Progress on Multi-Loop Calculations

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Based on: JHEP 02 (2021) 080 (arXiv:2010.06947 [hep-ph]), JHEP 01 (2021) 199 (arXiv: 2009.13917 [hep-ph]) & ongoing work

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Motivation for Precision Physics

Although many cosmological (and not only) results indicate the need of BSM physics:

- *Dark Matter and Dark Energy*
- *Neutrino Oscillations*
- *Matter - Antimatter asymmetry*
- *Quantum Gravity incorporation*

there is no striking manifestation of New Physics beyond the SM at the LHC, as this can be found by the comparison of the measurements with the theoretical predictions!

The upgrade of LHC and the establishment of future experiments will result to more accurate measurements and will require even more accurate theoretical predictions $\rightarrow$ Perturbative QCD! Current frontier:

- *NNLO for* $2 \rightarrow 3$ *processes*
- *N^3LO for* $2 \rightarrow 2$ *processes*
- *N^4LO for* $2 \rightarrow 1$ *processes*

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Cross sections

- The cross section for the collision of 2 initial hadrons (h_1, h_2) to some final state X is

$$d\sigma_{h_1 h_2 \to X} = \sum_{a,b=q,\bar{q},g} \int_{x_{1,min}}^{1} dx_1 \int_{x_{2,min}}^{1} dx_2 \, \mathcal{F}_{a/h_1}(x_1, \mu^2) \mathcal{F}_{b/h_2}(x_2, \mu^2) \, \hat{\sigma}_{ab \to X}(\mu^2)$$

where $\mathcal{F}_{a/h_1}$ and $\mathcal{F}_{b/h_2}$ are the Parton Distribution Functions, $\hat{\sigma}_{ab\to X}$ is the hard-part cross section, and μ^2 is the factorization scale.

- $\hat{\sigma}_{ab\to X}$ at NNLO receives contributions from three different sources (virtual, mixed real-virtual, and doubly-real corrections)

$$\begin{aligned} d\hat{\sigma}^{NNLO}_{ab\to X} \sim\ & |\mathcal{M}_{tree}|^2 + \alpha_S \left(2\,\mathrm{Re}\left[\mathcal{M}_{tree}\mathcal{M}^*_{loop}\right] + |\mathcal{M}_{+1up}|^2\right) \\ & + \alpha_S^2 \left(|\mathcal{M}_{loop}|^2 + 2\,\mathrm{Re}\left[\mathcal{M}_{tree}\mathcal{M}^*_{2loops}\right] + |\mathcal{M}_{+2up}|^2 + 2\,\mathrm{Re}\left[\mathcal{M}_{+1up+loop}\mathcal{M}^*_{+1up}\right]\right) \end{aligned}$$

Each of these contributions is individually divergent, and the divergences cancel in the sum (after renormalization for the UV and IR divergences) leaving behind the finite result for the cross section.

A lot of effort is needed in all the steps for the calculation of the cross section. Nonetheless, from the above expression the most difficult part to be calculated has been proved to be the $2-loop$ amplitude, $\mathcal{M}_{2loops}$!

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Recent Results for 2-loop $2 \to 3$ Amplitudes

- Leading color: $gg/q\bar{q} \to ggg$, $q\bar{q} \to q\bar{q}g/\gamma\gamma\gamma$, $u\bar{d} \to b\bar{b}W^+$, $gg/q\bar{q}/b\bar{b} \to b\bar{b}H$.
- Full color: $gg \to ggg$ (all-plus helicities) and $q\bar{q} \to g\gamma\gamma$, $gg \to gg\gamma$.

Some benchmark references:

- S. Badger, H. B. Hartanto, J. Kryś and S. Zoia, [arXiv:2107.14733 [hep-ph]].
- S. Badger et al, [arXiv:2106.08664 [hep-ph]].
- S. Badger, H. B. Hartanto and S. Zoia, Phys. Rev. Lett. **127** (2021) [arXiv:2102.02516 [hep-ph]].
- B. Agarwal, F. Buccioni, A. von Manteuffel and L. Tancredi, arXiv:2105.04585 [hep-ph].
- B. Agarwal, F. Buccioni, A. von Manteuffel and L. Tancredi, JHEP 2104 (2021) 201.
- H. A. Chawdhry, M. Czakon, A. Mitov and R. Poncelet, [arXiv:2105.06940 [hep-ph]].
- H. A. Chawdhry, M. Czakon, A. Mitov and R. Poncelet, arXiv:2103.04319 [hep-ph].
- H. A. Chawdhry, M. Czakon, A. Mitov and R. Poncelet, arXiv:2012.13553 [hep-ph].
- H. A. Chawdhry, M. A. Lim and A. Mitov, Phys. Rev. D **99** (2019) no.7, 076011.
- S. Abreu, F. Febres Cordero, H. Ita, B. Page and V. Sotnikov, arXiv:2102.13609 [hep-ph].
- S. Abreu, B. Page, E. Pascual and V. Sotnikov, JHEP 2101 (2021) 078.
- S. Abreu et al, arXiv:2009.11957 [hep-ph].
- S. Badger, H. B. Hartanto, C. Brønnum-Hansen and T. Peraro, JHEP 1909 (2019) 119.
- S. Badger et al, Phys. Rev. Lett. **123** (2019) no.7, 071601.
- S. Badger, C. Brønnum-Hansen, H. B. Hartanto and T. Peraro, JHEP **01** (2019), 186.
- T. Gehrmann, J. M. Henn and N. A. Lo Presti, Phys. Rev. Lett. **116** (2016) no.6, 062001.
- T. Peraro, JHEP **12** (2016), 030.
- G. De Laurentis and D. Maître, JHEP **02** (2021), 016.

Amplitude Reduction using Numerical Unitarity, Projectors etc.

Structure of an (2-loop) Amplitude

The construction and calculation of a (2-loop) Amplitude, $\mathcal{A}$, contains the following steps

1. *Use of SM Feynman rules to generate the Feynman graphs contributing to the process at hand.*
2. *Collect all the above contributions and create the Amplitude.*
3. *Integrand/Integral reduce the Amplitude in to a set of Master (Feynman) integrals, determining their coefficients.*
4. *Calculate analytically or numerically the Master Integrals.*

The final result is of the form

$$\mathcal{A} = \sum_i c_i(\mathbf{s}, \varepsilon) F_i(\mathbf{s}, \varepsilon) \tag{2.1}$$

where c_I are rational/algebraic coefficients obtained by the Amplitude reduction and they depend by the process at hand, F_i can be Master Integrals or special functions (Multiple Polylogarithms[1], Pentagon Functions[2], Elliptic Integrals, etc) that depend from the kinematics and are process-independent, and $\mathbf{s}$ are the Mandlestam variables.

[1] A. B. Goncharov, Math. Res. Lett. **5** (1998), 497-516
[2] D. Chicherin and V. Sotnikov, JHEP **12** (2020), 167

Quick Review of HELAC-1LOOP

Any 1−loop n−particle (color-stripped) amplitude can be written in the form

$$\mathcal{A} = \int d^d k\, A = \sum_{I \subset \{1,\ldots,n\}} \int \frac{\mu^{(4-d)}}{(2\pi)^d} \frac{N_I(k, p_1, \ldots, p_{n-1}, \gamma^\mu, \epsilon^\mu)}{\prod_{i\in I} D_i}$$

where N_I is the numerator and $D_i = (k + p_i)^2 + m_i^2$ the propagators. The loop momentum "lives" in d dimensions and can be decomposed as

$$k = \bar{k} + k^* \quad \text{with} \quad \bar{k} : 4-\text{dimensional} \quad \text{and} \quad k^* : \varepsilon-\text{dimensional}\,.$$

In order to compute $\mathcal{A}$ we need to cast it in to the following well-known form at $d \to 4$

$$\mathcal{A} = \sum_i d_i\, \text{Box}_i + \sum_i c_i\, \text{Triangle}_i + \sum_i b_i\, \text{Bubble}_i + \sum_i a_i\, \text{Tadpole}_i + (R_1) + R_2$$

where Box, …, Tadpole refer to the one-loop Feynman integrals with 4, …, 1 external leg, (R_1 is the rational part originating from the reduction process of a 4−dimensional numerator in the OPP method[3]) and R_2 is the rational part originating by the explicit dependence of the numerator on the ε−dimension and can be reproduced by tree-like Feynman rules involving up to 4 particles[4].

[3] G. Ossola, C. G. Papadopoulos and R. Pittau, Nucl. Phys. B **763** (2007), 147-169

[4] G. Ossola, C. G. Papadopoulos and R. Pittau, JHEP **05** (2008), 004

In HELAC-1LOOP[5] the OPP method (*integrand level*) is used for the amplitude reduction. The main idea is that for any numerator its 4−dimensional part can be written as

$$\begin{aligned}\bar{N}(\bar{k}) = & \sum_{i_0<i_1<i_2<i_3}^{I} \left[d(i_0,i_1,i_2,i_3) + \tilde{d}(\bar{k},i_0,i_1,i_2,i_3)\right] \prod_{i\neq i_0,i_1,i_2,i_3}^{I} \bar{D}_i \\ & + \sum_{i_0<i_1<i_2}^{I} \left[c(i_0,i_1,i_2) + \tilde{c}(\bar{k},i_0,i_1,i_2)\right] \prod_{i\neq i_0,i_1,i_2}^{I} \bar{D}_i \\ & + \sum_{i_0<i_1}^{I} \left[b(i_0,i_1) + \tilde{b}(\bar{k},i_0,i_1)\right] \prod_{i\neq i_0,i_1}^{I} \bar{D}_i \\ & + \sum_{i_0}^{I} \left[a(i_0) + \tilde{a}(\bar{k},i_0)\right] \prod_{i\neq i_0}^{I} \bar{D}_i\end{aligned}$$

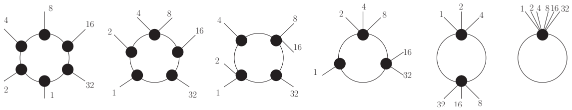
where $d_i = d(i_0,i_1,i_2,i_3)$, $c_i = c(i_0,i_1,i_2)$, $b_i = b(i_0,i_1)$, $a_i = a(i_0)$, and $\tilde{d}$, $\tilde{c}$, $\tilde{b}$, $\tilde{a}$ integrate to zero (*spurious terms*). The coefficients are determined by solving (iteratively) systems of equations by evaluating $\bar{N}(\bar{k})$ for values of $\bar{k}$, that are solutions of

$$\bar{D}_i(\bar{k}) = 0, \quad \text{for } i = 0,\dots,M-1, \quad \text{and } M = 1,\dots,4.$$

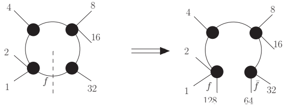
[5] G. Bevilacqua, M. Czakon, M. V. Garzelli, A. van Hameren, A. Kardos, C. G. Papadopoulos, R. Pittau and M. Worek, Comput. Phys. Commun. **184** (2013), 986-997

$\bar{N}(\bar{k})$ is numerically calculated by HELAC[6], which calculates tree-level amplitudes checking for all possible flavor, spin and color configurations using SM couplings and the color-connection representation!

In this set-up, a binary representation is used for the external particles (lv_1 blobs) and a generation of all topologically inequivalent partitions of $n, n-1, n-2, \ldots, 1$ blobs attached to the loop is done. For example, for $n = 6$ we could have



Each numerator contribution is calculated (by HELAC) by cutting the propagator-line connecting the first and the last blob and calculating the resulted $n+2$ tree-level amplitude without using denominators for the internal loop propagators



HELAC-1LOOP: Completely automated framework for the calculation of 1−loop amplitudes for n−particle processes!!!

[6] A. Kanaki and C. G. Papadopoulos, Comput. Phys. Commun. **132** (2000), 306-315

HELAC-2LOOP on the making

Construction of HELAC-2LOOP: Reuse as much as more from HELAC-1LOOP + new concepts for the amplitude reduction at 2−loops!

For the 2−loop Amplitude at the *integrand level*, we expect for $d \to 4$

$$A \equiv \sum_{I \subseteq T} \frac{N_I(k_1, k_2, p_1, \ldots, p_n, \gamma^\mu, \epsilon^\mu)}{\prod_{\{i_1,i_2,i_3\}\in I} D_{i_1}(k_1) D_{i_2}(k_2) D_{i_3}(k_1,k_2)} = \sum_i c_i(\mathbf{s}) F_i + \sum_j \tilde{c}_j(\mathbf{s}) S_j + R_1 + R_2$$

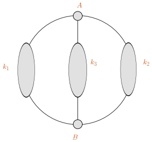
where T is the set containing the 2−loop graph topologies of the corresponding process, F_i are the *master integrands* that will integrate to *master integrals*, S_j are the *spurious* terms that will integrate to zero, and $\{R_1, R_2\}$ are the $2-$*loop* generalization of the 1−loop rational terms[7].

Calculation of $\bar{N}_I$: Generation of the 2-loop amplitude graphs in the "blob"-binary representation is needed (used internally by HELAC) → Creation of a generator for two-loop graph topologies with massless particles running within the loop in a list representation!

[7] The 2−loop R_2 terms have been computed in J. N. Lang, S. Pozzorini, H. Zhang and M. F. Zoller, JHEP **10** (2020), 016 and S. Pozzorini, H. Zhang and M. F. Zoller, JHEP **05** (2020), 077.

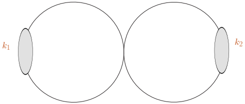
There exist three graph topologies for 2-loop amplitudes:

1) Theta−topologies:



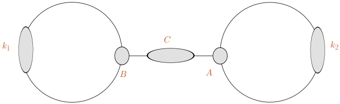
$$\equiv \{\{k_1\},\{k_2\},\{k_3\},\{A\},\{B\}\}$$

2) Infinity−topologies:



$$\equiv \{\{k_1\},\{k_2\}\}$$

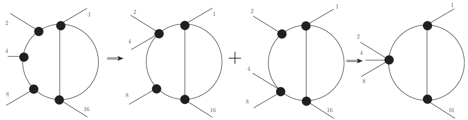
3) Dumbbell−topologies:



$$\equiv \{\{k_1\},\{k_2\},\{C\},\{A\},\{B\}\}$$

We have created two generators, one implemented in Mathematica (BlobMod) and one implemented in Fortran (GENTOOLS), using two different approaches in the generation:

- BlobMod starts by creating all the possible sets of putting the external particles in the sublists. Then in order to create the sub-topologies[8] if there exist lists with $Length[sublist] \geq 2$ takes for every list all the possible combinations of summing at most 2 neighboring elements of the same sublist[9]. After the generation of all the topologies, graph-symmetries are applied in order to remove identical lists.



$\{\{8,4,2\},\{\},\{\},\{1\},\{16\}\} \to \{\{8,6\},\{\},\{\},\{1\},\{16\}\} + \{\{12,2\},\{\},\{\},\{1\},\{16\}\} \to \{\{14\},\{\},\{\},\{1\},\{16\}\}$

- GENTOOLS generates the topologies exactly in the opposite way! Starts by taking all the possible sets of putting the higher level blobs in the sublists (lower-topologies) and creates the higher topologies by taking all the possible splittings of the blobs. In order to remove the identical lists graph-symmetries are applied also in this case.

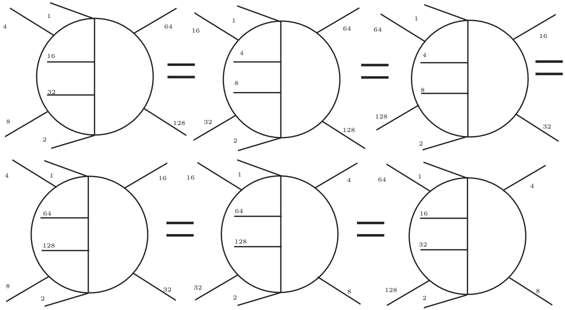
Perfect agreement found between the results of the two generators and q-graf (P. Nogueira, J. Comput. Phys. **105** (1993)). For $n \geq 6$ GENTOOLS is a lot faster from BlobMod!!!

[8] Meaning topologies where more than one particles shrink into a vertex, $lvl_2, \ldots, lvl_n$ blobs.

[9] The elements of $\{A\}$ and $\{B\}$ are always summed from the beginning.

Graph symmetries

The graphs are symmetric on (combined or individual) mirror transformations on the vertical and the horizontal axis (swap of the three loop lines). For example



All the symmetries of the graphs can be expressed in symmetries of the lists using one or both of the following 2 actions:

- Swap: corresponds to the swap of two sublists. E.g. $\{\{k_1\},\{k_2\}\} \rightarrow \{\{k_2\},\{k_1\}\}$.
- Reversion: corresponds to the reversion of the elements of a sublist. E.g. $\{1,2,4\} \rightarrow \{4,2,1\}$.

Introduction to the Differential Equations method for computing FIs

• For the calculation of the scalar FIs we define families of integrals for specific kinematic process. For an $L-$loop (k_i) FI with $E+1$ external legs (p_i) we have

$$F_{\alpha_1,\ldots,\alpha_N} = \int \left(\prod_{i=1}^{L} \frac{d^d k_i}{i\pi^{d/2}} \right) \frac{1}{D_1^{\alpha_1} \ldots D_N^{\alpha_N}}$$

with α_i arbitrary integers, $N = L(L+1)/2 + LE$ the number of linear scalar independent propagators $D_a = (k_i + p_j)^2 + m_a^2$ of the family.

• The fact that the total derivatives vanish in Dimensional Regularization ($d = 4 - 2\varepsilon$) give rise to the Integration by Parts Relations[10] (IBPs)[11]

$$\int \prod_{i=1}^{L} \frac{d^d k_i}{i\pi^{d/2}} \frac{\partial}{\partial k_i} \left(\frac{l_j}{D_1^{\alpha_1} \ldots D_N^{\alpha_N}} \right) = 0 \quad \text{with} \quad l_j = k_j \text{ or } p_j \,.$$

which imply the existence of a finite basis[12], master integrals (MI), in terms of which can be written any FI of the family, with some algebraic coefficients of $s_{ij} = (p_i + p_j)^2$ and d. There is a freedom in the choice of this basis!

[10] K.G. Chetyrkin and F.V. Tkachov, Nucl.Phys. B192 (1981) 159

[11] The reduction to MI using the IBP's is implemented in many programs some of whom are Kira, FIRE.

[12] A. V. Smirnov and A. V. Petukhov, Lett.Math.Phys. 97 (2011) 37-44

• FI are functions of external momenta and internal masses and fulfil differential equations (DE) in the kinematic invariants[13], S_k

$$\frac{\partial}{\partial S_k} G_i(\varepsilon, \{S_k\}) = \sum_{j=1}^{I} B_{ij}^k(\varepsilon, \{S_k\}) G_j(\varepsilon, \{S_k\}) \Rightarrow \partial^k \vec{G} = B^k \vec{G}$$

where I is the number of MI, and for us to solve this DE we need boundary conditions. In practise we want to solve the above DE in a Laurent expansion around $\varepsilon = 0$.

• Canonical DE[14]: Making a change of the basis $\vec{G} \to U\vec{G}$, B^k changes as $B^k \to UB^kU^{-1} + U\partial^kU^{-1}$. For a suitable choice of U we can obtain

$$\frac{\partial}{\partial S_k} \vec{G}'(\varepsilon, \{S_k\}) = \varepsilon \sum_i \frac{M_{ki}}{S_k - l_i} \vec{G}'(\varepsilon, \{S_k\})$$

which is ϵ-factorized, *Fuchsian* and M_{ki} are purely numerical. To obtain such a DE the chosen basis should be UT (functions with uniform degree of transcendentality).

Until now there does not exist an algorithmic way to find the UT Basis or to prove its existence!

[13] A.V.Kotikov, Phys.Lett. B254 (1991) 158
[14] J. Henn, Phys.Rev.Lett. 110 (2013) 251601

Quick Review of Simplified Differential Equations approach

1) Parametrize the external momenta in terms of an dimensionless parameter, x, in such a way that captures the off-shellness of an external leg[15].

2) Take derivatives of the MIs with respect to x and create, using IBPs, a system of DE in one independent variable

$$\partial_x \mathbf{G}(\{s_{ij}\}, x, \varepsilon) = \mathbf{H}(\{s_{ij}\}, x, \varepsilon)\mathbf{G}(\{s_{ij}\}, x, \varepsilon)$$

3) Find boundary conditions at $x \to 0$ and solve the differential equation:

- Use boundaries for already known integrals in closed form.
- Comparing the asymptotic regions obtained for the MIs from expansion-by-regions (asy[16]) with the ones obtained by the DE, using the resummation matrix at $x = 0$

$$\mathbf{M}_0 = \mathbf{S}_0\mathbf{D}_0\mathbf{S}_0^{-1} \quad \longrightarrow \quad \mathbf{R}_0 = \mathbf{S}_0 e^{\varepsilon \mathbf{D}_0 \log(x)}\mathbf{S}_0^{-1} \quad \longrightarrow \quad \mathbf{F}_{x\to 0} = \mathbf{T}^{-1}\mathbf{R}_0\mathbf{g}_{\text{bound}}\,.$$

  obtain relations between different boundaries of the family.
- In the end there are left some asymptotic regions to be calculated, which is done using the standard expansion-by-region approach (integrate in Feynman-parameters).

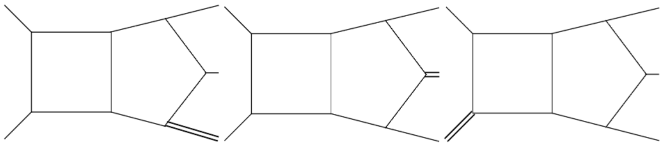
4) Take the $x \to 1$ limit to obtain also for free the solution for the same family with one external massive leg less.

[15] C. G. Papadopoulos, JHEP **07** (2014), 088

[16] B. Jantzen, A. V. Smirnov and V. A. Smirnov, Eur. Phys. J. C **72** (2012), 2139

Recent Computed families @ 2/3-loops

i) The planar families of two-loop massless Penta-box families with one off-shell leg



The P_1 (74 MI), P_2 (75 MI) and P_3 (86 MI) families. All external momenta are incoming.

Topology: 8 propagators + 3 numerators (ISPs).

Kinematics: $\{s_{12}, s_{23}, s_{34}, s_{45}, s_{15}, m^2\}$ with $s_{ij} = (q_i + q_j)^2$, $q_1^2 = m^2$ and else $q_j^2 = 0$.

SDE: $q_1 \to p_{123} - x p_{12}$, $q_2 \to p_4$, $q_3 \to -p_{1234}$, $q_4 \to x p_1$, with $p_i^2 = 0$ for $i = 1, ..., 5$[17].

Independent Variables: $\{s_{12}, s_{23}, s_{34}, s_{45}, s_{15}, m^2\} \to \{S_{12}, S_{23}, S_{34}, S_{45}, S_{15}, x\}$.

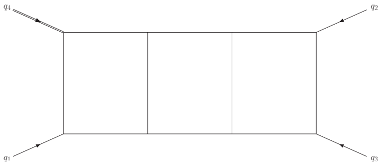
Canonical Basis: Obtained from S. Abreu et al, JHEP 11 (2020) 117.

Analytic results[18] for the Euclidean region in terms of Goncharov Polylogarithms (GPLs)!

[17] $p_{i...l} = p_i + \ldots + p_l$, and $S_{ij} = (p_i + p_j)^2$

[18] DC, C.G. Papadopoulos and N. Syrrakos, JHEP 01 (2021) 199.

ii) The three-loop massless Ladder-box family with one off-shell leg.



The Ladder-box family (83 MI). All external momenta are incoming.

Topology: 10 propagators + 5 numerators (ISPs).

Kinematics: $\{s, t, m^2\}$ with $s = (q_2 + q_3)^2$, $t = (q_1 + q_3)^2$, $q_4^2 = m^2$ and else $q_j^2 = 0$.

SDE: $q_1 \to xp_1$, $q_2 \to p_3$, $q_3 \to -p_{123}$, $q_4 \to p_{12} - xp_1$, with $p_i^2 = 0$ for $i = 1, ..., 4$.

Independent Variables: $\{s, t, m^2\} \to \{S_{12}, S_{23}, x\}$.

Canonical Basis: Obtained from S. Di Vita et al, JHEP 09 (2014) 148.

Analytic results[19] for the Euclidean region in terms of GPLs!

[19] DC and N. Syrrakos, JHEP 02 (2021) 080.

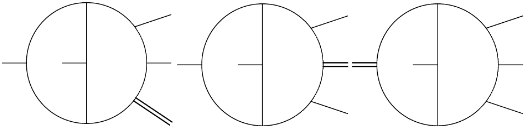
Conclusion: Ongoing Work

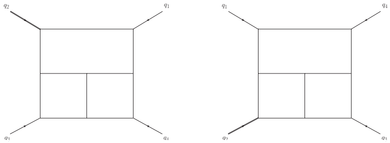
Amplitudes:

• Currently working on the upgrade of HELAC code such that to be able to numerically compute the 4−dimensional part of the numerator from the tree-level $n+4$ amplitude.

• Next step: creation of a general $\{$*master integrand* $+$ spurious terms$\}$ basis, and calculation of the 2−loop R_1 rational terms.

Feynman Integrals:





From left to right and from up to down, N_1 (86 MI), N_2 (86 MI), N_3 (135 MI), F2 (117 MI) and F3 (166 MI). All external momenta are incoming.

Thank you!

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H.F.R.I.
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Color connection representation

- In the color connection representation, the gluons are represented by a pair of color/anti-color indices (i,j) and the quarks (anti-quarks) by a single color $(i,0)$ (anti-color $(0,j)$) index, with $i_{,j} \in (1,...,N_C)$. All the other particles that do not carry color have $(0,0)$.

- The amplitude takes the following form

$$\mathcal{M}^{i_1,i_2,\ldots,i_k}_{j_1,j_2,\ldots,j_k} = \sum_{\sigma} \delta_{i_{\sigma_1},j_1}\delta_{i_{\sigma_2},j_2}\ldots\delta_{i_{\sigma_k},j_k} A_\sigma$$

with $k = n_g + n_q$ and the sum is running over all the permutations (equal to $k!$). The color-stripped amplitudes, A_σ, are calculated using properly defined Feynman rules [A. Cafarella, C. G. Papadopoulos and M. Worek, Comput. Phys. Commun. **180** (2009), 1941-1955].

- The total color factor is a product of δ's, and thus the color summed squared amplitude takes the form

$$\sum_{\{i\},\{j\}} \left|\mathcal{M}^{i_1,i_2,\ldots,i_k}_{j_1,j_2,\ldots,j_k}\right|^2 = \sum_{\sigma,\sigma'} A^*_{\sigma'} \mathcal{C}_{\sigma',\sigma} A_\sigma$$

where the color matrix $C_{\sigma',\sigma}$ is given by

$$C_{\sigma',\sigma} = \sum_{\{i\},\{j\}} \delta_{i_{\sigma'_1},j_1}\delta_{i_{\sigma'_2},j_2}\ldots\delta_{i_{\sigma'_k},j_k}\delta_{i_{\sigma_1},j_1}\delta_{i_{\sigma_2},j_2}\ldots\delta_{i_{\sigma_k},j_k} = N_C^{m(\sigma',\sigma)}$$

with $m(\sigma',\sigma)$ counting the number of common cycles of the 2 permutations.

Example of Pure and Impure relations

1) We call *pure* the relations that contain only boundaries of UT basis elements. As an illustrated example we consider the master integral F_{71}, from the Ladder-box family:

i) Expansion-by-regions method yields for $x \to 0 : x^{-1-3\varepsilon}$.

ii) The resummation matrix has produced two additional regions: $x^{-1-2\varepsilon}$ and x^{-1}.

iii) We proceed by setting the extra regions to zero since they are not predicted by asy.

From the second one, we obtain a relation which connects the boundary condition of g_{71} with the boundary condition of lower sector basis elements:

$$gb_{71} = \left(-12gb_2 + 4gb_{13} + 32gb_{16} + 48gb_{41} + 36gb_{42} - 45gb_{43}\right)/30\,.$$

2) We call *impure* the relations between boundaries and asymptotic limits, which are obtained by equating the result of the asy with that of the resummation matrix. E.g.

$$gb_{41} = F_{41}^{\text{soft}} s_{12}\varepsilon^5 + gb_2/9 - gb_{13}/12 - 2gb_{16}/3\,.$$

where F_{41}^{soft} is the $x^{-3\varepsilon}$ region of F_{41}.

As expected, in the *pure* relations between the boundaries the prefactors are just numbers $\longrightarrow$ Working perfectly even when a full analytic reduction is a bottleneck!!!

DEMOKRITOS

Procedure for taking the $x \to 1$

Briefly the procedure for taking the $x \to 1$ limit is:

1) Rewrite the solution as an expansion in $\log(1-x)$:

$$\mathbf{g} = \sum_{n \geq 0} \epsilon^n \sum_{i=0}^{n} \frac{1}{i!} \mathbf{c}_i^{(n)} \log^i(1-x)$$

2) Define the regular part of $\mathbf{g}$ at $x = 1$ and from it the truncated part:

$$\mathbf{g}_{reg} = \sum \epsilon^n \mathbf{c}_0^{(n)} \quad \text{and} \quad \mathbf{g}_{trunc} = \mathbf{g}_{reg}\Big|_{x=1}$$

3) Define the resummation matrix $\mathbf{R}_1$ and from it the purely numerical matrix $\mathbf{R}_{10}$:

$$\mathbf{R}_1 = e^{\epsilon \mathbf{M}_1 \log(1-x)} = \mathbf{S}_1 e^{\epsilon \mathbf{D}_1 \log(1-x)} \mathbf{S}_1^{-1} \quad \text{and} \quad \mathbf{R}_1 \xrightarrow{(1-x)^{a_i \epsilon} \to 0} \mathbf{R}_{10}$$

4) Find the $x \to 1$ limit by acting $\mathbf{R}_{10}$ to $\mathbf{g}_{trunc}$:

$$\mathbf{g}_{x \to 1} = \mathbf{R}_{10} \mathbf{g}_{trunc}$$

5) Reduce the number of the basis elements to the number of the MI of the massless problem using the property $\mathbf{R}_{10}^2 = \mathbf{R}_{10} \Rightarrow \mathbf{R}_{10}\mathbf{g}_{x \to 1} = \mathbf{g}_{x \to 1}$ and/or IBPs.