} eV^2$
- NAQC is a stronger nonclassical correlation than Bell non-locality and quantum steering.

§F. Ming, X-K. Song, D. Wang, Eur. Phys. J., (2020)

Quantumness in neutrino oscillations (MINOS) [¶]





- MINOS: $\sin^2 2\theta_{23} = 0.95$ and $\Delta m_{32}^2 = 2.32 \times 10^{-3} eV^2$.
- The NAQC is a stronger nonclassical correlation than Bell non-locality and quantum steering.

[¶] F. Ming, X-K. Song, D. Wang, Eur. Phys. J., (2020)

NAQC & Bell nonlocality in the wave packet approach

• We have extended the studies on quantumness of neutrino oscillations through NAQC using the wave packet approach.[‖]

Neutrino with definite flavor:

$$|\nu_\alpha(x,t)\rangle = \sum_j U^*_{\alpha j}\psi_j(x,t)|\nu_j\rangle$$

where:

$$\psi_j(x,t) = \frac{1}{\sqrt{2\pi}}\int dp\ \psi_j(p)e^{ipx-iE_j(p)t}$$

with:

$$\psi_j(p) = \left(2\pi{\sigma_p^P}^2\right)^{-\frac{1}{4}}\exp-\frac{(p-p_j)^2}{4{\sigma_p^P}^2}$$

[‖] M.B., S. De Siena and C. Matrella, Eur. Phys. J. C (2021)

Wave packet description of neutrino oscillations

Assume the condition $\sigma_p^P \ll E_j^2(p_j)/m_j$. Then we have:

$$E_j(p) \simeq E_j + v_j(p - p_j)$$

Integrating on p, one gets the wave packet in coordinate space:

$$\psi_j(x,t) = \left(2\pi {\sigma_x^P}^2\right)^{-\frac{1}{4}} \exp\left[-iE_j t + ip_j x - \frac{(x - v_j t)^2}{4{\sigma_x^P}^2}\right]$$

Write density matrix operator $\rho_\alpha(x,t) = |\nu_\alpha(x,t)\rangle\langle\nu_\alpha(x,t)|$. After time integration, one gets the oscillation formula in space

$$P_{\alpha\beta}(L) = \sum_{j,k} U_{\alpha j}^* U_{\alpha k} U_{\beta j}^* U_{\beta k} \exp\left[-2\pi i \frac{L}{L_{jk}^{osc}} - \left(\frac{L}{L_{jk}^{coh}}\right)^2 - 2\pi^2(1-\xi)^2\left(\frac{\sigma_x}{L_{jk}^{osc}}\right)^2\right]$$

Wave packet description of neutrino oscillations**

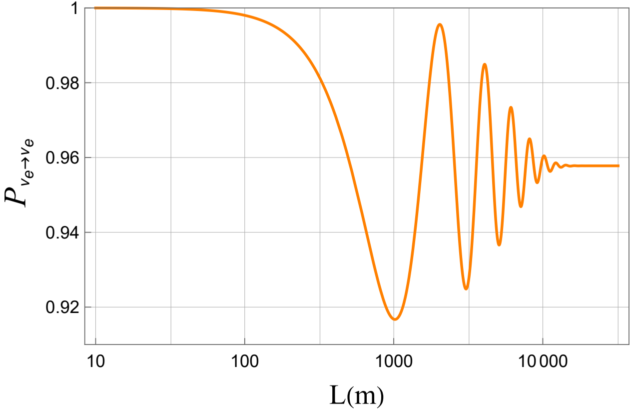
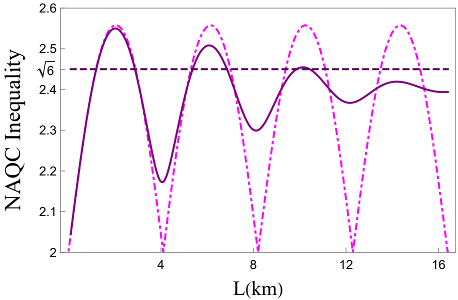
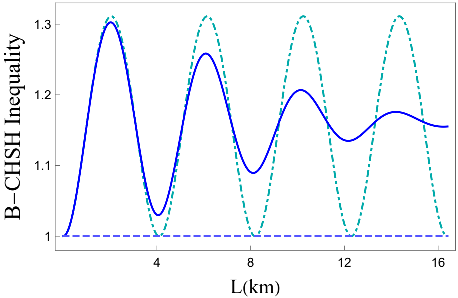


Figure 1: Survival probability in the wave packet approach. $E = 2\,MeV$, $\xi = 0$, $\sin^2 2\theta_{13} = 0.084$ and $\Delta m^2_{ee} = 2.42 \times 10^{-3} eV^2$ and $\sigma_x = 3.3 \times 10^{-6} m$.

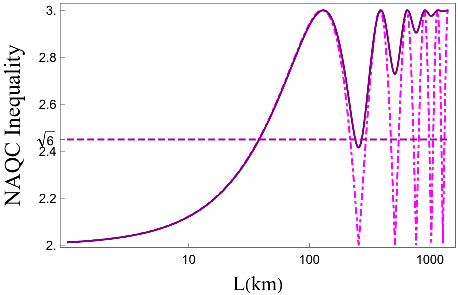
**C. Giunti, C.W. Kim, *Fundamentals of Neutrino Physics and Astrophysics*, Oxford University Press (2007)

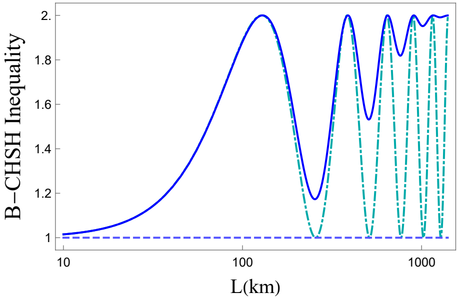
NAQC in the wave packet approach (Daya Bay)





NAQC in the wave packet approach (MINOS)





Results and perspectives

- Our treatment based on wave packets leads to a better agreement with experimental data in the case of MINOS.*

- NAQC has a different long-distance behaviour for the two experiments, due to the different values of the mixing angle.

- Existence of a "critical" angle for which NAQC exceeds the bound.

* M.B., S. De Siena and C. Matrella, Eur. Phys. J. C (2021)

Complete Complementarity Relations in neutrino oscillations

Complete Complementarity Relations

To better understand the above results, we resort to the recently introduced concept of CCR.

- N.Bohr (1928): complementarity principle
- W.K.Wootters and W.H.Zurek,*Complementarity in the double-slit experiment: quantum nonseparability and a quantitative statement of Bohr's principle*, Phys. Rev. D (1979)
- M.Jakob and J.A.Bergou, *Quantitative complementarity relations in bipartite systems: entanglement as a physical reality*, Opt. Comm. (2010)
- M.L.W.Basso and J.Maziero, *Complete complementarity relations for multipartite pure states*, J. Phys. A (2020)

N.Bohr (1928)[†]: Complementarity Principle

• Complementarity: a quantum system may possess properties which are equally real but mutually exclusive.

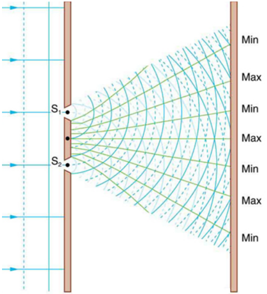
It is often associated with wave-particle duality, the complementarity aspect between propagation and detection.

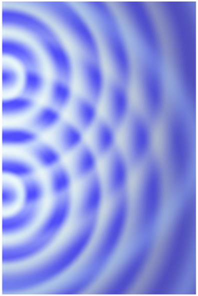
In the double-slit interferometer, the wave aspect is characterized by the **interference fringes visibility**, while the particle nature is given by the **which-way information** of the path along the interferometer.

[†] N. Bohr, *The quantum postulate and the recent development of atomic theory*, Nature (1928)

Double-slit

Usual view on complementarity: The complete knowledge of the path destroys the interference pattern visibility and vice-versa.





Quantitative wave-particle duality

- Wootters and Zurek *: first quantitative version of the wave-particle duality. A path-detecting device can give incomplete which-way information and a sharply interference pattern can still be retained.

Their work was then extended and formulated in terms of a complementarity relation†

$$P^2 + V^2 \leq 1$$

where P is the predictability and V is the visibility.

- A "**quanton**"‡ may behave partially as a wave or as a particle at the same time.

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* W.K.Wootters and W.H.Zurek, Phys. Rev. D (1979)

† D.M.Greenberger and A.Yasin, Phys.Lett. A (1988); B.-G. Englert, Phys. Rev. Lett. (1996).

‡ J.-M.Lévy-Leblond, Physica (1988)

Triality relation

- For bipartite systems a complete complementarity relation (CCR) can be obtained by including the correlations between A and B subsystems[§]:

$$V_k^2 + P_k^2 + C^2 = 1$$

  - V_k and P_k, $k = 1, 2$, generate *local* single-partite realities which can be related to wave-particle duality.
  - C is the entanglement measure **concurrence** which generate an exclusive bipartite *nonlocal* reality.

[§] M.Jakob and J.A.Bergou, Opt. Comm. (2010)

Concurrence[¶]

The concurrence for a generic qubit system described by the density matrix ρ is given by

$$C(\rho) \equiv \max\{0, \lambda_1 - \lambda_2 - \lambda_3 - \lambda_4\}$$

where the λ_i are the square root of the eigenvalues λ_i^2 of the operator $\rho\tilde{\rho}$ in decreasing order, with

$$\tilde{\rho} = (\sigma_y \otimes \sigma_y)\rho^*(\sigma_y \otimes \sigma_y)$$

[¶]S. A. Hill, W. K. Wootters, Phys. Rev. Lett. (1997)

Triality relation

Consider the most general bipartite state of two qubits:

$$|\Theta\rangle = a\,|00\rangle + b\,|01\rangle + c\,|10\rangle + d\,|11\rangle$$

One obtains:

$$C = |\langle\Theta|\tilde{\Theta}\rangle| = 2|ad - bc|$$

$$V_k = 2|\langle\Theta|\sigma_k^\dagger|\Theta\rangle| \rightarrow \begin{cases} V_1 = 2|ac^* + bd^*| \\ V_2 = 2|ab^* + cd^*| \end{cases}$$

$$P_k = |\langle\Theta|\sigma_{z,k}|\Theta\rangle| \rightarrow \begin{cases} P_1 = |(|c|^2 + |d|^2) - (|a|^2 + |b|^2)| \\ P_2 = |(|b|^2 + |d|^2) - (|a|^2 + |c|^2)| \end{cases}$$

where: $|\tilde{\Theta}\rangle = (\sigma_y \otimes \sigma_y)\,|\Theta^*\rangle, \quad \sigma_k^\dagger = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad \sigma_{z,k} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$

The complementarity relation is satisfied, since the left hand side is just the square norm of the general *pure* bipartite state $|\Theta\rangle$:
$(|a|^2 + |b|^2 + |c|^2 + |d|^2)^2 = 1.$

Examples

- Bell states (maximally entangled states)

$$\Phi^{\pm} = \frac{1}{\sqrt{2}}\left(|00\rangle \pm |11\rangle\right), \quad \Psi^{\pm} = \frac{1}{\sqrt{2}}\left(|01\rangle \pm |10\rangle\right)$$

We have $C = 1, \; V_1 = V_2 = P_1 = P_2 = 0.$

- Separable state

$$|\Theta_1\rangle = \frac{1}{\sqrt{2}}\left(|00\rangle + |01\rangle\right) = \frac{1}{\sqrt{2}}|0\rangle\left(|0\rangle + |1\rangle\right)$$

In this case $C = 0, \; V_1 = P_2 = 0, \; V_2 = P_1 = 1.$

- Unbalanced state

$$|\Theta_2\rangle = \frac{1}{2}|00\rangle + \frac{\sqrt{3}}{2}|11\rangle.$$

In this case $C = \frac{\sqrt{3}}{2}, \; V_1 = V_2 = 0, \; P_1 = P_2 = \frac{1}{2}.$

Examples

• A separable state with all four terms

$$|\Theta_3\rangle = \frac{1}{2}\left(|00\rangle + |01\rangle + |10\rangle + |11\rangle\right) = \frac{1}{2}(|0\rangle + |1\rangle)(|0\rangle + |1\rangle).$$

We have $C = 0, \;\; V_1 = V_2 = 1, \;\; P_2 = P_2 = 0.$

• Unbalanced state with all four terms

$$|\Theta_4\rangle = \frac{1}{2}|00\rangle + \frac{1}{2}|01\rangle + \frac{1}{2\sqrt{2}}|10\rangle + \frac{\sqrt{3}}{2\sqrt{2}}|11\rangle.$$

In this case we have $C = \frac{\sqrt{3}-1}{2\sqrt{2}}, V_1 = \frac{\sqrt{3}+1}{2\sqrt{2}}, \;\; V_2 = \frac{\sqrt{3}+2}{4}, \;\; P_1 = 0, \;\; P_2 = \frac{1}{4}.$

Complete Complementarity Relation for pure states

Alternative form of CCR for multipartite states*.

Consider a bipartite pure state in the Hilbert space $\mathcal{H}_A \otimes \mathcal{H}_B$:

$$\rho_{A,B} = \sum_{i,k=0}^{d_A-1} \sum_{j,l=0}^{d_B-1} \rho_{ij,kl} \left|i,j\right\rangle \left\langle k,l\right|.$$

If the state of subsystem A is mixed:

$$P_{hs}(\rho_A) + C_{hs}(\rho_A) < \frac{d_A - 1}{d_A}.$$

where $P_{hs}(\rho_A)$ and $C_{hs}(\rho_A)$ are the predictability and the Hilbert-Schmidt quantum coherence (generalization of the visibility†).

*M.L.W.Basso and J.Maziero, J. Phys. A (2020)

†T. Qureshi, Quanta (2019).

CCR for pure states

• The missing information about subsystem A is being shared via correlations with the subsystem B:

$$P_{hs}(\rho_A) + C_{hs}(\rho_A) + C_{hs}^{nl}(\rho_{A|B}) = \frac{d_A - 1}{d_A}$$

– Predictability

$$P_{hs}(\rho_A) \equiv \sum_{i=0}^{d_A-1} (\rho_{ii}^A)^2 - \frac{1}{d_A}\,,$$

– Quantum coherence (visibility)

$$C_{hs}(\rho_A) \equiv \sum_{i \neq k}^{d_A-1} |\rho_{ik}^A|^2$$

– Non-local quantum coherence (entanglement)

$$C_{hs}^{nl}(\rho_{A|B}) = \sum_{i \neq k, j \neq l} |\rho_{ij,kl}|^2 - 2 \sum_{i \neq k, j < l} \Re(\rho_{ij,kj}\rho_{il,kl}^*)$$

$C_{hs}^{nl}(\rho_{A|B})$ is equivalent to the linear entropy of subsystem A.

CCR for pure states - entropic formulation

- Another form of CCR can be obtained by defining the predictability and the coherence measures in terms of the von Neumann entropy:

$$C_{\mathrm{re}}(\rho_A) + P_{vn}(\rho_A) + S_{vn}(\rho_A) = \log_2 d_A$$

where

$$C_{\mathrm{re}}(\rho_A) = S_{vn}(\rho_{A\mathrm{diag}}) - S_{vn}(\rho_A)$$

$$P_{vn}(\rho_A) \equiv \log_2 d_A - S_{vn}(\rho_{A\mathrm{diag}})$$

For pure states $S_{vn}(\rho_A) = -\mathrm{Tr}\,(\rho_A \log_2 \rho_A)$ is a measure of entanglement between A and B.

CCR for mixed states*

- For mixed states, $S_{vn}(\rho_A)$ does not quantify entanglement, but it is just a measure of mixedness of A. CCR have to be modified:

$$P_{vn}(\rho_A) + C_{re}(\rho_A) + I_{A:B}(\rho_{AB}) + S_{A|B}(\rho_{AB}) = \log_2 d_A,$$

where:

- $P_{vn}(\rho_A) \equiv \ln d_A - S_{vn}(\rho_{Adiag})$ is the **predictability** measure;
- $C_{re}(\rho_A) = S_{vn}(\rho_{Adiag}) - S_{vn}(\rho_A)$ is the **relative entropy of coherence**;
- $I_{A:B}(\rho_{AB}) = S_{vn}(\rho_A) + S_{vn}(\rho_B) - S_{vn}(\rho_{AB})$ is the **mutual information** of A and B;
- $S_{A|B}(\rho_{AB}) = S_{vn}(\rho_{AB}) - S_{vn}(\rho_B)$ is the **conditional entropy**: It tells how much it is convenient knowing about subsystem A with respect to the whole system.

* M.L.W.Basso and J.Maziero, EPL (2021)

CCR for oscillating neutrinos[†]

- We now consider the CCR for neutrino oscillations, both for pure and mixed states.

Let us consider a two-flavor neutrino state:

$$|\nu_\alpha(t)\rangle = a_{\alpha\alpha}(t)\,|\nu_\alpha\rangle + a_{\alpha\beta}(t)\,|\nu_\beta\rangle$$

We can use the following correspondence:

$$|\nu_\alpha\rangle = |1\rangle_\alpha \otimes |0\rangle_\beta = |10\rangle$$
$$|\nu_\beta\rangle = |0\rangle_\alpha \otimes |1\rangle_\beta = |01\rangle$$

For an initial electronic neutrino, we have:

$$|\nu_e(t)\rangle = a_{ee}\,|10\rangle + a_{e\mu}\,|01\rangle$$

[†] V.Bittencourt, M.B., S.De Siena and C.Matrella, Eur. Phys. J. C. (2022)

CCR for oscillating neutrinos

The corresponding density matrix is:

$$\rho_{e\mu} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & |a_{e\mu}|^2 & a_{ee}a^*_{e\mu} & 0 \\ 0 & a_{e\mu}a^*_{ee} & |a_{ee}|^2 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

The state of subsystems e and μ are:

$$\rho_e = \begin{pmatrix} |a_{ee}|^2 & 0 \\ 0 & |a_{e\mu}|^2 \end{pmatrix}; \qquad \rho_\mu = \begin{pmatrix} |a_{e\mu}|^2 & 0 \\ 0 & |a_{ee}|^2 \end{pmatrix}$$

CCR for oscillating neutrinos

We verify that the CCRs for pure states are verified in the case of neutrino. We find:[‡]

$$
\begin{aligned}
P_{hs}(\rho_e) &= P_{ee}^2 + P_{e\mu}^2 - \frac{1}{2} \\
C_{hs}(\rho_e) &= 0 \\
C_{hs}^{nl}(\rho_{e\mu}) &= 2P_{ee}P_{e\mu}
\end{aligned}
$$

where $|a_{ee}|^2 = P_{ee}$, $|a_{e\mu}|^2 = P_{e\mu}$ and $P_{ee} + P_{e\mu} = 1$.

Thus:

$$P_{hs}(\rho_e) + C_{hs}(\rho_e) + C_{hs}^{nl}(\rho_{e\mu}) = \frac{1}{2}$$

as expected.

[‡] V.Bittencourt, M.B., S.De Siena and C.Matrella, Eur. Phys. J. C. (2022)

CCR for oscillating neutrinos

Analogously:

$$
\begin{aligned}
P_{vn}(\rho_e) &= 1 + |a_{ee}|^2 \log_2 |a_{ee}|^2 + |a_{e\mu}|^2 \log_2 |a_{e\mu}|^2 \\
C_{re}(\rho_e) &= 0 \\
S_{vn}(\rho_e) &= -|a_{ee}|^2 \log_2 |a_{ee}|^2 - |a_{e\mu}|^2 \log_2 |a_{e\mu}|^2
\end{aligned}
$$

and the CCR is verified:

$$P_{vn}(\rho_e) + C_{re}(\rho_e) + S_{vn}(\rho_e) = 1$$

CCR for neutrino mixed state

In a wave-packet description of neutrino oscillations, one starts with a pure state $\rho_\alpha(x,t)$ which become mixed after time integration:

$$\rho_\alpha(x) = \sum_{k,j} U_{\alpha k} U^*_{\alpha j} f_{jk}(x) \left|\nu_j\right\rangle \left\langle \nu_k\right| ,$$

where:

$$f_{jk}(x) = \exp\left[-i\frac{\Delta m^2_{jk} x}{2E} - \left(\frac{\Delta m^2_{jk} x}{4\sqrt{2}E^2\sigma_x}\right)^2\right]$$

By considering:

$$\left|\nu_i\right\rangle = \sum_\alpha U_{\alpha i} \left|\nu_\alpha\right\rangle , \qquad \left|\nu_\alpha\right\rangle = \left|\delta_{\alpha e}\right\rangle_e \left|\delta_{\alpha\mu}\right\rangle_\mu \left|\delta_{\alpha\tau}\right\rangle_\tau$$

we can write:

$$\rho_\alpha(x) = \sum_{\beta\gamma} F^\alpha_{\beta\gamma}(x) \left|\delta_{\beta e}\delta_{\beta\mu}\delta_{\beta\tau}\right\rangle \left\langle \delta_{\gamma e}\delta_{\gamma\mu}\delta_{\gamma\tau}\right|$$

where:

$$F^\alpha_{\beta\gamma}(x) = \sum_{kj} U^*_{\alpha j} U_{\alpha k} f_{jk}(x) U_{\beta j} U^*_{\gamma k}$$

CCR for neutrino mixed state

- We consider the CCR in the case of a two-flavor neutrino oscillation, for an initial electron neutrino

$$P_{vn}(\rho_e) + C_{re}(\rho_e) + I_{A:B}(\rho_{e\mu}) + S_{e|\mu}(\rho_{e\mu}) = \log_2 d_e,$$

where:

$$\begin{aligned}
P_{vn}(\rho_e) &= \log_{d_e} - S_{vn}(\rho_{e_{diag}}) \\
C_{re}(\rho_e) &= S_{vn}(\rho_{e_{diag}}) - S_{vn}(\rho_e) \\
I_{A:B}(\rho_{e\mu}) &= S_{vn}(\rho_e) + S_{vn}(\rho_\mu) - S_{vn}(\rho_{e\mu}) \\
S_{e|\mu}(\rho_{e\mu}) &= S_{vn}(\rho_{e\mu}) - S_{vn}(\rho_\mu)
\end{aligned}$$

For a generic matrix ρ, the von Neumann entropy is defined as $S_{vn}(\rho) = -\sum_i \lambda_i \log_2 \lambda_i$, where λ_i are the eigenvalues of ρ.

CCR for neutrino mixed state

The starting density matrix is:

$$\rho_{e\mu}(x) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & F^e_{ee} & F^e_{e\mu} & 0 \\ 0 & F^e_{\mu e} & F^e_{\mu\mu} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

and the reduced density matrices are:

$$\rho_e(x) = \begin{pmatrix} F^e_{ee} & 0 \\ 0 & F^e_{\mu\mu} \end{pmatrix} \qquad \rho_\mu(x) = \begin{pmatrix} F^e_{\mu\mu} & 0 \\ 0 & F^e_{ee} \end{pmatrix}$$

By evaluating the eigenvalues of these matrices, we obtain:

$$P_{vn}(\rho_e) = 1 + F^e_{ee} \log_2 F^e_{ee} + F^e_{\mu\mu} \log_2 F^e_{\mu\mu}$$

$$C_{re}(\rho_e) = 0$$

$$I_{e:\mu}(\rho_{e\mu}) + S_{e|\mu}(\rho_{e\mu}) = -F^e_{ee} \log_2 F^e_{ee} - F^e_{\mu\mu} \log_2 F^e_{\mu\mu}$$

By adding all the terms we find that the CCR for mixed states is satisfied for a neutrino state.

CCR for neutrino mixed state

We[§] find that the sum of the non-local terms of the CCR is equal to the Quantum Discord, defined as:

$$QD(\rho_{AB}) = I(\rho_{AB}) - CC(\rho_{AB}),$$

where $I(\rho_{AB})$ is the total correlations between the subsystems A and B; and $CC(\rho_{AB})$ quantifies the classical correlations. We have

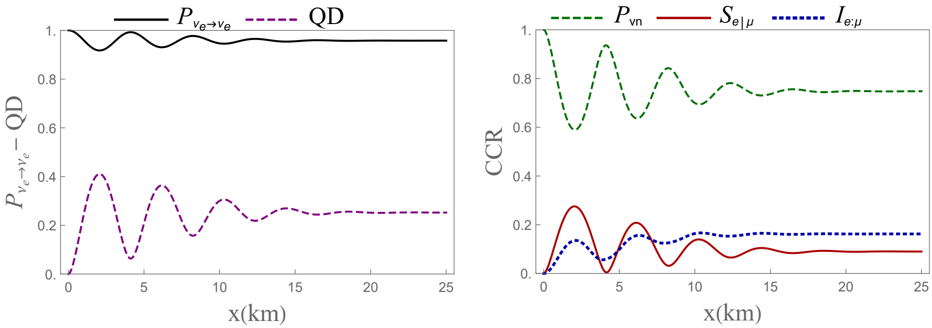
$$QD(\rho_{AB}) = S_{\mathrm{vn}}(\rho_B) - S_{\mathrm{vn}}(\rho_{AB}) + \min_{\{\Pi_i^b\}} S_{\mathrm{vn},\{\Pi_i^b\}}(\rho_{A|B})$$

that, for the neutrino density matrix under consideration, gives

$$QD(\rho_{e\mu}) = -F_{ee}^{e} \log_2 F_{ee}^{e} - F_{\mu\mu}^{e} \log_2 F_{\mu\mu}^{e}$$

[§] V.Bittencourt, M.B., S.De Siena and C.Matrella, Eur. Phys. J. C. (2022)

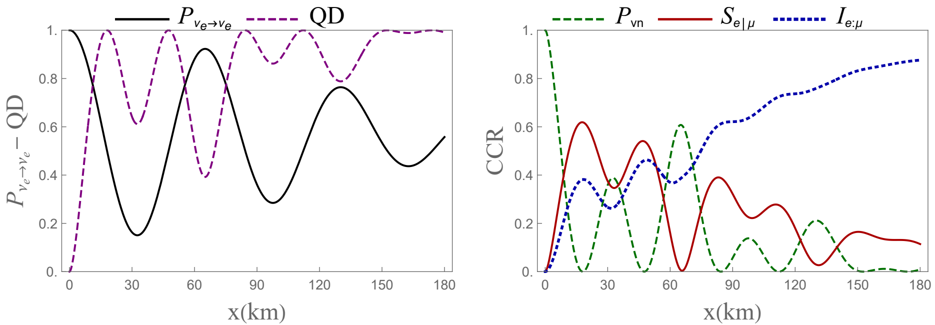
CCR for neutrino oscillations - DAYA BAY



(a) *DAYA BAY* $(L \in [364m, 1912m])$

$$\Delta m^2_{ee} = 2.42 \times 10^{-3} eV^2,\ \sin^2 2\theta_{13} = 0.084,\ E = 4MeV$$

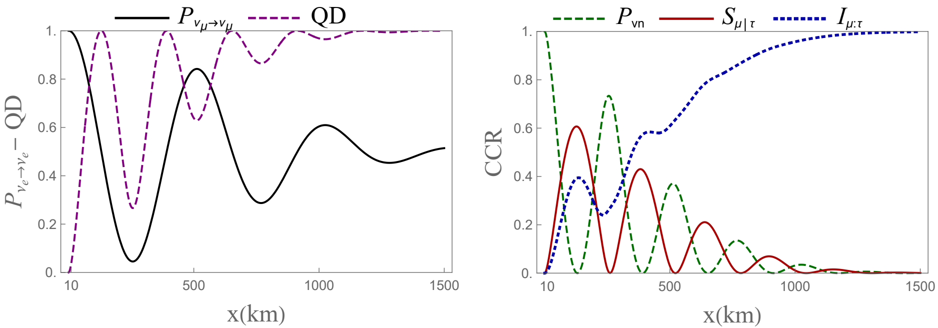
CCR for neutrino oscillations - KamLAND



(b) *KamLAND (L = 180 Km)*

$$\Delta m_{12}^2 = 7.49 \times 10^{-5} eV^2,\ \tan^2 2\theta_{12} = 0.47,\ E = 2MeV$$

CCR for neutrino oscillations - MINOS



(c) *MINOS ($L = 735$ km)*

$$\Delta m^2_{32} = 2.32 \times 10^{-3} eV^2,\ \sin^2 2\theta_{23} = 0.95,\ E = 0.5 GeV$$

Observations

In the case of the wave-packet approach, the terms included in the CCR show nontrivial characteristics:

- Different values of the mixing angle lead to very different behaviors: In the KamLand and Minos experiments (higher values of the mixing angle) the mutual information grows "engulfing" the other two terms. This aspect is not present for the Daya Bay parameters.
- By looking to the left panels for KamLand and Minos experiments, it is very difficult to recognize in the mutual information a behaviour exclusively dependent on the oscillation probability.

Results and perspectives

- We have studied CCR for the oscillating neutrino systems, both in the pure and in the mixed case.
- Complete characterization of quantum correlations in neutrino oscillations.
- Interesting long-distance behaviour of the correlations, depending on the mixing angle.
- To be done: Extension to three flavors, multipartite entanglement.

Quantum Field Theory of neutrino mixing and oscillations

Mixing of neutrino fields

– Mixing relations for two Dirac fields

$$\nu_e(x) = \cos\theta\,\nu_1(x) + \sin\theta\,\nu_2(x)$$

$$\nu_\mu(x) = -\sin\theta\,\nu_1(x) + \cos\theta\,\nu_2(x)$$

ν_1, ν_2 are fields with definite masses.

– Mixing transformations connect the two quadratic forms:

$$\mathcal{L} = \bar{\nu}_1\,(i\,\not\partial - m_1)\,\nu_1 + \bar{\nu}_2\,(i\,\not\partial - m_2)\,\nu_2$$

and

$$\mathcal{L} = \bar{\nu}_e\,(i\,\not\partial - m_e)\,\nu_e + \bar{\nu}_\mu\,(i\,\not\partial - m_\mu)\,\nu_\mu - m_{e\mu}\,(\bar{\nu}_e\,\nu_\mu + \bar{\nu}_\mu\,\nu_e)$$

with

$$m_e = m_1\cos^2\theta + m_2\sin^2\theta,\; m_\mu = m_1\sin^2\theta + m_2\cos^2\theta,\; m_{e\mu} = (m_2 - m_1)\sin\theta\cos\theta.$$

– ν_i are free Dirac field operators:

$$\nu_i(x) = \sum_{\mathbf{k},r} \frac{e^{i\mathbf{k}\cdot\mathbf{x}}}{\sqrt{V}} \left[u^r_{\mathbf{k},i}(t)\, \alpha^r_{\mathbf{k},i} \, + \, v^r_{-\mathbf{k},i}(t)\, \beta^{r\dagger}_{-\mathbf{k},i} \right], \quad i = 1, 2.$$

– Anticommutation relations:

$$\{\nu^\alpha_i(x), \nu^{\beta\dagger}_j(y)\}_{t=t'} = \delta^3(\mathbf{x}-\mathbf{y})\delta_{\alpha\beta}\delta_{ij}\,; \quad \{\alpha^r_{\mathbf{k},i}, \alpha^{s\dagger}_{\mathbf{q},j}\} = \{\beta^r_{\mathbf{k},i}, \beta^{s\dagger}_{\mathbf{q},j}\} = \delta^3(\mathbf{k}-\mathbf{q})\delta_{rs}\delta_{ij}$$

– Orthonormality and completeness relations:

$$u^r_{\mathbf{k},i}(t) \,=\, e^{-i\omega_{k,i}t}\, u^r_{\mathbf{k},i} \quad ; \quad v^r_{\mathbf{k},i}(t) \,=\, e^{i\omega_{k,i}t}\, v^r_{\mathbf{k},i} \quad ; \quad \omega_{k,i} = \sqrt{k^2 + m_i^2}$$

$$u^{r\dagger}_{\mathbf{k},i} u^s_{\mathbf{k},i} = v^{r\dagger}_{\mathbf{k},i} v^s_{\mathbf{k},i} = \delta_{rs} \quad , \quad u^{r\dagger}_{\mathbf{k},i} v^s_{-\mathbf{k},i} = 0 \quad , \quad \sum_r (u^{r\alpha *}_{\mathbf{k},i} u^{r\beta}_{\mathbf{k},i} + v^{r\alpha *}_{-\mathbf{k},i} v^{r\beta}_{-\mathbf{k},i}) = \delta_{\alpha\beta}\ .$$

– Fock space for ν_1, ν_2:

$$\mathcal{H}_{1,2} = \left\{ \alpha^\dagger_{1,2}\ ,\ \beta^\dagger_{1,2}\ ,\ |0\rangle_{1,2} \right\}.$$

– Vacuum state $|0\rangle_{1,2} \equiv |0\rangle_1 \otimes |0\rangle_2$.

Neutrino mixing in QFT

- Mixing relations for two Dirac fields

$$\nu_e(x) = \cos\theta\,\nu_1(x) + \sin\theta\,\nu_2(x)$$

$$\nu_\mu(x) = -\sin\theta\,\nu_1(x) + \cos\theta\,\nu_2(x)$$

can be written as*

$$\nu_e^\alpha(x) = G_\theta^{-1}(t)\,\nu_1^\alpha(x)\,G_\theta(t)$$

$$\nu_\mu^\alpha(x) = G_\theta^{-1}(t)\,\nu_2^\alpha(x)\,G_\theta(t)$$

– Mixing generator:

$$G_\theta(t) = \exp\left[\theta\int d^3\mathbf{x}\,\left(\nu_1^\dagger(x)\nu_2(x) - \nu_2^\dagger(x)\nu_1(x)\right)\right]$$

For ν_e, we get $\frac{d^2}{d\theta^2}\,\nu_e^\alpha = -\nu_e^\alpha$ with i.c. $\nu_e^\alpha\big|_{\theta=0} = \nu_1^\alpha$, $\frac{d}{d\theta}\nu_e^\alpha\big|_{\theta=0} = \nu_2^\alpha$.

*M.B. and G.Vitiello, Annals Phys. (1995)

• The vacuum $|0\rangle_{1,2}$ is not invariant under the action of $G_\theta(t)$:

$$|0(t)\rangle_{e,\mu} \equiv G_\theta^{-1}(t)\;|0\rangle_{1,2}$$

• Relation between $|0\rangle_{1,2}$ and $|0(t)\rangle_{e,\mu}$: orthogonality! (for $V \to \infty$)

$$\lim_{V\to\infty}\ {}_{1,2}\langle 0|0(t)\rangle_{e,\mu} = \lim_{V\to\infty}\ e^{V\int \frac{d^3\mathbf{k}}{(2\pi)^3}\ \ln\left(1-\sin^2\theta\,|V_\mathbf{k}|^2\right)^2} = 0$$

with

$$|V_\mathbf{k}|^2 \;\;\equiv\;\; \sum_{r,s} |\ v^{r\dagger}_{-\mathbf{k},1} u^s_{\mathbf{k},2}\ |^2 \neq 0 \quad for \quad m_1 \neq m_2$$

.

Quantum Field Theory vs. Quantum Mechanics

- Quantum Mechanics:

- finite $\sharp$ of degrees of freedom.

- unitary equivalence of the representations of the canonical commutation relations (von Neumann theorem).

- Quantum Field Theory:

- infinite $\sharp$ of degrees of freedom.

- ∞ many unitarily inequivalent representations of the field algebra $\Leftrightarrow$ many vacua .

- The mapping between interacting and free fields is "weak", i.e. representation dependent (LSZ formalism)*. Example: theories with spontaneous symmetry breaking.

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*F.Strocchi, *Elements of Quantum Mechanics of Infinite Systems* (W. Sc., 1985).

- The "flavor vacuum" $|0(t)\rangle_{e,\mu}$ is a $SU(2)$ generalized coherent state[†]:

$$|0\rangle_{e,\mu} = \prod_{\mathbf{k},r} \Big[(1-\sin^2\theta\,|V_{\mathbf{k}}|^2) - \epsilon^r \sin\theta\,\cos\theta\,|V_{\mathbf{k}}|\,(\alpha^{r\dagger}_{\mathbf{k},1}\beta^{r\dagger}_{-\mathbf{k},2} + \alpha^{r\dagger}_{\mathbf{k},2}\beta^{r\dagger}_{-\mathbf{k},1})$$

$$+\,\epsilon^r \sin^2\theta\,|V_{\mathbf{k}}||U_{\mathbf{k}}|\,(\alpha^{r\dagger}_{\mathbf{k},1}\beta^{r\dagger}_{-\mathbf{k},1} - \alpha^{r\dagger}_{\mathbf{k},2}\beta^{r\dagger}_{-\mathbf{k},2}) + \sin^2\theta\,|V_{\mathbf{k}}|^2\,\alpha^{r\dagger}_{\mathbf{k},1}\beta^{r\dagger}_{-\mathbf{k},2}\alpha^{r\dagger}_{\mathbf{k},2}\beta^{r\dagger}_{-\mathbf{k},1}\Big]\,|0\rangle_{1,2}$$

- Condensation density:

$${}_{e,\mu}\langle 0(t)|\alpha^{r\dagger}_{\mathbf{k},i}\alpha^{r}_{\mathbf{k},i}|0(t)\rangle_{e,\mu} \;=\; {}_{e,\mu}\langle 0(t)|\beta^{r\dagger}_{\mathbf{k},i}\beta^{r}_{\mathbf{k},i}|0(t)\rangle_{e,\mu} \;=\; \sin^2\theta\,|V_{\mathbf{k}}|^2$$

vanishing for $m_1 = m_2$ and/or $\theta = 0$ (in both cases no mixing).

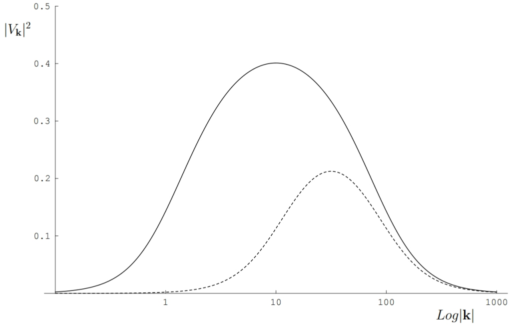
– Condensate structure as in systems with SSB (e.g. superconductors)

– Exotic condensates: mixed pairs

– Note that $|0\rangle_{e\,\mu} \neq |a\rangle_1 \otimes |b\rangle_2 \Rightarrow$ entanglement.

[†] A. Perelomov, *Generalized Coherent States,* (Springer V., 1986)

Condensation density for mixed fermions



Solid line: $m_1 = 1$, $m_2 = 100$; Dashed line: $m_1 = 10$, $m_2 = 100$.

- $V_{\mathbf{k}} = 0$ when $m_1 = m_2$ and/or $\theta = 0$.
- Max. at $k = \sqrt{m_1 m_2}$ with $V_{max} \to \frac{1}{2}$ for $\frac{(m_2 - m_1)^2}{m_1 m_2} \to \infty$.
- $|V_{\mathbf{k}}|^2 \simeq \frac{(m_2 - m_1)^2}{4k^2}$ for $k \gg \sqrt{m_1 m_2}$.

• Structure of the annihilation operators for $|0(t)\rangle_{e,\mu}$:

$$\alpha^r_{\mathbf{k},e}(t) = \cos\theta\,\alpha^r_{\mathbf{k},1} + \sin\theta\left(U^*_{\mathbf{k}}(t)\,\alpha^r_{\mathbf{k},2} + \epsilon^r V_{\mathbf{k}}(t)\,\beta^{r\dagger}_{-\mathbf{k},2}\right)$$

$$\alpha^r_{\mathbf{k},\mu}(t) = \cos\theta\,\alpha^r_{\mathbf{k},2} - \sin\theta\left(U_{\mathbf{k}}(t)\,\alpha^r_{\mathbf{k},1} - \epsilon^r V_{\mathbf{k}}(t)\,\beta^{r\dagger}_{-\mathbf{k},1}\right)$$

$$\beta^r_{-\mathbf{k},e}(t) = \cos\theta\,\beta^r_{-\mathbf{k},1} + \sin\theta\left(U^*_{\mathbf{k}}(t)\,\beta^r_{-\mathbf{k},2} - \epsilon^r V_{\mathbf{k}}(t)\,\alpha^{r\dagger}_{\mathbf{k},2}\right)$$

$$\beta^r_{-\mathbf{k},\mu}(t) = \cos\theta\,\beta^r_{-\mathbf{k},2} - \sin\theta\left(U_{\mathbf{k}}(t)\,\beta^r_{-\mathbf{k},1} + \epsilon^r V_{\mathbf{k}}(t)\,\alpha^{r\dagger}_{\mathbf{k},1}\right)$$

• Mixing transformation = Rotation + Bogoliubov transformation .

– Bogoliubov coefficients:

$$U_{\mathbf{k}}(t) \;=\; u^{r\dagger}_{\mathbf{k},2} u^r_{\mathbf{k},1}\, e^{i(\omega_{k,2}-\omega_{k,1})t} \quad ; \qquad V_{\mathbf{k}}(t) = \epsilon^r\, u^{r\dagger}_{\mathbf{k},1} v^r_{-\mathbf{k},2}\, e^{i(\omega_{k,2}+\omega_{k,1})t}$$

$$|U_{\mathbf{k}}|^2 + |V_{\mathbf{k}}|^2 = 1$$

Decomposition of mixing generator *

Mixing generator function of m_1, m_2, and θ. Try to disentangle the mass dependence from the one by the mixing angle.

Let us define:

$$R(\theta) \equiv \exp\left\{\theta \sum_{\mathbf{k},r} \left[\left(\alpha^{r\dagger}_{\mathbf{k},1}\alpha^{r}_{\mathbf{k},2} + \beta^{r\dagger}_{\mathbf{k},1}\beta^{r}_{\mathbf{k},2}\right)e^{i\psi_k} - \left(\alpha^{r\dagger}_{\mathbf{k},2}\alpha^{r}_{\mathbf{k},1} + \beta^{r\dagger}_{\mathbf{k},2}\beta^{r}_{\mathbf{k},1}\right)e^{-i\psi_k}\right]\right\},$$

$$B_i(\Theta_i) \equiv \exp\left\{\sum_{\mathbf{k},r} \Theta_{\mathbf{k},i}\ \epsilon^r \left[\alpha^{r}_{\mathbf{k},i}\beta^{r}_{-\mathbf{k},i}e^{-i\phi_{k,i}} - \beta^{r\dagger}_{-\mathbf{k},i}\alpha^{r\dagger}_{\mathbf{k},i}e^{i\phi_{k,i}}\right]\right\}, \qquad i = 1, 2$$

Since $[B_1, B_2] = 0$ we put

$$B(\Theta_1, \Theta_2) \equiv B_1(\Theta_1)\, B_2(\Theta_2)$$

*M.B., M.V.Gargiulo and G.Vitiello, Phys. Lett. B (2017)

• We find:

$$G_\theta \;=\; B(\Theta_1,\Theta_2)\;\; R(\theta)\;\; B^{-1}(\Theta_1,\Theta_2)$$

which is realized when the $\Theta_{\mathbf{k},i}$ are chosen as:

$$U_{\mathbf{k}} = e^{-i\psi_k}\;\cos(\Theta_{\mathbf{k},1}-\Theta_{\mathbf{k},2})\,; \qquad V_{\mathbf{k}} = e^{\frac{(\phi_{k,1}+\phi_{k,2})}{2}}\;\sin(\Theta_{\mathbf{k},1}-\Theta_{\mathbf{k},2})$$

The $B_i(\Theta_{\mathbf{k},i})$, $i=1,2$ are Bogoliubov transformations implementing a mass shift, and $R(\theta)$ is a rotation.

– Their action on the vacuum is given by:

$$|\widetilde{0}\rangle_{1,2} \equiv B^{-1}(\Theta_1,\Theta_2)|0\rangle_{1,2} \;=\; \prod_{\mathbf{k},r,i}\Big[\cos\Theta_{\mathbf{k},i}+\epsilon^r\sin\Theta_{\mathbf{k},i}\alpha^{r\dagger}_{\mathbf{k},i}\beta^{r\dagger}_{-\mathbf{k},i}\Big]|0\rangle_{1,2}$$

$$R^{-1}(\theta)|0\rangle_{1,2} \;=\; |0\rangle_{1,2}\;.$$

Bogoliubov vs Pontecorvo

Bogoliubov and Pontecorvo do not commute!

[ , ] ≠ 0

As a result, flavor vacuum gets a non-trivial term:

$$|0\rangle_{e,\mu} \equiv G_\theta^{-1}|0\rangle_{1,2} \;=\; |0\rangle_{1,2} \;+\; \left[B(m_1, m_2)\;,\; R^{-1}(\theta)\right]\; |\widetilde{0}\rangle_{1,2}$$

- Non-diagonal Bogoliubov transformation

$$|0\rangle_{e,\mu} \cong \left[\mathbb{1} + \theta\, a \int \frac{d^3\mathbf{k}}{(2\pi)^{\frac{3}{2}}}\; \tilde{V}_{\mathbf{k}} \sum_r \epsilon^r \left(\alpha^{r\dagger}_{\mathbf{k},1}\beta^{r\dagger}_{-\mathbf{k},2} + \alpha^{r\dagger}_{\mathbf{k},2}\beta^{r\dagger}_{-\mathbf{k},1}\right)\right]|0\rangle_{1,2}\,,$$

with $a \equiv \frac{(m_2-m_1)^2}{m_1 m_2}$.

Currents and charges for mixed fermions *

– Lagrangian in the mass basis:

$$\mathcal{L} = \bar{\nu}_m \left(i \not\partial - M_d\right) \nu_m$$

where $\nu_m^T = (\nu_1, \nu_2)$ and $M_d = \begin{pmatrix} m_1 & 0 \\ 0 & m_2 \end{pmatrix}$.

- $\mathcal{L}$ invariant under global $U(1)$ with conserved charge Q= total charge.

  – Consider now the $SU(2)$ transformation:

$$\nu_m' = e^{i\alpha_j \tau_j} \nu_m \qquad ; \qquad j = 1, 2, 3.$$

with $\tau_j = \sigma_j/2$ and σ_j being the Pauli matrices.

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* M. B., P. Jizba and G. Vitiello, Phys. Lett. B (2001)

The associated currents are:

$$\delta\mathcal{L} = i\alpha_j \, \bar{\nu}_m \, [\tau_j, M_d] \, \nu_m = -\alpha_j \, \partial_\mu J^\mu_{m,j}$$

$$J^\mu_{m,j} = \bar{\nu}_m \, \gamma^\mu \, \tau_j \, \nu_m$$

– The charges $Q_{m,j}(t) \equiv \int d^3\mathbf{x} \, J^0_{m,j}(x)$, satisfy the $su(2)$ algebra:

$$[Q_{m,j}(t), Q_{m,k}(t)] = i \, \epsilon_{jkl} \, Q_{m,l}(t) \, .$$

– Casimir operator proportional to the total charge: $C_m = \frac{1}{2} Q$.

- $Q_{m,3}$ is conserved $\Rightarrow$ charge conserved separately for ν_1 and ν_2:

$$Q_1 = \frac{1}{2} Q + Q_{m,3} = \int d^3\mathbf{x} \, \nu_1^\dagger(x) \, \nu_1(x)$$

$$Q_2 = \frac{1}{2} Q - Q_{m,3} = \int d^3\mathbf{x} \, \nu_2^\dagger(x) \, \nu_2(x).$$

These are the flavor charges in the absence of mixing.

The currents in the flavor basis

– Lagrangian in the flavor basis:

$$\mathcal{L} = \bar{\nu}_f \left(i \not\partial - M\right) \nu_f$$

where $\nu_f^T = (\nu_e, \nu_\mu)$ and $M = \begin{pmatrix} m_e & m_{e\mu} \\ m_{e\mu} & m_\mu \end{pmatrix}$.

– Consider the $SU(2)$ transformation:

$$\nu'_f = e^{i\alpha_j \tau_j} \nu_f \qquad ; \qquad j = 1, 2, 3.$$

with $\tau_j = \sigma_j/2$ and σ_j being the Pauli matrices.

– The charges $Q_{f,j} \equiv \int d^3\mathbf{x}\, J^0_{f,j}$ satisfy the $su(2)$ algebra:

$$[Q_{f,j}(t), Q_{f,k}(t)] = i\,\epsilon_{jkl}\, Q_{f,l}(t).$$

– Casimir operator proportional to the total charge $C_f = C_m = \frac{1}{2}Q$.

- $Q_{f,3}$ is not conserved $\Rightarrow$ exchange of charge between ν_e and ν_μ.

Define the flavor charges as:

$$Q_e(t) \;\equiv\; \frac{1}{2}Q + Q_{f,3}(t) = \int d^3\mathbf{x}\, \nu_e^\dagger(x)\, \nu_e(x)$$

$$Q_\mu(t) \;\equiv\; \frac{1}{2}Q - Q_{f,3}(t) = \int d^3\mathbf{x}\, \nu_\mu^\dagger(x)\, \nu_\mu(x)$$

where $Q_e(t) + Q_\mu(t) = Q$.

– We have:

$$Q_e(t) \;=\; \cos^2\theta\, Q_1 + \sin^2\theta\, Q_2 + \sin\theta\cos\theta \int d^3\mathbf{x} \left[\nu_1^\dagger \nu_2 + \nu_2^\dagger \nu_1\right]$$

$$Q_\mu(t) \;=\; \sin^2\theta\, Q_1 + \cos^2\theta\, Q_2 - \sin\theta\cos\theta \int d^3\mathbf{x} \left[\nu_1^\dagger \nu_2 + \nu_2^\dagger \nu_1\right]$$

In conclusion:

– In presence of mixing, neutrino flavor charges are defined as

$$Q_e(t) \equiv \int d^3\mathbf{x}\, \nu_e^\dagger(x)\, \nu_e(x) \quad ; \qquad Q_\mu(t) \equiv \int d^3\mathbf{x}\, \nu_\mu^\dagger(x)\, \nu_\mu(x)$$

– They are not conserved charges $\Rightarrow$ flavor oscillations.

– They are still (approximately) conserved in the vertex $\Rightarrow$ define flavor neutrinos as their eigenstates

- Problem: find the eigenstates of the above charges.

- Flavor charge operators are diagonal in the flavor ladder operators:

$$
\begin{aligned}
:: Q_\sigma(t) :: &\equiv \int d^3\mathbf{x} :: \nu_\sigma^\dagger(x)\, \nu_\sigma(x) :: \\
&= \sum_r \int d^3\mathbf{k} \left(\alpha^{r\dagger}_{\mathbf{k},\sigma}(t) \alpha^r_{\mathbf{k},\sigma}(t) - \beta^{r\dagger}_{-\mathbf{k},\sigma}(t) \beta^r_{-\mathbf{k},\sigma}(t) \right), \quad \sigma = e, \mu.
\end{aligned}
$$

Here $:: \ldots ::$ denotes normal ordering w.r.t. flavor vacuum:

$$
:: A :: \equiv A - {}_{e,\mu}\langle 0|A|0\rangle_{e,\mu}
$$

- Define flavor neutrino states with definite momentum and helicity:

$$
|\nu^r_{\mathbf{k},\sigma}\rangle \equiv \alpha^{r\dagger}_{\mathbf{k},\sigma}(0)\, |0\rangle_{e,\mu}
$$

– Such states are eigenstates of the flavor charges (at t=0):

$$
:: Q_\sigma :: |\nu^r_{\mathbf{k},\sigma}\rangle = |\nu^r_{\mathbf{k},\sigma}\rangle
$$

Neutrino oscillation formula (QFT)

– We have, for an electron neutrino state:

$$\mathcal{Q}_{\mathbf{k},\sigma}(t) \equiv \langle \nu^r_{\mathbf{k},e}| :: Q_\sigma(t) :: |\nu^r_{\mathbf{k},e}\rangle$$
$$= \left|\left\{\alpha^r_{\mathbf{k},\sigma}(t), \alpha^{r\dagger}_{\mathbf{k},e}(0)\right\}\right|^2 + \left|\left\{\beta^{r\dagger}_{-\mathbf{k},\sigma}(t), \alpha^{r\dagger}_{\mathbf{k},e}(0)\right\}\right|^2$$

• Neutrino oscillation formula (exact result)*:

$$\mathcal{Q}_{\mathbf{k},e}(t) = 1 - |U_{\mathbf{k}}|^2 \sin^2(2\theta) \sin^2\left(\frac{\omega_{k,2}-\omega_{k,1}}{2}\, t\right) - |V_{\mathbf{k}}|^2 \sin^2(2\theta) \sin^2\left(\frac{\omega_{k,2}+\omega_{k,1}}{2}\, t\right)$$

$$\mathcal{Q}_{\mathbf{k},\mu}(t) = |U_{\mathbf{k}}|^2 \sin^2(2\theta) \sin^2\left(\frac{\omega_{k,2}-\omega_{k,1}}{2}\, t\right) + |V_{\mathbf{k}}|^2 \sin^2(2\theta) \sin^2\left(\frac{\omega_{k,2}+\omega_{k,1}}{2}\, t\right)$$

- For $k \gg \sqrt{m_1 m_2}$, $|U_{\mathbf{k}}|^2 \to 1$ and $|V_{\mathbf{k}}|^2 \to 0$.

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*M.B., P.Henning and G.Vitiello, Phys. Lett. **B** (1999).

Lepton charge violation for Pontecorvo states[†]

– Pontecorvo states:

$$|\nu^r_{\mathbf{k},e}\rangle_P = \cos\theta\,|\nu^r_{\mathbf{k},1}\rangle + \sin\theta\,|\nu^r_{\mathbf{k},2}\rangle$$

$$|\nu^r_{\mathbf{k},\mu}\rangle_P = -\sin\theta\,|\nu^r_{\mathbf{k},1}\rangle + \cos\theta\,|\nu^r_{\mathbf{k},2}\rangle\,,$$

are *not* eigenstates of the flavor charges.

⇒ *violation of lepton charge conservation* in the production/detection vertices, at tree level:

$${}_P\langle\nu^r_{\mathbf{k},e}| :Q_e(0): |\nu^r_{\mathbf{k},e}\rangle_P = \cos^4\theta + \sin^4\theta + 2|U_{\mathbf{k}}|\,\sin^2\theta\cos^2\theta < 1,$$

for any $\theta \neq 0$, $\mathbf{k} \neq 0$ and for $m_1 \neq m_2$.

[†] M. B., A. Capolupo, F. Terranova and G. Vitiello, Phys. Rev. **D** (2005)
C. C. Nishi, Phys. Rev. **D** (2008).

Other results

- Rigorous mathematical treatment for any number of flavors *
- Three flavor fermion mixing: CP violation†;
- QFT spacetime dependent neutrino oscillation formula‡;
- Boson mixing§;   Majorana neutrinos¶;
- Flavor vacuum and cosmological constant‖
- Flavor vacuum induced by condensation of D-particles.**
- Geometric phase for mixed particles††.

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* K. C. Hannabuss and D. C. Latimer, J. Phys. A (2000); J. Phys. A (2003);

† M.B., A.Capolupo and G.Vitiello, Phys. Rev. **D** (2002)

‡ M.B., P. Pires Pachêco and H. Wan Chan Tseung, Phys. Rev. **D**, (2003).

§ M.B., A.Capolupo, O.Romei and G.Vitiello, Phys. Rev. **D**(2001); M.Binger and C.R.Ji. Phys. Rev. **D**(1999); C.R.Ji and Y.Mishchenko, Phys. Rev. **D**(2001); Phys. Rev. **D**(2002).

¶ M.B. and J.Palmer, Phys. Rev. **D** (2004)

‖ M.B., A.Capolupo, S.Capozziello, S.Carloni and G.Vitiello Phys. Lett. A (2004);

** N.E.Mavromatos, S.Sarkar and W.Tarantino, Phys. Rev. D (2008); Phys. Rev. D (2011).

†† M.B., P.Henning and G.Vitiello, Phys. Lett. **B** (1999)

- Dynamical generation of fermion mixing*.
- Flavor-energy uncertainty relations for mixed states†.
- Poincaré invariance for flavor neutrinos‡.
- Violation of equivalence principle§.

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*M.B., P.Jizba, N.E.Mavromatos and L.Smaldone, Phys. Rev. D (2019)
†M. B., P. Jizba and L. Smaldone, Phys. Rev. D (2019)
‡M.B., P.Jizba, N.E.Mavromatos and L.Smaldone, Phys. Rev. D (2020) ; A. E. Lobanov, Ann. Phys. (2019)
§M.B., P.Jizba, G.Lambiase and L.Petruzziello, Phys. Lett. B (2020)

Neutrino Ontology: flavor or mass?

- In view of the unitary inequivalence of mass and flavor representations, we have the problem of the fundamental (ontological) nature of neutrino.

Flavor or mass, that is the question...

Neutrino ontology: research directions

- How to verify the fundamental nature of neutrino states?

Two directions:

- Investigate the phenomenology of flavor neutrinos, with corrections expected in the non-relativistic regime: oscillations, beta decay endpoint, quantum correlations, ...
- Use the formal consistency of QFT on curved backgrounds, by comparing neutrino processes in two different frames (inertial and comoving).