Celestial twistor models and $w_{1+\infty}$ symmetry.

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Celestial Amplitudes and Flat Space Holography,
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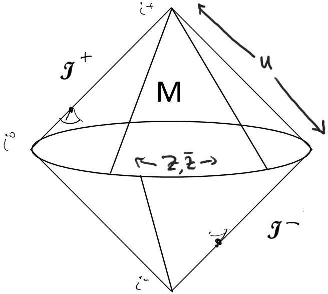
Work with: Tim Adamo & Atul Sharma 2103.16984, 21??...

Revisits: M & Wolf CMP, 288, '09, CMP, and M & Skinner CMP 294, '10 in light of recent developments.

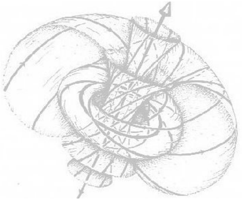
Compact gravity scattering formulae from twistors at $\mathscr{I}$.

Twistors at null infinity, and amplitudes

- Massless amplitudes are defined in terms of data at $\mathscr{I}^{\pm}$.
- Newman's good cuts attempt to rebuild space-time from $\mathscr{I}$ data.
- Yields instead '$\mathcal{H}$-space' a complex self-dual space-time.



- Penrose's nonlinear graviton arose from Newman's $\mathcal{H}$-space.
- Twistor's non-locality $\rightsquigarrow$ extend out to $\mathscr{I}^{\pm}$.
- Asymptotic Twistor space $\mathbb{PT} = \mathbb{CP}^3$ is 3d space describing 4d physics.



Outline

1. (Generating functional for the gravity MHV amplitude from the Plebanski action & scalar for SD background.)
2. Good cuts of $\mathscr{I}$ & Newman's $\mathcal{H}$-space.
3. Lift to asymptotic twistor space and nonlinear graviton.
4. Twistor sigma model for Plebanski scalar and tree formulae.
5. (Extension to full gravity tree S-matrix.)
6. $Lw_{1+\infty}$-symmetry.

Gravity amplitudes at MHV ($--+\ldots+$ helicity)

- In spinor helicity notation, momenta $k_{i\alpha\dot\alpha} = \kappa_{i\alpha}\kappa_{i\dot\alpha}$ and

$$\langle 1\,2\rangle := \kappa_{1\alpha}\kappa_2^{\alpha}\,,\ [1\,2] := \kappa_{1\dot\alpha}\kappa_2^{\dot\alpha}\,,\ \langle 1|2|3] = \kappa_{1\alpha}k_2^{\alpha\dot\alpha}\kappa_{3\dot\alpha}\,.$$

- Hodges 2012 MHV formula:

$$\mathcal{M} = \langle 12\rangle^6 \det{}'\mathbb{H}\ \delta^4(\textstyle\sum_i k_i)$$

$$\mathbb{H}_{ij} = \begin{cases} \frac{[ij]}{\langle ij\rangle} & i \neq j \\ -\sum_k \frac{[ik]}{\langle ik\rangle} & i = j\,. \end{cases}$$

- $\mathbb{H}$ is Laplace matrix & matrix-tree theorem $\rightsquigarrow$[Feng,He 2012]
- Sum of tree diagrams [Bern,Dixon,Perelstein,Rosowski '98, Nguyen, Spradlin, Volovich, Wen '10]

Plebanski gravity action

Expanding about the SD sector, Abou-Zeid, Hull hep-th/0511189

- Use Plebanski-Palatini formulation with variables on M^4:
  - $\mathbf{e}^{\alpha\dot{\alpha}}$ = tetrad of 1-forms s.t.

$$\mathrm{d}s^2 = \varepsilon_{\alpha\beta}\varepsilon_{\dot{\alpha}\dot{\beta}}\,\mathbf{e}^{\alpha\dot{\alpha}}\mathbf{e}^{\beta\dot{\beta}}\,, \qquad \mathbf{e}^{\alpha\dot{\alpha}} = \frac{1}{\sqrt{2}}\begin{pmatrix} T+iZ & X+iY \\ X-iY & T-iZ \end{pmatrix}$$

  - $\Gamma_{\alpha\beta} = \Gamma_{(\alpha\beta)}$ the ASD spin connection 1-forms.
- Action uses ASD two-forms $\Sigma^{\alpha\beta} = \mathbf{e}^{(\alpha}_{\dot{\alpha}} \wedge \mathbf{e}^{\beta)\dot{\alpha}}$

$$S = \int_M R d^4x = \int_M \Sigma^{\alpha\beta}\left(d\Gamma_{\alpha\beta} + \kappa^2\,\Gamma^{\gamma}_{\alpha}\wedge\Gamma_{\beta\gamma}\right)\,,$$

- Field equations:

$$d\Sigma^{\alpha\beta} = 2\kappa^2\Gamma^{(\alpha}_{\gamma}\wedge\Sigma^{\beta)\gamma}\,, \qquad d\Gamma_{\alpha\beta} + \kappa^2\,\Gamma^{\gamma}_{\alpha}\wedge\Gamma_{\beta\gamma} = \Psi_{\alpha\beta\gamma\delta}\Sigma^{\gamma\delta}\,.$$

- $\Rightarrow \kappa^2\Gamma_{\alpha\beta} =$ ASD spin connection 1-form, $\qquad \mathrm{Ricci} = 0.$

The SD sector and MHV amplitudes

SD sector: Set $\kappa = 0$, $S_{SD} = \int_M \Sigma^{\alpha\beta} \mathrm{d}\Gamma_{\alpha\beta}$, $\rightsquigarrow$ field equs

$$\mathrm{d}\Sigma^{\alpha\beta} = 0 \Rightarrow \text{ metric is SD, and}$$

$$\mathrm{d}\Gamma_{\alpha\beta} \wedge \mathbf{e}^{\alpha\dot\alpha} = 0\,, \Rightarrow \mathrm{d}\Gamma_{\alpha\beta} = \psi_{\alpha\beta\gamma\delta}\Sigma^{\gamma\delta}$$

and $\psi_{\alpha\beta\gamma\delta}$ is linearized ASD Weyl spinor on SD background.

- All $+$ amplitude $= 0 \leftrightarrow$ consistency of SD sector.
- One $-$, rest $+$ amplitude $= 0 \leftrightarrow$ integrability of SD sector.

MHV interactions:

$$\mathcal{M}(1^-, 2^-, e^+) = \int_M \kappa^2 \Sigma^{\alpha\beta} \wedge \Gamma_{1\alpha\gamma} \wedge \Gamma^{\gamma}_{2\beta}\,.$$

MHV amplitude $\leftrightarrow$, probability of helicity flip of $-$ helicity particle on SD background $\Sigma^{\alpha\beta}$.

Plebanski scalar as MHV generating function

Eliminating gauge choice in '08 paper with Skinner.

- An ASD linear field of momentum $k_{\alpha\dot\alpha} = \kappa_\alpha \kappa_{\dot\alpha}$ is

$$\Gamma_{\alpha\beta} = \mathbf{e}^{\gamma\dot\gamma} b_{\dot\gamma} \kappa_\gamma \kappa_\alpha \kappa_\beta e^{ik\cdot x} \quad \text{with} \quad [b, \kappa] = 1 .$$

  At MHV have two of these with momenta k_1, k_2.
- Plebanski scalar is Kahler scalar with respect to coords:

$$x^{\dot\alpha} = x^{\alpha\dot\alpha}\kappa_{1\alpha} , \qquad \tilde{x}^{\dot\alpha} = x^{\alpha\dot\alpha}\kappa_{2\alpha} ,$$

- The general SD metric is determined by $\Omega(x^{\dot\alpha}, \tilde{x}^{\dot\alpha})$ subject to Monge-Ampere:

$$\Sigma^{\alpha\beta} = \kappa_1^\alpha \kappa_1^\beta d^2x + \kappa_2^\alpha \kappa_2^\beta d^2\tilde{x} + \kappa_1^{(\alpha}\kappa_2^{\beta)} \partial\tilde\partial\Omega , \qquad \det \partial\tilde\partial\Omega = 1 .$$

- Then can integrate by parts twice to obtain

$$\mathcal{M}(1^-, 2^-, \Omega) = \langle 1\, 2 \rangle^4 \int_M d^4x \; \Omega \, \mathrm{e}^{[\kappa_1 x] + [\tilde\kappa_2 \tilde{x}]} .$$

How can we generate Ω from twistor space?

Cuts of null infinity

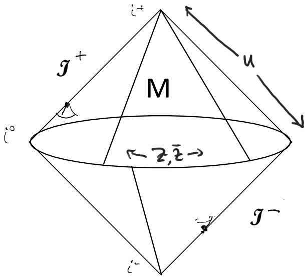
Asymptotically simple (M, g) has $\mathscr{I}^{\pm}$ and light rays meet both.

- Bondi coordinates $(u = t - r, z, \bar{z})$.
- Flat space conformal to

$$2dudR - dzd\bar{z} + O(R^0), \qquad R = \frac{1 + |z|^2}{r}.$$

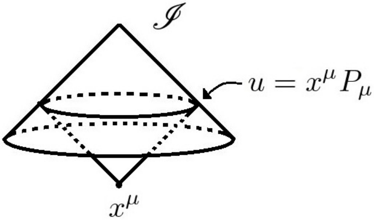
- Use spinor coordinates on on $S^2 = \mathbb{CP}^1$

$$\lambda_\alpha = (1, z)\,, \quad \alpha = 0, 1.$$



Light-cone of $x^{\alpha\dot{\alpha}} = (x^{\dot{\alpha}}, \tilde{x}^{\dot{\alpha}})$ *cuts* $\mathscr{I}$ at

$$u = \tilde{x}^{\dot{1}}|z|^2 + x^{\dot{1}}\bar{z} + \tilde{x}^{\dot{0}}z + x^{\dot{0}}$$
$$= x^{\alpha\dot{\alpha}}\lambda_\alpha\bar{\lambda}_{\dot{\alpha}}\,.$$



Newman's good cut equation and $\mathcal{H}$-space

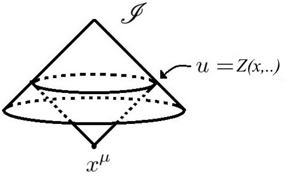
Curved space-time (M, g) rescales near $\mathscr{I}$ to:

$$2dudR - dzd\bar{z} + R(\sigma^0 d\bar{z}^2 + c.c.) + O(1)\,, \qquad R = \frac{1}{r} \to 0$$

where $\sigma^0(u, z, \bar{z})$ is *asymptotic shear*; gravitational data at $\mathscr{I}$.

- *Good cut equation* for $u = Z(z, \bar{z})$ is

$$\partial_{\bar{z}}^2 Z = \sigma^0(Z, z, \bar{z})\,.$$



$\leftrightarrow$ '$\mathbb{C}$-Null geodesic' in $\bar{z}$-direction.

- For $Z \in \mathbb{C}$, $\exists$ $4_{\mathbb{C}}$dim solution space $\mathcal{H}$,

$$Z(x, z, \bar{z}) = mz^2 + \tilde{x}^{\dot{1}}|z|^2 + \tilde{x}^{\dot{0}}z + x^{\dot{1}}\bar{z} + x^{\dot{0}}\,.$$

Theorem (Newman 1976)

'H-space', $\mathcal{H} = \{$space of solutions $Z(x, z, \bar{z})$, $x \in \mathcal{H}\}$ is $4d_{\mathbb{C}}$ with a holomorphic self-dual, Ricci-flat metric g_+.

Twistor space $\mathcal{T} = \mathbb{C}^4$ or projective $\mathbb{P}\mathcal{T}$, homogeneous coords:

$$W = (\lambda_\alpha, \mu^{\dot\alpha}) \in \mathbb{T}, \qquad W \sim aW\,, a \neq 0\,.$$

Poisson bracket:

$$\{f, g\} = \varepsilon^{\dot\alpha\dot\beta} \frac{\partial f}{\partial \mu^{\dot\alpha}} \frac{\partial g}{\partial \mu^{\dot\beta}} = \left[\frac{\partial f}{\partial \mu} \frac{\partial g}{\partial \mu}\right].$$

- $\mathcal{T}$ is $\mathbb{C}$-deformed from Hamiltonian h

$$\bar\partial_h f := \bar\partial_0 f + \{h, f\}\,, \qquad h = D\bar\lambda \int^u \boldsymbol{\sigma}^0 du' \in \Omega^{0,1}_{\mathbb{PT}}\,.$$

- Cuts lift to $\mathbb{CP}^1_{x,\sigma} \subset \mathbb{P}\mathcal{T}$, hgs coords (σ_0, σ_1) by

$$\lambda_\alpha = \left(\frac{1}{\sigma_0}, \frac{1}{\sigma_1}\right) = \frac{(1, z)}{\sigma_0}, \qquad \mu^{\dot\alpha} = \frac{x^{\dot\alpha}}{\sigma_0} + \frac{\tilde{x}^{\dot\alpha}}{\sigma_1} + M^{\dot\alpha}\,,$$

- Good cut eq $\rightsquigarrow$ d-bar eq for $\mathbb{C}$-curves in deformed $\mathbb{P}\mathcal{T}$:

$$\bar\partial_\sigma \mu^{\dot\alpha} = \{\mu^{\dot\alpha}, h\} = \varepsilon^{\dot\alpha\dot\beta} \frac{\partial h}{\partial \mu^{\dot\beta}}\,,$$

Sigma model action at $\mathscr{I}$ & $\mathbb{PT}$ for cuts and curves

- For good cut, $Z = mz^2 + \ldots$, with $\partial_{\bar{z}}^2 m = \boldsymbol{\sigma}^0(Z, z, \bar{z})/z^2$.
- Good cut equation has action

$$S_{\mathscr{I}}[m] = \int d^2z \left((\partial_{\bar{z}} m)^2 + \frac{2h}{z^2} \right) , \quad h = D\bar{\lambda} \int^u \boldsymbol{\sigma}^0 du$$

- Lift to $\mathbb{PT}$ by $\mu^{\dot{\alpha}} = x^{\dot{\alpha}}/\sigma_0 + \tilde{x}^{\dot{\alpha}}/\sigma_1 + M^{\dot{\alpha}}$, action

$$S_{\mathbb{PT}}[M] = \int D\sigma \left([M\bar{\partial}_\sigma M] + 2h \right)$$

Key proposition: These on-shell actions compute Plebanski scalar $\Omega =$ Kahler scalar for g_+:

$$\Omega(x, \tilde{x}) = S_{\mathscr{I}}[m] = S_{\mathbb{PT}}[M] \, .$$

MHV generating function, trees and Hodge formula

- MHV generating function becomes

$$\mathcal{M}(1,2,h) = \langle 1\,2\rangle^4 \int_M d^4x\, \mathrm{e}^{[\kappa_1 x]+[\kappa_2 \tilde{x}]}\, S_{\mathbb{PT}}[M,h]$$

- Now perturbatively expand in h in momentum eigenstates

$$h = \sum_{i=3}^{n} h_i\,, \qquad h_i = \int \frac{ds}{s^3}\bar{\delta}^2(s\lambda_\alpha - \kappa_{i\alpha})\,\mathrm{e}^{is[\mu\,\kappa_i]}\,.$$

- On-shell action has tree expansion (ignoring $O(h_i^2)$)

$$S_{\mathbb{PT}}[M,h] = \langle V_{h_3}\dots V_{h_n}\rangle_{\mathrm{tree}}$$

with vertex operators $V_h = \int_{\mathbb{CP}^1} hD\lambda$ and propagators

$$\frac{[\partial_\mu h_i\,\partial_\mu h_j]}{\langle i\,j\rangle} = \frac{[i\,j]}{\langle i\,j\rangle} h_i h_j$$

- Yields tree-diagram formalism of Bern et. al. 1998.

Matrix-tree theorem gives $\langle V_{h_3}\dots V_{h_n}\rangle_{\mathrm{tree}} = \det{}'\mathbb{H}$

$\rightsquigarrow$ Hodges reduced determinant formula, [cf Feng-He'12].

Sigma model at higher MHV degree

For N^{k-2}MHV need k ASD particles:

- ASD wave functions as momentum eigenstates

$$\tilde{h}_r(W_r) = \int s^5 ds\, \bar{\delta}^2(s\lambda - \kappa) e^{is[\mu\kappa]} \in H^1(\mathcal{O}(-6))\,.$$

- Insert ASD particles at $W_r \in \mathbb{T}$ and $\sigma_r \in \mathbb{CP}^1$, $r = 1, \ldots, k$:

$$W(\sigma) = \sum_{r=1}^{k} \frac{W_r}{\sigma - \sigma_r} + (0, M^{\dot{\alpha}}) : \mathbb{CP}^1 \to \mathbb{PT}\,.$$

- This is rigid at degree $k-1$, with M of weight $(-1, 0)$.
- Action is now simply

$$S[W(\sigma), W_r, \sigma_r, h] = \int_{\mathbb{CP}^1} d\sigma \left([M\, \bar{\partial} M] + 2h \right)$$

Claim: On-shell action gives gravity N^{k-2}MHV tree-amplitudes

The formula on background h is:

$$\mathcal{M}(1^-,\dots,k^-,h) = \int_{(\mathbb{CP}^1\times\mathbb{PT})^k} S[W(\sigma),W_r,\sigma_r,h]\,\det{}'\tilde{\mathbb{H}}\prod_{r=1}^k \tilde{h}_r D^3 W_r d\sigma_r$$

here we have inserted $\det{}'\tilde{\mathbb{H}}$, for the 'conjugate' Hodge matrix

$$\tilde{\mathbb{H}}_{ij} = \begin{cases} \frac{\langle\lambda_r\lambda_s\rangle}{\sigma_r-\sigma_s} & r\neq s \\ -\sum_q \frac{\langle\lambda_r\lambda_q\rangle}{\sigma_r-\sigma_q}\,, & r=s\,. \end{cases}$$

Expanding $h=\sum_{i=k+1}^n h_i$ as before gives

$$\mathcal{M} = \int_{(\mathbb{CP}^1\times\mathbb{PT})^k} \langle h_{k+1}\dots h_n\rangle_{\rm tree}\det{}'\tilde{\mathbb{H}}\prod_{r=1}^k \tilde{h}_r D^3 W_r d\sigma_r\,,$$

$$= \int_{(\mathbb{CP}^1)^n\times\mathbb{PT}^k} \det{}'\mathbb{H}\det{}'\tilde{\mathbb{H}}\prod_{r=1}^k \tilde{h}_r D^3 W_r d\sigma_r$$

proved by reduction to Cachazo-Skinner formula.

Informally, $w_{1+\infty}$ = Poisson diffeos of $\mu^{\dot\alpha}$-plane.

- With $\mu^{\dot\alpha} = x^{\dot\alpha}/\sigma_0 + \tilde{x}^{\dot\alpha}/\sigma_1 + M^{\dot\alpha}$, sigma model action

$$S_{\mathbb{PT}}[\mu^{\dot\alpha}] = \int D\sigma \left([\mu\bar\partial_\sigma\mu] + 2h\right) + [x\,\tilde{x}]\,.$$

- OPE of $f(W), g(W)$ is

$$f(W(\sigma))\cdot g(W(\sigma')) \sim \frac{\{f,g\}}{\sigma-\sigma'} + \dots\,.$$

- Gauge symmetry under Poisson diffeos $\leftrightarrow g(W,\bar\lambda)$ gives

$$\delta\mu^{\dot\alpha} = \{g,\mu^{\dot\alpha}\} = \varepsilon^{\dot\alpha\dot\beta}\frac{\partial g}{\partial\mu^{\dot\beta}}\,, \qquad \delta h = \bar\partial_0 g + \{h,g\} =: \bar\partial_h g\,.$$

- $\delta h = 0 \Rightarrow g$ holomorphic, $\bar\partial_h g = 0$; if g global $\rightsquigarrow$ Poincaré.
- If local, gives Čech version of vertex operators

$$Q_g := \oint g D\sigma = \int h D\sigma = V_h\,.$$

- i.e.: Local symmetries = gravity vertex operators.

Soft expansion versus mode expansion for $w_{1+\infty}$

- Čech momentum eigenstate is

$$g_{\omega,\kappa,\tilde{\kappa}} = \frac{\langle \lambda o\rangle^3}{\langle \lambda \kappa\rangle} \mathrm{e}^{\frac{i\omega_j[\mu\tilde{\kappa}]}{\langle \lambda o\rangle}} \, ,$$

- On $\langle \lambda 0\rangle \neq 0$, $\bar{\partial} g_{\omega,\kappa,\tilde{\kappa}} = h_\kappa$ so $Q_{g_\kappa} = V_{h_\kappa}$.
- Mellin transform to $\Delta = k \in \mathbb{Z}_{\leq 2} \leftrightarrow$ soft expansion in $\omega \rightsquigarrow$

$$g^k = \frac{\mathrm{i}^{2-k}}{(2-k)!} \frac{[\mu\,\tilde{\kappa}]^{2-k} \langle \lambda o\rangle^{k+1}}{\langle \lambda\,\kappa\rangle} \, .$$

Now directly exands to give Strominger's $Lw_{1+\infty}$-algebra of symmetries complete with combinatoric factors:

- $Lw_{1+\infty} \leftrightarrow$ Hamiltonians on the $\mu^{\dot\alpha}$-plane, Laurent in λ:

$$w^p_{m,r} = (\mu^{\dot 0})^{p-m-1} (\mu^{\dot 1})^{p+m-1} \frac{\langle \lambda \iota\rangle^r}{\langle \lambda o\rangle^{r+2p-4}} \, , \qquad |m| \leq p-1$$

- Poisson brackets

$$\{w^p_{m,r}, w^q_{n,s}\} = (2(p-1)n - 2(q-1)m) w^{p+q-2}_{m+n,r+s} \, . \qquad (1)$$

Conclusions & discussion

- Gravity tree amplitudes generated by on-shell action of sigma model for curves in $\mathbb{PT}$ or cuts of $\mathscr{I}$.
- degree of map $= k - 1$ at N^{k-2}MHV corresponds to rational approximation of true light cone cut.
- Geometric action of $Lw_{1+\infty}$ on $\mathbb{PT}$ is by Čech vertex operators for SD gravitons.
- Penrose's nonlinear graviton realizes SD graviton phase space as loop group $Lw_{1+\infty}$.
- Soft graviton expansion $\leftrightarrow$ mode expansion for $Lw_{1+\infty}$.
- Beyond SD sector, ideas embed into 4d ambitwistor-string.
- Story extends to $\Lambda \neq 0$, YM and nonlinear backgrounds.
- Gives value of Einstein-Hilbert action at MHV.
- Atul Sharma has full off-shell twistor action for GR [2104.07031].

- Einstein gravity tree $=$ tree sigma model correlator (MHV).
- Does full quantum sigma model correlator $\leftrightarrow$ gravity loops?

$$\langle 1\,2\rangle^{2n}\prod_{i=3}^{n}\frac{1}{\langle 1\,i\rangle^2\,\langle 2\,i\rangle^2}\;\exp\left[-\frac{\mathrm{i}\,\alpha}{8\pi}\sum_{j\neq i}\frac{[i\,j]}{\langle i\,j\rangle}\,\frac{\langle 1\,i\rangle^2\,\langle 2\,j\rangle^2}{\langle 1\,2\rangle^2}\right]\,.$$

- Does quantum sigma model realize $W_{1+\infty}$ or W-gravity?
- Moyal quantization of $\mu^{\dot\alpha}$-plane and ‘palatial twistors’?

Thank You!