Spin $\frac{1}{2}$ from Colour and a Little More

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Remarks on QCD θ-vacua

- Gauge Group $\mathcal{G} =$ Maps from S^3 to $SU(N)$.
- There are winding number transformations g^n in $\mathcal{G}$, $n \in \mathbb{Z}$ since
$$\pi_3(SU(N)) = \mathbb{Z}.$$
- If $U(g)$ is the operator for winding number 1 , its action on θ-vacuum $|e^{i\theta}\rangle$, is then
$$U(g)|e^{i\theta}\rangle = e^{i\theta}|e^{i\theta}\rangle.$$
- But the questions remain :

1. What is $|e^{i\theta}\rangle$?
2. What is $U(g)$?

The answer to Q 1 is given by the "Chern-Simons twisted" vacuum :

$$|e^{i\theta}\rangle = e^{i\theta \int K(A) d^3x} |0\rangle$$

where $\int d^3x\, K(A) = \frac{1}{8\pi^2} \int d^3x\, \mathrm{tr}(AdA + \frac{2}{3}A^3)$.

One can easily see : If $A^g \equiv gDg^{-1}$:

$$\begin{aligned} \int d^3x\, K(A^g) &= \int d^3x\, K(A) + \frac{1}{24\pi^2} \int \mathrm{tr}(dgg^{-1})^3 \\ &= \int d^3x\, K(A) + \text{winding number } n \text{ of } g. \end{aligned}$$

But what is the operator $U(g)$ such that

$$U(g) A U(g)^{-1} = gDg^{-1}?$$

REMARKS ON GAUSS LAW

For finding U(g), we turn to Gauss law.

Gauss law is typically written as

$$(D \cdot E + J_0)|\cdot\rangle = 0.$$

It is best to write this using Lie-algebra-valued test functions $\Xi(x)$:

$$\Xi(x) \to 0 \quad \text{as} \quad |\mathbf{x}| \to \infty.$$

Then

$$\int d^3x \,\mathrm{tr}(-D_i \Xi E_i + \Xi J_0)|\cdot\rangle \equiv Q(\Xi)|\cdot\rangle = 0$$

where J_0 is the colour charge density.

A Large Gauge Transformation

One checks that $Q(\Xi)$ generates gauge transformations :

$$[Q(\Xi), A_i(x)] = D_i\Xi(x).$$

The particular transformation for theta vacuum has winding number equal to 1 .

Let

$$h(\vec{x}) = (\boldsymbol{\tau} \cdot \hat{\mathbf{x}})\hat{h}(r)$$

with

$$\hat{h}(r = 0) = 0, \qquad \lim_{r\to\infty} \hat{h}(r) = \pi \neq 0.$$

Using this, we consider

$$Q(h) = \int d^3x \ [-D_i h(\vec{x}) E_i + h(\vec{x}) J_0(\vec{x})] .$$

Let Ψ be a coloured field. A finite transformation on Ψ is then given by

$$e^{iQ(h)}\Psi(\vec{x})e^{-iQ(h)} = \sum_n \frac{i^n}{n!}\left[Q(h),\left[Q(h),\cdots\left[Q(h),\Psi\right]\cdots\right]\right]$$

$$= \sum_n \frac{i^n}{n!}\left((\boldsymbol{\tau}\cdot\hat{\mathbf{x}})\hat{h}(r)\right)^n\Psi(\vec{x}) = e^{i(\boldsymbol{\tau}\cdot\hat{\mathbf{x}})\hat{h}(r)}\Psi(\vec{x}) \equiv g(h)\Psi(\vec{x})$$

Here g is a Skyrmion configuration which is well-defined :

$$g(h) = \cos\hat{h}(r) + i\,(\boldsymbol{\tau}\cdot\hat{\mathbf{x}})\sin\hat{h}(r).$$

However this configuration g is not in the Gauss law group : $h \neq 0$ at infinity and hence $Q(h)$ is not a Gauss law generator.

Instead, it is a member of "large" gauge transformations.

$U(g)$ is $\exp[iQ(h)]$.

Observables commute with it.

The set of such operators with h not vanishing at infinity generate the Sky group.

It was introduced by Balachandran and S. Vaidya (arXiv:1302.3406 [hep-th]).

Next consider spatial rotation generators L_i .

They only rotate $\hat{\mathbf{x}}$, and so do not commute with Sky generators.

They cannot be implemented on theta vacuum and its folium.

They are "spontaneously broken".

Instead, the operators

$$J_i = L_i + Q\left(\frac{\tau_i}{2}\mathbb{I}\right)$$

($\mathbb{I}$ is the constant function) augmented by global colour rotation do commute with $Q\left((\boldsymbol{\tau}\cdot\hat{x})\hat{h}(r)\right)$.

They can be implemented in the θ sector.

This is an analogue of the ‘spin from isospin’ phenomenon discovered by Hasenfratz and ’t Hooft, and Jackiw and Rebbi, as well as the ’spin $\frac{1}{2}$ from gravity’ found by Friedman and Sorkin.

This is a remarkable result : one gets spin $1/2$ without spin $1/2$ fields acting on the Chern-Simons -twisted vacuum.

This phenomenon also occurs in fuzzy physics as mentioned in the abstract.

The Poincaré generators implementable in the θ sector can be written down as well.

The modified boosts are now

$$K_i = \text{standard boosts} + Q\left(\frac{i}{2}\tau_i \mathbb{I}\right)$$

giving a Majorana representation.

Note : The Majorana field breaks parity just like the Chern-Simons twist.

The axion phenomenology if built on the theta sector will be affected by this new Poincaré representation..

Remark:

The choice of the quantum state determines the representation of observables.

(GNS construction.)

We see that states can lead to different superselection sectors , a known result.

There are many more states leading to different superselection sectors.

There are in fact an uncountably many such states.

They are created by infrared effects as is known. We now explain them.

In QED, they break Lorentz invariance.

In QCD, they also break colour transformations. (Balachandran and Vaidya, Balachandran, Balachandran and V.P.Nair)

Back to the Sky Group:

For general Ξ, the generators $Q(\Xi)$ have

$$\lim_{r\to\infty} \Xi(\vec{x}) \to \Xi(\hat{x}) \neq 0$$

which gives us the Bal-Vaidya Sky Group.

Question: What are the states to excite them?

First, we look at the abelian case of QED.

Consider the photon field $A_\mu(x)$ in the Feynman gauge.

We have:

$$\langle A_\mu(x)A_\nu(y)\rangle = i\eta_{\mu\nu}D(x-y)$$

where $D(x-y)$ is the causal propagator.

The field operators act on an indefinite metric (Krein) space.

Let

$$e \cdot e = e_\lambda e^\lambda < 0$$

Note : $\eta^{00} = +1$.

$\{e\}$ is the union of de Sitter spaces.

Following Mund, Rehren and Schroer, we define *the escort infra-field*:

$$\Phi(x, e) \equiv \int_0^\infty d\tau A_\lambda(x + e\tau)e^\lambda.$$

Under a gauge transformation generated by $Q(\Xi)$,

$$\Phi(x, e) \to \Phi(x, e) + \lim_{\tau \to \infty} \Xi(x + e\tau) - \Xi(x).$$

So $e^{i\Phi(x,e)}$ creates a charge on sky in the direction e and its anti-charge at x.

In particular, for electron field ψ of charge q, one has the vector state

$$e^{iq\Phi(x,e)}\psi(x)|0\rangle$$

which is

i) Gauss law invariant (Dirac),

ii) and has a charged blip in sky along e.

The infrafield, or the 'escort' field $\Phi(x, e)$ can be written as a quantum field as follows:

$$\begin{aligned}\Phi(x,e) &= \int_0^\infty d\tau \left(\int d\mu(k) e^{ik\cdot(x+e\tau)} a_\mu(k) + * \right) e^\mu \\ &= i \int d\mu(k) \left[\frac{a_\mu(k)}{(k.e)_+} e^{ik\cdot x} - * \right] e^\mu\end{aligned}$$

where $(k.e)_+ \equiv \lim_{\epsilon\to 0_+}(k.e + i\epsilon)$.

Now

$$e^{iq\Phi(x,e)}\psi(x)|0\rangle, \qquad e^{iq\Phi(x,e')}\psi(x)|0\rangle$$

give different values for sky generators $\Rightarrow$ they are orthogonal (as explicitly shown by Mund et. al.).

But under the Lorentz group action by $U(\Lambda)$:

$$U(\Lambda)\Phi(x,e)U^{-1}(\Lambda) = \Phi(\Lambda x, \Lambda e)$$

as Λ changes e. So it changes the superselection sector.

Accordingly

Lorentz group is spontaneously broken.

$\Phi(x,e)$ is the order parameter field for this breaking.

In papers by Balachandran and S. Vaidya , Balachandran , and Balachandran and V.P. Nair, it was suggested that QCD breaks Lorentz group and colour.

Infrafield approach also confirms this.

Now,

$$V(x, e) \equiv P \exp[i \int_{x}^{x+e\,\infty} dy^{\lambda} A_{\lambda}(y)]$$

replaces $e^{i\Phi(x,e)}$.

(Here $e\,\infty$ means limit to ∞ in direction e.)

Under gauge transformations $U(g)$ (which can be written using $Q(\Xi)$),

$$V(x, e) \to g(e\,\infty)V(x, e)g^{-1}(x).$$

So if Ψ is a coloured field,

$$V(x, e)\Psi(x)|0\rangle$$

is Gauss law invariant. It transforms under $U(g)$ as per

$$V(x, e)\Psi(x)|0\rangle \to g(e\,\infty)V(x, e)\Psi(x)|0\rangle$$

so that

$$V(x, e)\Psi(x)|0\rangle,\ V(x, e')\Psi(x)|0\rangle$$

are in different superselected sectors $\Rightarrow$ they are orthogonal.

More On the Infrafield

In QED. $\Phi(x, e)$ couples to a current J^μ as:

$$\partial_\mu \Phi(x, e) J^\mu(x) \tag{1}$$

But $\partial_\mu J^\mu(x) = 0 \Rightarrow$ (1) leads to no scattering. It dresses the field ψ :

$$\psi(x) \to e^{i\Phi(x,e)} \psi(x)$$

that captures its infrared dressing completely ! (Mund et. al.)

As remarked earlier, we can regard $\Phi(x, e)$ as the Higgs field for Lorentz breaking.

The Scale of This Breaking

The short answer: it depends on what is observed.

For massive charged fields :

if

$$\hat{\psi}(x) \equiv e^{i\Phi(x.e)}\psi(x)$$

then the effect of Φ on

$$\langle\hat{\psi}(x)\hat{\psi}(y)\rangle$$

depends on the mass of ψ.

That should determine the scale where infrared effects become significant.

Goldstone Modes

For QED, they are the vibrations of de Sitter strings (!!).

The infrafield $\Phi(x, e)$ should be Lie algebra valued and should lead to no scattering.

Now if J^μ represents the matter current , one has

$$D_\mu J^\mu = 0. \tag{2}$$

Also, $V(x, e)J^\mu(x)$ is gauge invariant .

Let us consider the gauge-invariant coupling

$$\mathrm{tr}\left(\{D_\mu V(x, e)\}\, V^{-1}(x, e)\right) \times \left(V(x, e)J^\mu(x)\right)$$

By (2), this is

$$\partial_\mu \mathrm{tr}\,\{V(x, e)J^\mu(x)\}$$

and so leads to no scattering.

So

$$[D_\mu V(x,e)]\, V^{-1}(x,e) \tag{3}$$

substitutes for abelian infrafield derivative $\partial_\mu \Phi(x,e)$.

This infrafield is group valued.

But note that Lorentz transformations change e.

So Lorentz transformations are spontaneously broken.

As shown before, so is the colour group.

V is a string localised quantum field.

The analysis suggests a σ-model for Lorentz and QCD symmetry breaking Goldstone modes from fluctuations of e.

Thanks to all for listening.

The End