Spectral Metric and Einstein Functionals

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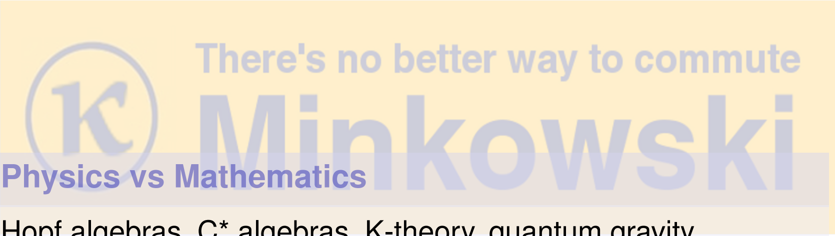
Quantum Geometry



Physics vs Mathematics

Hopf algebras, C* algebras, K-theory, quantum gravity, quantum space-time, deformation quantization, quantum Poincaré algebra, κ-deformation, Moyal deformation...

$$[x_0, x_i] = \frac{i}{\kappa} x_i,$$

$$[x_i, x_j] = 0,$$

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Questions

What is **space** - how to describe it allowing for all **different approaches** - how to get **numbers**.

Spectral Geometry

Can one hear the shape of a drum?

An eminent spectral scheme that generates geometric objects on manifolds such as volume, scalar curvature, and other scalar combinations of curvature tensors and their derivatives *prima facie* is the small-time asymptotic expansion of the (localised) trace of heat kernel.

$$\operatorname{Tr} e^{-t\Delta} = \sum_{n=0}^{\infty} t^{\frac{n-d}{2}} a_n(\Delta)$$

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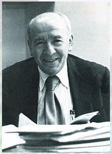


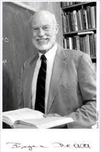

Spectral Geometry

Geometry from residues

Using the Mellin transform, the coefficients of this expansion can be transmuted into certain values or residues of the (localised) zeta function of the Laplacian. In turn, they can be expressed using the Wodzicki residue $\mathscr{W}$ (also known as noncommutative residue),

$$\mathscr{W}(P) := \int_M \left(\int_{|\xi|=1} tr\, \sigma_{-n}(P)(x,\xi)\, \mathscr{V}_\xi \right) d^n x,$$

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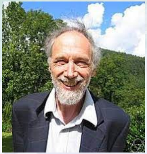

Spectral Geometry

The Weyl's law

In this case, for a Riemannian manifold M (of dimension $2m$) equipped with a metric tensor g and the (scalar) Laplacian Δ one has,

$$\mathscr{W}(\Delta^{-m}) = v_{n-1}\, vol(M),$$

and a *localized* form, a functional of $f \in C^\infty(M)$,

$$\upsilon(f) := \mathscr{W}(f\Delta^{-m}) = v_{n-1} \int_M f\, vol_g,$$

$$v_{n-1} := vol(S^{n-1}) = \frac{2\pi^m}{\Gamma(m)},$$

is the volume of the unit sphere S^{n-1} in $\mathbb{R}^n$.

THIS IS RELATED TO THE ASYMPTOTIC BEHAVIOR OF EIGENVALUES OF THE LAPLACE OPERATOR.

Spectral Geometry

The scalar curvature

A startling result regarding a higher power of the Laplacian was divulged by Connes in the early 1990s and explicitly confirmed independently by Kastler and Kalau-Walze.

$$\mathscr{W}(\Delta^{-m+1}) = \frac{n-2}{12} v_{n-1} \int_M R(g)\, vol_g,$$

Here $R = R(g)$ is the scalar curvature, that is the g-trace $R = g^{jk} R_{jk}$ of the Ricci tensor with components R_{jk} in local coordinates, where g^{jk} are the raised components of the metric g. A localised form of this functional for $n > 2$ is a functional in $C^\infty(M)$,

$$\mathscr{R}(f) := \mathscr{W}(f\Delta^{-m+1}) = \frac{n-2}{12} v_{n-1} \int_M f\, R(g) vol_g.$$

The metric and Einstein functionals

Spectral functional over vectors fields

Let V, W be a pair of vector fields on a compact Riemannian manifold M, of dimension $n = 2m$. Using the Laplace operator, we define two functionals $\mathcal{g}^{\Delta}(V, W)$ and $\mathcal{G}^{\Delta}(V, W)$.

Theorem

The functional:

$$\mathcal{g}^{\Delta}(V, W) := \mathscr{W}\left(VW\Delta^{-m-1}\right),$$

is a bilinear, symmetric map, whose density is proportional to the metric evaluated on the vector fields:

$$\mathcal{g}^{\Delta}(V, W) = -\frac{v_{n-1}}{n}\int_M g(V, W)\, vol_g.$$

The metric and Einstein functionals

Next:

Theorem

The functional:

$$\mathcal{G}^{\Delta}(V,W) := \mathscr{W}\left(VW\Delta^{-m}\right),$$

is a bilinear, symmetric map, whose density is proportional to the Einstein tensor G evaluated on the two vector fields:

$$\mathcal{G}^{\Delta}(V,W) = \frac{v_{n-1}}{6}\int_M G(V,W)\, vol_g.$$

where:

$$G(V,W) = Ric(V,W) - \frac{1}{2}R(g)\, g(V,W),$$

is the Einstein tensor.

The proof

Pseudodifferential calculus

Suppose that P and Q are two pseudodifferential operators with symbols,

$$\sigma(P)(x,\xi) = \sum_{\alpha} \sigma(P)_{\alpha}(x)\xi^{\alpha}, \qquad \sigma(Q)(x,\xi) = \sum_{\beta} \sigma(Q)_{\beta}(x)\xi^{\beta},$$

respectively, where α, β are multiindices. The composition rule for the symbols of their product takes the form:

$$\sigma(PQ)(x,\xi) = \sum_{\beta} \frac{(-i)^{|\beta|}}{|\beta|!} \partial^{\xi}_{\beta}\sigma(P)(x,\xi)\partial_{\beta}\sigma(Q)(x,\xi),$$

where ∂^{ξ}_{a} denotes the partial derivative with respect to the coordinate of the cotangent bundle.

Normal coordinates

Expanding the metric

In the normal coordinates the metric has a Taylor expansion:

$$g_{ab} = \delta_{ab} - \frac{1}{3} R_{acbd} x^c x^d + o(\mathbf{x^2}),$$

and

$$\sqrt{\det(g)} = 1 - \frac{1}{6} \mathrm{Ric}_{ab} x^a x^b + o(\mathbf{x^2}),$$

where R_{acbd} and Ric_{ab} are the components of the Riemann and Ricci tensor, respectively, at the point with $\mathbf{x} = 0$ and we use the notation $o(\mathbf{x^k})$ to denote that we expand a function up to the polynomial of order k in the normal coordinates. The inverse metric is

$$g^{ab} = \delta_{ab} + \frac{1}{3} R_{acbd} x^c x^d + o(\mathbf{x^2}),$$

Normal coordinates

The Laplace and its inverse

Consequently, the symbols of the Laplace operator in normal coordinates are

$$\begin{aligned}\mathfrak{a}_2 &= (\delta_{ab} + \frac{1}{3} R_{acbd} x^c x^d)\xi_a \xi_b + o(\mathbf{x}^2),\\ \mathfrak{a}_1 &= \frac{2i}{3}\mathrm{Ric}_{ab} x^a \xi_b + o(\mathbf{x}^2).\end{aligned}$$

The first three leading symbols of the operator Δ^{-k}, $k > 0$

$$\sigma(\Delta^{-k}) = \mathfrak{c}_{2k} + \mathfrak{c}_{2k+1} + \mathfrak{c}_{2k+2} + \dots,$$

are..

Normal coordinates

The symbols of the inverse of Laplace operator

$$\mathfrak{c}_{2k} = ||\xi||^{-2k-2} \left(\delta_{ab} - \frac{k}{3} R_{acbc} x^c x^d \right) \xi_a \xi_b + o(\mathbf{x}^2),$$

$$\mathfrak{c}_{2k+1} = \frac{-2ki}{3||\xi||^{2k+2}} \mathrm{Ric}_{ab} x^b \xi_a + o(\mathbf{x}),$$

$$\mathfrak{c}_{2k+2} = \frac{k(k+1)}{3||\xi||^{2k+4}} \mathrm{Ric}_{ab} \xi_a \xi_b + o(\mathbf{1}).$$

Laplace-type operators

Theorem: Laplace operator on a vector bundle

We assume that there is a connection ∇ on the vector bundle V, i.e. for any vector field X on M, we have a covariant derivative ∇_X on the module of smooth sections of V. The functional

$$\mathcal{G}^{\Delta_\nabla}(V,W) := \mathscr{W}(\nabla_V \nabla_W \Delta_\nabla^{-n})$$

is equal to

$$\mathcal{G}^{\Delta_T}(V,W) = \frac{v_{n-1}}{6}\mathrm{rk}(V) \int_M G(V,W)\, vol_g + \frac{v_{n-1}}{2} \int_M F(V,W)\, vol_g,$$

where

$$F(V,W) = \mathrm{Tr}\ V^a W^b F_{ab},$$

and F_{ab} is the curvature tensor of the connection ∇.

Spinors and differential forms

Differential forms as operators

In differential geometry besides functionals on vector fields, one can consider functionals over the dual bimodule of one-forms. To investigate whether the Einstein tensor (or, more precisely, its contravariant version) can be obtained from such functional using the spectral methods we need to represent differential forms as differential operators, and a suitable way is to employ Clifford modules. We assume thus that M is a $n = 2m$ dimensional spin$_c$ manifold and use the Clifford representation of one-forms as 0-order differential operators, that is, endomorphisms of a rank 2^m spinor bundle.

Why spin?

We want to represent *differential forms* using the Dirac operator, not using the Laplace operator (or spinor Laplacian).

Spectral functionals over forms

Theorem

The following spectral functionals of one-forms on a spin-c manifold M of dimension n

$$\begin{aligned} \mathcal{g}_D(v,w) &:= \mathscr{W}\big(\hat{v}\hat{w}D^{-n}\big), \\ \mathscr{G}_D(v,w) &:= \mathscr{W}\big(\hat{v}(D\hat{w}+\hat{w}D)D^{-n+1}\big) \\ &= \mathscr{W}\big((D\hat{v}+\hat{v}D)\hat{w}D^{-n+1}\big), \end{aligned}$$

read

$$\begin{aligned} \mathcal{g}_D(v,w) &= 2^m v_{n-1}\int_M g(v,w)\, vol_g, \\ \mathscr{G}_D(v,w) &= 2^m \frac{v_{n-1}}{6}\int_M G(v,w)\, vol_g, \end{aligned}$$

where $g(v,w) = g^{ab}v_a w_b$ and $G(v,w) = \big(Ric^{ab} - \frac{1}{2}Rg^{ab}\big)v_a w_b$, using the expression in any local coordinates.

Generalizations

Noncommutative toric manifolds

The noncommutative tori are prominent examples of noncommutative manifolds. In particular, there are external derivations that act on the smooth algebra $\mathscr{A} = C^\infty(\mathbb{T}^n_\theta)$ that can be interpreted as noncommutative vector fields. It is then straightforward to identify a noncommutative counterpart of the flat-metric Laplace operator, which can also be generalised to the case of conformally rescaled geometry.

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Spectral triples

The spectral metric functional on differential forms is

$$g_D(v, w) = \mathscr{W}(vw|D|^{-n}),$$

where $v, w \in \Omega^1_D(\mathscr{A})$, and the Einstein functional is

$$\mathcal{G}_D(v, w) = \mathscr{W}(v\{D, w\}D|D|^{-n}).$$

Noncommutative 2-torus: vector fields

We take

$$\Delta_h = h^{-1} \Delta \, h^{-1}$$

Proposition

For the conformally rescaled Laplace operator on a noncommutative 2-torus the metric functional reads

$$g^{\Delta_h}(V_h, W_h) = \mathscr{W}\left(V_h W_h \Delta_h^{-2} \right) = \pi \tau(h^4) V^a W^b \delta_{ab},$$

whereas the spectral Einstein functional and its density vanish identically

$$\mathcal{G}^{\Delta_h}(V_h, W_h) = \mathscr{W}\left(V_h W_h \Delta_h^{-1} \right) = 0.$$

Noncommutative 2-torus: differential forms

We take a spectral triple is given by $(\mathscr{A}, D_k = kDk, \mathscr{H} \otimes \mathbb{C}^2)$

Proposition

For the conformally rescaled spectral triple over the noncommutative 2-torus the metric functional for $v = k^2 V^a \sigma^a$ and $w = k^2 W^a \sigma^a$, $V^a, W^a \in \mathscr{A}$, reads

$$g_{D_k}(v, w) = \tau(V^a W^a),$$

whereas the spectral Einstein functional vanishes identically,

$$\mathscr{G}_{D_k}(v, w) = 0.$$

Noncommutative 4-torus

Differential forms

The metric and the Einstein functionals for the conformally rescaled spectral triple over the noncommutative 4-torus are,

$$
\begin{aligned}
g_{D_k}(v,w) &= \tau\left(W^a V^b k^{-4}\right),\\
\mathcal{G}_{D_k}(v,w) &= \tau\Bigg(V^a W^b \Bigg(\frac{1}{3}k^{-4}(\delta_a k)k^2(\delta_b k) + \frac{2}{3}k^{-3}(\delta_a k)k^1(\delta_b k)\\
&\qquad + k^{-2}(\delta_a k)(\delta_b k) + \frac{2}{3}k^{-1}(\delta_a k)k^{-1}(\delta_b k)\\
&\qquad - \frac{4}{3}k(\delta_a k)k^{-3}(\delta_b k) - \frac{2}{3}k^2(\delta_a k)k^{-4}(\delta_b k)\\
&\qquad + \frac{2}{3}k^{-1}(\delta_a\delta_b k) + \delta_{ab}\Bigg(\frac{1}{3}k^{-1}(\delta_c k)k^{-1}(\delta_c k)\\
&\qquad + \frac{1}{3}k^2(\delta_c k)k^{-4}(\delta_c k) + \frac{2}{3}k^1(\delta_c k)k^{-3}(\delta_c k)\\
&\qquad - \frac{2}{3}k^{-1}(\Delta k)\Bigg)\Bigg)\Bigg).
\end{aligned}
$$

Outlook

The concept that various geometric objects like tensors (metric, torsion and curvature tensors) can be expressed using spectral methods provides an invaluable possibility to study them globally both for the manifolds as well as for different extensions of geometries like noncommutative geometry.

Applications

- study torsion and Levi-Civita connection
- quantum metric spaces
- orbifolds and manifolds with singularities
- flat manifolds and noncommutative flat manifolds

Outlook

Noncommutative Einstein manifolds

This opens a possibility to study spectral Einstein manifolds:

Definition

A spectral triple is called an Einstein spectral triple if the spectral Einstein functional is proportional to the metric functional.

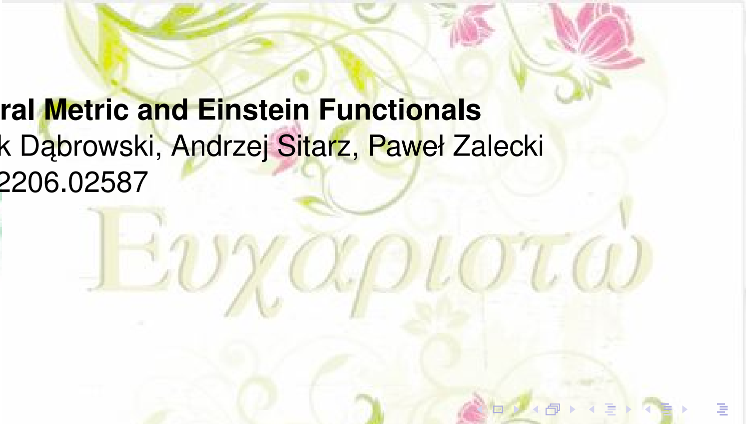
A particularly interesting case is of 2-dimensional geometries:

Conjecture

A suitably regular spectral triple of dimension 2 has a vanishing Einstein spectral functional.

Of course, we expect that regularity alone will not be the only condition, yet such a result will show the robustness of the noncommutative generalisation of manifolds.

THANK YOU !



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