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# IIB matrix model, bosonic master field, and emergent spacetime


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# 1. Algebraic equation

The master field of the title is essentially determined by an **algebraic equation** for $D$ traceless Hermitian matrices $\widehat{a}^{\,\mu}$ of dimension $N \times N$:

$$i\,\big(\widehat{p}_k - \widehat{p}_l\big)\,\widehat{a}^{\,\mu}_{\;kl} \;=\; \Big[\widehat{a}^{\,\nu},\,\big[\widehat{a}^{\,\nu},\,\widehat{a}^{\,\mu}\big]\Big]_{kl} - F\,\frac{1}{\mathcal{P}(\widehat{a})}\,\frac{\partial\,\mathcal{P}(\widehat{a})}{\partial\,\widehat{a}^{\,\mu}_{\;lk}} + \widehat{\eta}^{\,\mu}_{\;kl}\,, \tag{1a}$$

$$\mathcal{P}(\widehat{a}) \;=\; \text{homogeneous polynomial of degree } K\,, \tag{1b}$$

$$K \;\equiv\; \big(D-2\big)\,\big(N^2-1\big)\,, \tag{1c}$$

$$(D,\,F) \;=\; (10,\,1), \quad N \gg 1\,, \tag{1d}$$

with matrix indices $k$, $l$ running over $\{1,\,\ldots\,,\,N\}$ and directional indices $\mu$, $\nu$ running over $\{1,\,\ldots\,,\,D\}$, while $\nu$ in (1a) is implicitly summed over.

The $\widehat{p}_k$ are fixed uniform random numbers and the $\widehat{\eta}^{\,\mu}_{\;kl}$ fixed Gaussian random numbers.

There is an explicit expression for the Pfaffian $\mathcal{P}$ to be discussed later.

# 1. Algebraic equation

The algebraic equation (1) is quite a challenge for mathematics and computational science.

But why is that equation also of interest to physics?

Well, the answer is that its solution may contain information about

| the emergence of spacetime and the birth of the Universe. |
|---|

Here, I can only give some background in a ultrashort version, as the main focus will be on discussing **solutions** of this algebraic equation.

# 2. Background (ultrashort version)

We start from the IIB matrix model [1, 2], which reproduces the structure of the light-cone string field theory of type-IIB superstrings.

The IIB matrix model has a **finite number** of $N \times N$ traceless Hermitian matrices: ten bosonic matrices $A^\mu$ and eight fermionic (Majorana–Weyl) matrices $\Psi_\alpha$.

The partition function $Z$ of the IIB matrix model is defined by a “path” integral over $A$ and $\Psi$ with a weight factor $\exp[-S_{\text{bos}}(A) - S_{\text{ferm}}(\Psi,\, A)]$. The fermionic matrices $\Psi$ can be integrated out exactly (Gaussian integrals) and then give the Pfaffian $\mathcal{P}(A)$.

For strings of bosonic observables, the expectation values are defined by the same $A$-integral as $Z$, that is, involving the exponential weight factor with the bosonic action, $\exp[-S_{\text{bos}}(A)]$, and the Pfaffian $\mathcal{P}(A)$.

For large $N$, these expectation values can also be obtained by inserting the matrices $\widehat{A}^{\,\mu}$ of the so-called **master-field** [3, 4] directly into the observables, without need of any integration.

# 2. Background (ultrashort version)

Recently, we have suggested [5] that precisely the master-field matrices $\widehat{A}^{\,\mu}$ of the IIB matrix model may give rise to an emergent classical spacetime (some details are given in Appendix A).

**Assuming** that the matrices $\widehat{A}^{\,\mu}$ of the IIB-matrix-model master field are known and that they are approximately band-diagonal, it relatively easy [5] to extract a discrete set of spacetime points $\{\widehat{x}_k^{\,\mu}\}$ and an interpolating (inverse) metric $g^{\mu\nu}(x)$.

It has also been established that, in principle, it is possible to get, from appropriate distributions of the extracted spacetime points $\{\widehat{x}_k^{\,\mu}\}$, the metrics of the Minkowski and the spatially flat Robertson–Walker spacetimes. See the recent review [6] for further discussion.

But, instead of *assuming* the matrices $\widehat{A}^{\,\mu}$, we want to **calculate** them. And, for that, we need an equation . . .

## 3. Equation and solutions

GOOD NEWS:

the master-field equation has already been established, nearly 40 years ago, by Greensite and Halpern [4], who write in the first line of their abstract:

“We derive an exact algebraic (master) equation for the euclidean master field of any large-$N$ matrix theory, including quantum chromodynamics.”

Now, “any” means “any” and we may as well consider the large-$N$ IIB matrix theory of Kawai and collaborators [1, 2].

# 3.1 Bosonic master-field equation

Building on this work by Greensite and Halpern [4], we then have the IIB-matrix-model bosonic master field in "quenched" form [5]:

$$\widehat{A}^{\,\mu}_{\,kl} = e^{i\,(\widehat{p}_k\,-\,\widehat{p}_l)\,\tau_{\text{eq}}}\;\widehat{a}^{\,\mu}_{\,kl}\,. \tag{2a}$$

The $\widehat{p}_k$ are random constants (see below) and the dimensionless time $\tau_{\text{eq}}$ must have a sufficiently large value in order to represent an equilibrium situation ($\tau$ is the fictitious Langevin time of the stochastic-quantization procedure). The $\tau$-independent matrix $\widehat{a}^{\,\mu}$ on the right-hand side of (2a) solves the following algebraic equation [5]:

$$i\,\big(\widehat{p}_k - \widehat{p}_l\big)\,\widehat{a}^{\,\mu}_{\,kl} \;=\; -\frac{\partial\, S_{\text{eff}}}{\partial\,\widehat{a}_{\mu\;lk}} + \widehat{\eta}^{\,\mu}_{\,kl}\,, \tag{2b}$$

in terms of the master momenta $\widehat{p}_k$ (real uniform random numbers) and the master noise matrices $\widehat{\eta}^{\,\mu}_{\,kl}$ (real Gaussian random numbers).

The algebraic equation (2b) is, of course, precisely (1).

# 3.2 Solutions of the simplified equation

The algebraic equation (1) is rather formidable and it makes sense to first consider the simplified equation obtained by setting $F = 0$:

$$i \left(\widehat{p}_k - \widehat{p}_l\right) \widehat{a}\,^{\mu}_{kl} \quad = \quad \left[\widehat{a}\,^{\nu},\, \left[\widehat{a}\,^{\nu},\, \widehat{a}\,^{\mu}\right]\right]_{kl} + \widehat{\eta}\,^{\mu}_{kl}\,. \tag{3}$$

The matrices $\widehat{a}\,^{\mu}$ are $N \times N$ traceless Hermitian matrices and the number of variables is

$$N_{\text{var}} = D\left(N^2 - 1\right), \tag{4}$$

which grows rapidly with increasing $N$. Remark also that the simplified equation (3) is essentially a **cubic polynomial**.

It appears impossible to obtain a general analytic solution of (3) in terms of the master constants $\widehat{p}_k$ and $\widehat{\eta}\,^{\mu}_{kl}$. Instead, we will try to get solutions for an explicit choice for the random master constants.

## 3.2 Solutions of the simplified equation

For $(D, N) = (2, 6)$, consider the simplified equation (3) for 70 real variables, with a particular realization (the "alpha-realization") of the pseudorandom numbers entering the equation.

Other realizations give similar results.

Specifically, we take the following 6 pseudorandom numbers for the master momenta:

$$\widehat{p}_{\alpha\text{-realization}} = \left\{ \frac{53}{500},\; -\frac{9}{100},\; -\frac{441}{1000},\; \frac{217}{1000},\; \frac{371}{1000},\; \frac{19}{40} \right\}, \qquad (5)$$

and the following 70 pseudorandom numbers for the master noise (splitting the matrices into real and imaginary parts):

# 3.3 Solutions of the simplified equation

$$\mathrm{Re}\left[\widehat{\eta}^{\,1}_{\alpha\text{-realization}}\right] = \begin{pmatrix} -\frac{81}{125} & \frac{71}{1000} & -\frac{151}{500} & \frac{371}{500} & -\frac{83}{200} & \frac{491}{1000} \\ \frac{71}{1000} & -\frac{279}{1000} & -\frac{259}{500} & -\frac{13}{1000} & -\frac{493}{500} & \frac{449}{1000} \\ -\frac{151}{500} & -\frac{259}{500} & -\frac{413}{1000} & \frac{911}{1000} & \frac{203}{250} & \frac{299}{1000} \\ \frac{371}{500} & -\frac{13}{1000} & \frac{911}{1000} & \frac{671}{1000} & -\frac{417}{500} & -\frac{913}{1000} \\ -\frac{83}{200} & -\frac{493}{500} & \frac{203}{250} & -\frac{417}{500} & \frac{51}{125} & \frac{181}{250} \\ \frac{491}{1000} & \frac{449}{1000} & \frac{299}{1000} & -\frac{913}{1000} & \frac{181}{250} & \frac{261}{1000} \end{pmatrix}, \tag{6}$$

$$\mathrm{Im}\left[\widehat{\eta}^{\,1}_{\alpha\text{-realization}}\right] = \begin{pmatrix} 0 & -\frac{441}{1000} & -\frac{17}{250} & -\frac{87}{1000} & -\frac{127}{200} & -\frac{199}{500} \\ \frac{441}{1000} & 0 & -\frac{177}{250} & -\frac{783}{1000} & -\frac{303}{500} & \frac{969}{1000} \\ \frac{17}{250} & \frac{177}{250} & 0 & -\frac{14}{25} & \frac{259}{1000} & -\frac{711}{1000} \\ \frac{87}{1000} & \frac{783}{1000} & \frac{14}{25} & 0 & \frac{43}{250} & \frac{1}{125} \\ \frac{127}{200} & \frac{303}{500} & -\frac{259}{1000} & -\frac{43}{250} & 0 & -\frac{491}{1000} \\ \frac{199}{500} & -\frac{969}{1000} & \frac{711}{1000} & -\frac{1}{125} & \frac{491}{1000} & 0 \end{pmatrix}, \tag{7}$$

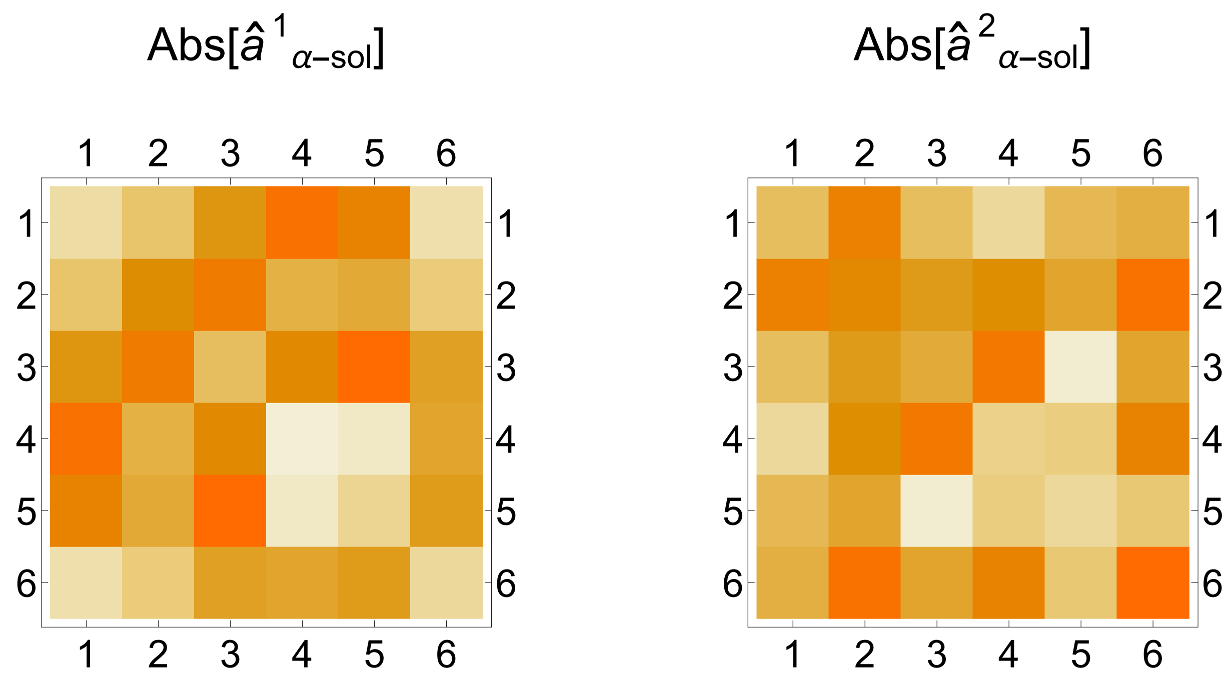
# 3.2 Solutions of the simplified equation

$$\mathrm{Re}\left[\widehat{\eta}^{\,2}_{\alpha\text{-realization}}\right] = \begin{pmatrix} \frac{41}{200} & \frac{53}{1000} & -\frac{241}{250} & \frac{621}{1000} & \frac{3}{20} & -\frac{51}{200} \\ \frac{53}{1000} & -\frac{139}{500} & \frac{23}{200} & -\frac{557}{1000} & \frac{7}{100} & -\frac{137}{200} \\ -\frac{241}{250} & \frac{23}{200} & -\frac{31}{500} & -\frac{22}{125} & \frac{14}{25} & -\frac{31}{100} \\ \frac{621}{1000} & -\frac{557}{1000} & -\frac{22}{125} & \frac{289}{1000} & -\frac{227}{1000} & -\frac{103}{200} \\ \frac{3}{20} & \frac{7}{100} & \frac{14}{25} & -\frac{227}{1000} & \frac{17}{1000} & \frac{369}{1000} \\ -\frac{51}{200} & -\frac{137}{200} & -\frac{31}{100} & -\frac{103}{200} & \frac{369}{1000} & -\frac{171}{1000} \end{pmatrix}, \quad (8)$$

$$\mathrm{Im}\left[\widehat{\eta}^{\,2}_{\alpha\text{-realization}}\right] = \begin{pmatrix} 0 & \frac{449}{500} & -\frac{31}{250} & \frac{233}{1000} & -\frac{413}{500} & -\frac{807}{1000} \\ -\frac{449}{500} & 0 & -\frac{56}{125} & \frac{7}{50} & -\frac{77}{200} & \frac{23}{500} \\ \frac{31}{250} & \frac{56}{125} & 0 & \frac{409}{500} & \frac{57}{250} & -\frac{689}{1000} \\ -\frac{233}{1000} & -\frac{7}{50} & -\frac{409}{500} & 0 & \frac{189}{500} & -\frac{953}{1000} \\ \frac{413}{500} & \frac{77}{200} & -\frac{57}{250} & -\frac{189}{500} & 0 & \frac{47}{200} \\ \frac{807}{1000} & -\frac{23}{500} & \frac{689}{1000} & \frac{953}{1000} & -\frac{47}{200} & 0 \end{pmatrix}, \quad (9)$$

# 3.2 Solutions of the simplified equation

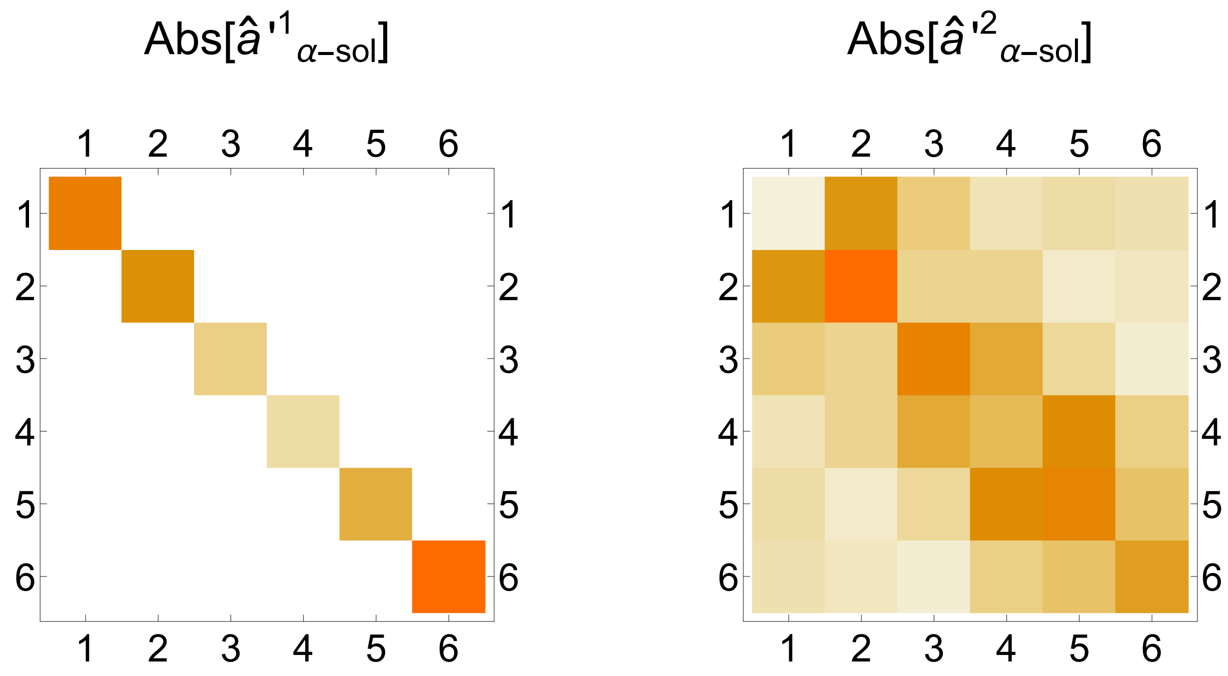
A particular solution [7] for the 70 variables of the simplified equation (3) with the alpha-realization of pseudorandom constants is given by $\widehat{a}^{1}_{\alpha\text{-sol}}$ and $\widehat{a}^{2}_{\alpha\text{-sol}}$ . Consider the absolute values of the matrix entries:




$\Rightarrow$ no obvious band-diagonal structure.

# 3.2 Solutions of the simplified equation

Now, change the basis, in order to diagonalize and order the $\mu = 1$ matrix. This gives the matrices $\widehat{a}\,'^{\,1}_{\alpha\text{-sol}}$ and $\widehat{a}\,'^{\,2}_{\alpha\text{-sol}}$. Consider the absolute values of the matrix entries:




$\Rightarrow$ a diagonal/band-diagonal structure, a highly nontrivial result !

## 3.2 Solutions of the simplified equation

The values $(D,\, N) = (2,\, 6)$ are, of course, rather small. But ...

... scientists at Google Research, Zürich, now have obtained numerical solutions of the simplified equation (3) with $(D,\, N) = (10,\, 50)$ and these solutions apparently also display a diagonal/band-diagonal structure [T. Fischbacher, private communication].

In short, work is in progress on solving and understanding the simplified algebraic equation ...

... but, first, we need to make sure that **dynamical fermions** do not spoil this structure.

## 3.3 Solutions of the full equation

We now look for solutions of the **full** bosonic master-field equation (1), with $F = 1$ to include the dynamic fermions, but, first, with rather small values of $D$ and $N$.

The Pfaffian is a $K$-th order polynomial, denoted symbolically by $P_K(A)$, with $K = (D-2)\,(N^2-1)$ according to (1c).

The basic structure of the algebraic equation (1) is then as follows:

$$P_1^{(\,\widehat{p}\,)}\,(\,\widehat{a}\,) \quad = \quad P_3\,(\,\widehat{a}\,) + F\,\frac{P_{K-1}\,(\,\widehat{a}\,)}{P_K\,(\,\widehat{a}\,)} + P_0^{(\,\widehat{\eta}\,)}\,(\,\widehat{a}\,)\ , \tag{10}$$

where the suffixes on $P_1$ and $P_0$ indicate their respective dependence on the master momenta $\widehat{p}_k$ and the master noise $\widehat{\eta}_{kl}^{\,\mu}$.

Multiplying (10) by $P_K\,(\,\widehat{a}\,)$, we get a polynomial equation of order $K+3$:

$$P_{K+1}^{(\,\widehat{p}\,)}\,(\,\widehat{a}\,) \quad = \quad P_{K+3}\,(\,\widehat{a}\,) + F\,P_{K-1}\,(\,\widehat{a}\,) + P_K^{(\,\widehat{\eta}\,)}\,(\,\widehat{a}\,)\ . \tag{11}$$

# 3.3 Solutions of the full equation

As a start, we have considered the case

$$\{D,\, N,\, F\} = \{3,\, 3,\, 1\}\,, \tag{12}$$

for which the model still has a supersymmetry invariance and the eight generators $T^I$ are proportional to the $3 \times 3$ Gell-Mann matrices $\lambda^I$. The bosonic matrices are expanded as $A_\mu = A_I^\mu\, T^I$, with real coefficients $A_I^\mu$.

Remarkably, there is now an explicit result for the Pfaffian [9],

$$\begin{aligned} \mathcal{P}_{3,\,3}[A] \;=\; & -\frac{3}{4}\,\text{Tr}\Big(\big[A^\mu,\, A^\nu\big]\,\big\{A^\rho,\, A^\sigma\big\}\Big)\,\text{Tr}\Big(\big[A^\mu,\, A^\nu\big]\,\big\{A^\rho,\, A^\sigma\big\}\Big) \\ & +\frac{6}{5}\,\text{Tr}\Big(A^\mu\,\big[A^\nu,\, A^\rho\big]\Big)\text{Tr}\Big(A^\mu\,\big[\{A^\nu,\, A^\sigma\},\, \{A^\rho,\, A^\sigma\}\big]\Big), \end{aligned} \tag{13}$$

which corresponds to a real homogenous eighth-order polynomial in the bosonic coefficients $A_I^\mu$.

# 3.3 Solutions of the full equation

Taking an explicit realization of the random constants, we have established the existence of several solutions of the full bosonic master-field equation (1) for the case $(D,\, N) = (3,\, 3)$.

Moreover, there is a suggested diagonal/band-diagonal structure, but the value $N = 3$ is too small for definitive statements.

Further details can be found in Ref. [8].

## 3.3 Solutions of the full equation

The $(D,\,N)=(3,\,3)$ result was obtained by a direct algebraic calculation. Perhaps there can be further progress by an indirect numerical approach.

The idea (emphasized to me by Jun Nishimura) is to use the fact that the square of the Pfaffian of the skew-symmetric matrix $\mathcal{M}=\mathcal{M}(\widehat{a})$ equals its determinant,

$$\Big[\mathsf{Pf}(\mathcal{M})\Big]^2=\det\mathcal{M}\,, \tag{14a}$$

so that we can write the variational term in the algebraic equation (1) as a trace,

$$\frac{\delta\,\mathsf{Pf}(\mathcal{M})}{\mathsf{Pf}(\mathcal{M})}=\frac{1}{2}\,\frac{\delta\det\mathcal{M}}{\det\mathcal{M}}=\frac{1}{2}\,\mathsf{Tr}\Big[\mathcal{M}^{-1}\delta\,\mathcal{M}\Big]\,, \tag{14b}$$

and this trace can be evaluated numerically (as used in Ref. [10] and earlier papers). This work is still in progress, see the next slide.

# 3.3 Solutions of the full equation

Table 1: Numerical solutions of the full ($F = 1$) bosonic master-field equation (1). The number of variables $N_\text{var}$ is given by (4) and the equation is a polynomial of order $N_\text{poly} = K + 3$, with $K$ given by (1c).

| | $N_\text{var}$ | $N_\text{poly}$ | status |
|---|---|---|---|
| $(D,\, N) = (3,\, 3)$ | 24 | 11 | done ($\sim 1/2$ hr) [a] |
| $(D,\, N) = (10,\, 3)$ | 80 | 67 | done ($\sim 76$ hrs) |
| $(D,\, N) = (10,\, 4)$ | 150 [b] | 123 | preliminary result ($\to$ App. B) |

[a] previous algebraic results reproduced

[b] complex variables in the solution, as the Pfaffian $\mathcal{P}(\widehat{a})$ is complex

# 4. Conclusions

It is conceivable that a **new physics phase** gives rise to classical spacetime, gravity, and matter, as described by our current theories (General Relativity and the Standard Model).

For an explicit calculation, we have considered the **IIB matrix model**, which has been proposed as a nonperturbative formulation of type-IIB superstring theory (M-theory).

The crucial insight is that the emergent classical spacetime may reside in the **large-N master field** $\widehat{A}^{\mu}$ of the IIB matrix model.

We have now started to **solve** the full bosonic master-field equation of the IIB matrix model: first results are in, but the road ahead is long and arduous ...

# 5. References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# A. Background (short version)

The algebraic equation of interest arises from the IKKT matrix model [1]. That model is also known as the IIB matrix model [2], as the matrix model reproduces the basic structure of the light-cone string field theory of type-IIB superstrings.

The IIB matrix model has a **finite number** of $N \times N$ traceless Hermitian matrices: ten bosonic matrices $A^\mu$ and eight fermionic (Majorana–Weyl) matrices $\Psi_\alpha$.

The partition function $Z$ of the IIB matrix model is defined by the following "path" integral:

$$Z = \int dA\, d\Psi\; e^{-\,S(A,\,\Psi)} = \int dA\, d\Psi\; e^{-S_{\text{bos}}(A)\,-\,S_{\text{ferm}}(\Psi,\,A)}, \quad \text{(A.1)}$$

where the bosonic action $S_{\text{bos}}(A)$ is quartic in $A$ and the fermionic action $S_{\text{ferm}}(\Psi,\, A)$ is quadratic in $\Psi$ and linear in $A$, i.e., $S_{\text{ferm}} = \overline{\Psi}\, \mathcal{M}(A)\, \Psi$.

# A. Background (short version)

The fermionic matrices $\Psi$ can be integrated out exactly (Gaussian integrals) and give the Pfaffian Pf$[\mathcal{M}(A)] \equiv \mathcal{P}(A)$:

$$Z \;=\; \int dA\, \mathcal{P}(A)\, e^{-S_{\text{bos}}(A)} \equiv \int dA\, e^{-S_{\text{eff}}(A)}\,. \qquad \text{(A.2)}$$

For the bosonic observable

$$w^{\mu_1\,\cdots\,\mu_m} = \text{Tr}\left(A^{\mu_1}\cdots\, A^{\mu_m}\right), \qquad \text{(A.3)}$$

and arbitrary strings thereof, the expectation values are defined by the same integral as in (A.2):

$$\begin{aligned}&\left\langle w^{\mu_1\,\cdots\,\mu_m}\, w^{\nu_1\,\cdots\,\nu_n}\;\cdots\; w^{\omega_1\,\cdots\,\omega_z}\right\rangle\\ &= \frac{1}{Z}\int dA\left(w^{\mu_1\,\cdots\,\mu_m}\, w^{\nu_1\,\cdots\,\nu_n}\;\cdots\; w^{\omega_1\,\cdots\,\omega_z}\right) e^{-\,S_{\text{eff}}}\,. \qquad \text{(A.4)}\end{aligned}$$

# A. Background (short version)

Now, the IIB matrix model just gives **numbers**, $Z$ and the expectation values $\langle w\, w\, \cdots \rangle$, while the matrices $A^\mu$ and $\Psi_\alpha$ in the "path" integral are merely integration variables.

Moreover, there is no obvious small dimensionless parameter to motivate a saddle-point approximation.

The **conceptual** question arises: where is the classical spacetime?

Recently, we have suggested to revisit an old idea, the large-$N$ master field of Witten [3, 4], for a possible origin of classical spacetime in the context of the IIB matrix model [5].

# A. Background (short version)

According to Witten [3], the large-$N$ factorization of the expectation values (A.4) implies that the path integrals are saturated by a <u>single</u> configuration, the so-called **master field** $\widehat{A}^{\mu}$.

To leading order in $N$, the expectation values are then given by

$$\langle w^{\mu_1 \cdots \mu_m}\, w^{\nu_1 \cdots \nu_n} \ldots w^{\omega_1 \cdots \omega_z} \rangle \stackrel{N}{=} \widehat{w}^{\,\mu_1 \cdots \mu_m}\, \widehat{w}^{\,\nu_1 \cdots \nu_n} \ldots \widehat{w}^{\,\omega_1 \cdots \omega_z}, \quad \text{(A.5a)}$$

$$\widehat{w}^{\,\mu_1 \cdots \mu_m} \equiv \text{Tr}\left(\widehat{A}^{\mu_1} \cdots \widehat{A}^{\mu_m}\right). \quad \text{(A.5b)}$$

Hence, we do not have to perform the path integrals on the right-hand side of (A.4): we just need ten traceless Hermitian matrices $\widehat{A}^{\mu}$ to get *all* these expectation values from the simple procedure of replacing each $A^{\mu}$ in the observables by the corresponding $\widehat{A}^{\mu}$.

# A. Background (short version)

Now, the meaning of the suggestion at the bottom of slide 25 is clear:

classical spacetime may reside in the bosonic master-field matrices $\widehat{A}^{\,\mu}$ of the IIB matrix model.

The heuristics is as follows [6]:

- The expectation values $\langle w^{\mu_1 \cdots \mu_m} \; \ldots \; w^{\omega_1 \cdots \omega_z} \rangle$ from (A.4), infinitely many numbers, correspond to a large part of the **information content** of the IIB matrix model (but, of course, not all the information).
- That **same** information is contained in the master-field matrices $\widehat{A}^{\,\mu}$, which, to leading order in $N$, give the same numbers $(\widehat{w}^{\,\mu_1 \cdots \mu_m} \; \ldots \; \widehat{w}^{\,\omega_1 \cdots \omega_z})$, where $\widehat{w}$ is the observable $w$ evaluated for $\widehat{A}$.
- From these master-field matrices $\widehat{A}^{\,\mu}$, it appears possible to **extract** the points and metric of an emergent classical spacetime (recall that the original matrices $A^{\,\mu}$ were merely integration variables).

# A. Background (short version)

**Assuming** that the matrices $\widehat{A}^{\,\mu}$ of the IIB-matrix-model master field are known and that they are approximately band-diagonal (as suggested by the numerical results of Ref. [11, 12, 10] and references therein), it is possible [5] to extract a discrete set of spacetime points $\{\widehat{x}_k^{\,\mu}\}$ and an interpolating (inverse) metric $g^{\mu\nu}(x)$.

It has been established that, in principle, it is possible to get, from appropriate distributions of the extracted spacetime points $\{\widehat{x}_k^{\,\mu}\}$, the metrics of the Minkowski and the spatially flat Robertson–Walker spacetimes. See the recent review [6] for further discussion.

But, instead of assuming the matrices $\widehat{A}^{\,\mu}$, we want to **calculate** them. And, for that, we need an equation. $\rightarrow$ Sec. 3

# B. Approximate solution

The numerical results of the first two rows in Table 1 were obtained from the `NMinimize` routine of MATHEMATICA 12.1, using the downhill-simplex method of Nelder and Mead (1965).

But this procedure is no longer suitable for the 300 real variables of the case of interest,

$$(D,\, N) = (10,\, 4)\,. \tag{B.1}$$

Instead, we have used a simple procedure, which could be partially parallelized. In this way, we have obtained an approximate solution.

Before we describe the obtained result, we need to specify a particular realization (the "$\kappa$-realization") of the pseudorandom numbers entering the algebraic equation (1). As this involves 154 real numbers, the details are rather cumbersome and are relegated to App. C.

# B. Approximate solution

We have obtained an approximate solution for the 300 real variables of the algebraic equation (1) with the $\kappa$-realization of pseudorandom constants:

$$\widehat{a}^{\,\mu}_{\kappa\text{-approx-sol}}\,, \quad \text{for} \quad \mu = 1,\,\ldots\,,10\,. \tag{B.2}$$

With complex coefficients $\widehat{a}^{\,\mu}_{I}$, these (approximate) master-field matrices are no longer Hermitian. The situation is perhaps analogous to that of complex saddle-points appearing for a real problem.

Our interpretation is that these (approximate) master-field matrices contain **information** both in their Hermitian and anti-Hermitian parts.

We suspect that the Hermitian parts of the master-field matrices (with real eigenvalues) contain information about the emerging spacetime. What the information in the anti-Hermitian parts corresponds to is not clear for the moment.

# B. Approximate solution

Consider, therefore, the Hermitian parts

$$\widehat{a}^{\,\mu}_{\kappa\text{-approx-sol-HERM}} \equiv \frac{1}{2}\,\left[\widehat{a}^{\,\mu}_{\kappa\text{-approx-sol}} + \left(\widehat{a}^{\,\mu}_{\kappa\text{-approx-sol}}\right)^{\dagger}\right], \qquad \text{(B.3)}$$

Getting the absolute values of these matrix entries, we observe no obvious band-diagonal structure.

See the next 3 slides with three improving approximate solutions (decreasing penalty function).

# B. Approximate solution

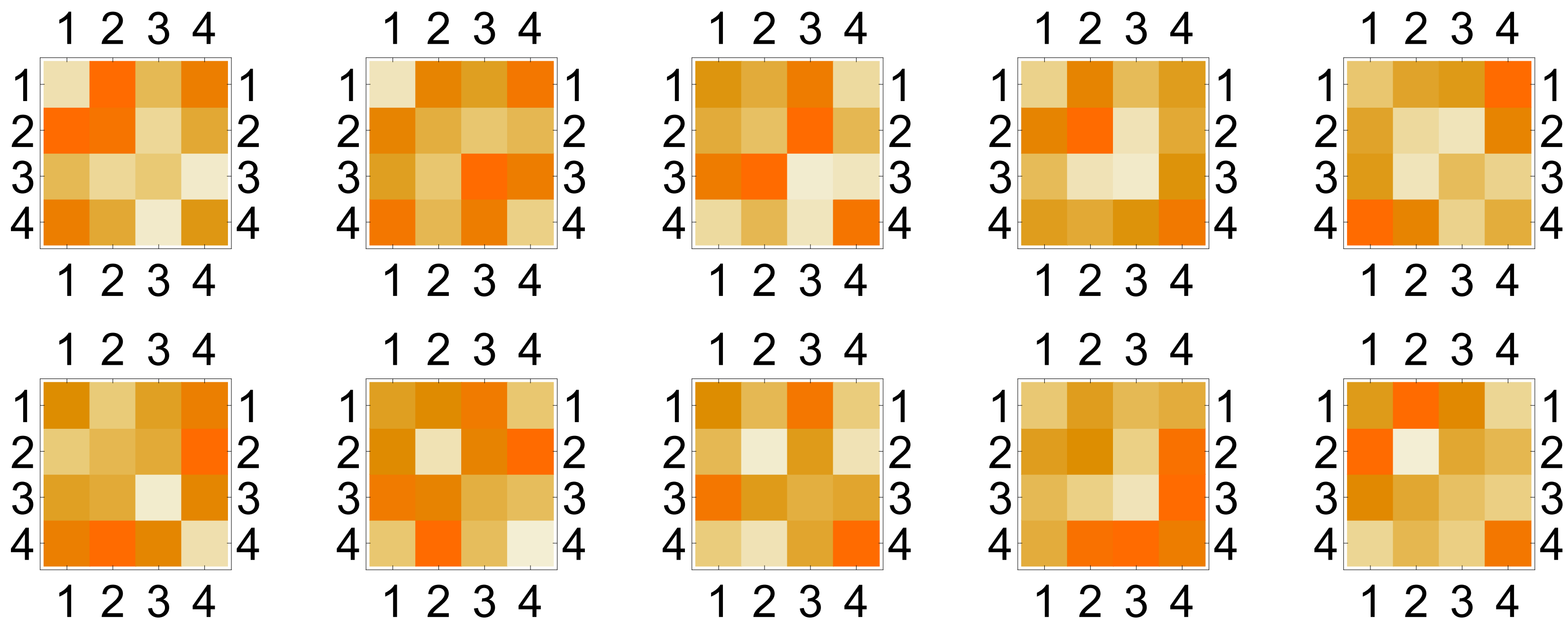


Figure 1: $(D,\, N) \;=\; (10,\, 4)$ approximate solution Abs $\left[\widehat{a}^{\,\mu}_{\kappa\text{-approx-sol-HERM}}\right]$ with $f_{\text{penalty}} = 422.468$. Shown are $\mu = 1, \ldots, 5$ on the top row and $\mu = 6, \ldots, 10$ on the bottom row.

# B. Approximate solution

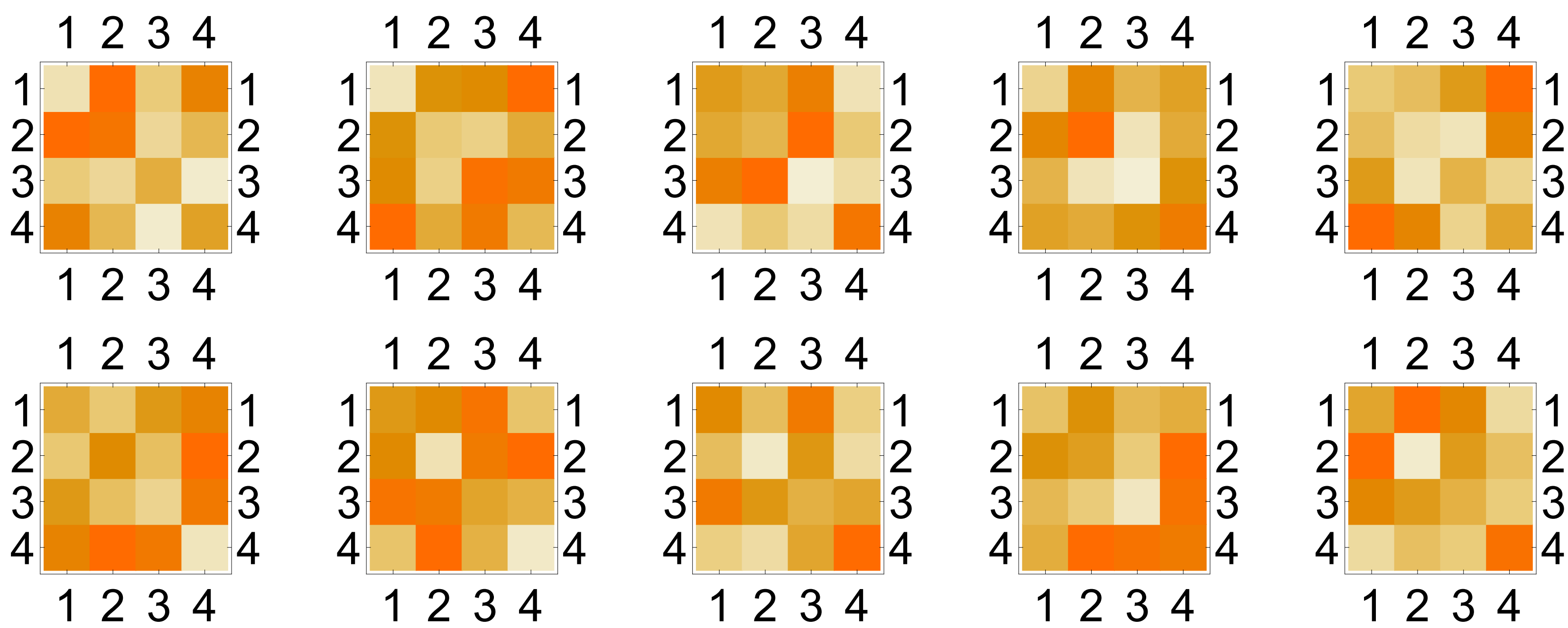

Figure 2: $(D,\, N) = (10,\, 4)$ approximate solution Abs $\left[\widehat{a}^{\,\mu}_{\kappa\text{-approx-sol-HERM}}\right]$ with $f_{\text{penalty}} = 310.932$. Shown are $\mu = 1,\, \ldots\, , 5$ on the top row and $\mu = 6,\, \ldots\, , 10$ on the bottom row.

# B. Approximate solution

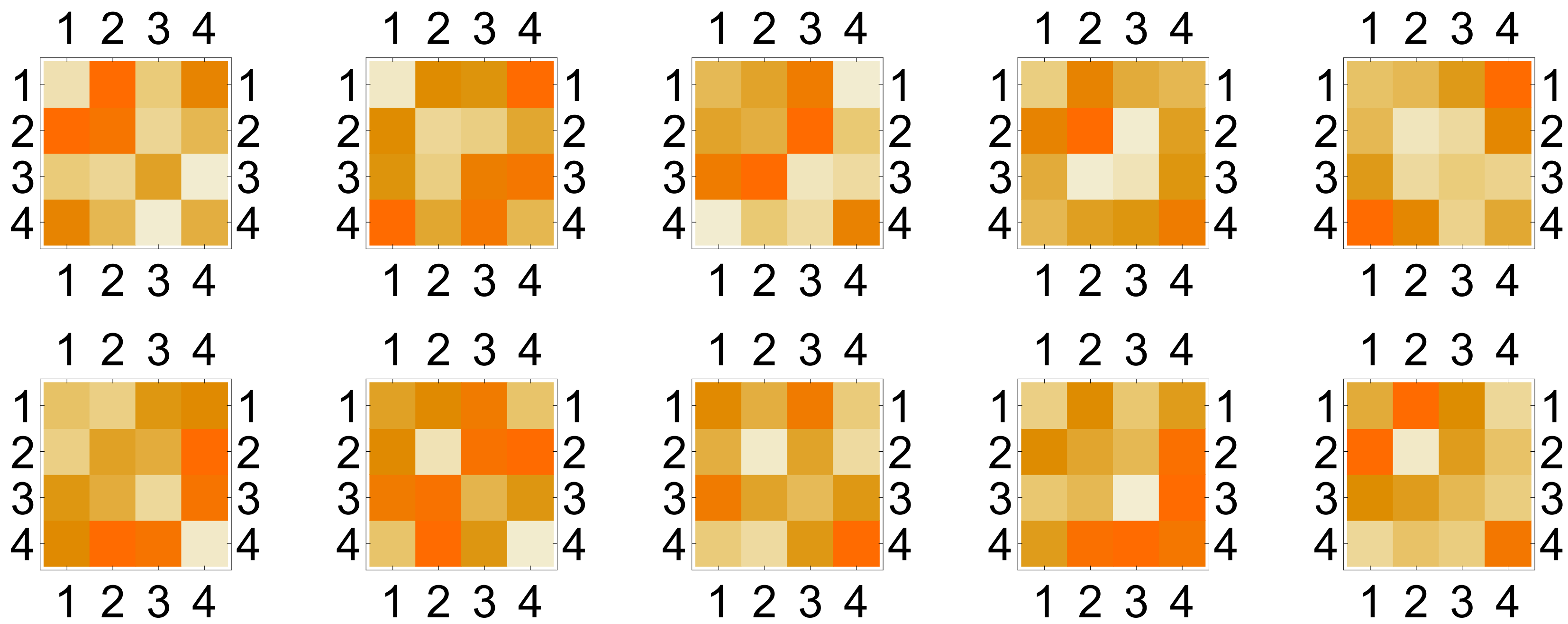


Figure 3: $(D,\, N) = (10,\, 4)$ approximate solution Abs $\left[\widehat{a}^{\,\mu}_{\kappa\text{-approx-sol-HERM}}\right]$ with $f_{\text{penalty}} = 209.33$. Shown are $\mu = 1,\, \ldots\, , 5$ on the top row and $\mu = 6,\, \ldots\, , 10$ on the bottom row.

# B. Approximate solution

Now, change the basis, in order to diagonalize and order the $\mu = 1$ matrix. This gives the matrices

$$\widehat{a}^{\,\prime\,\mu}_{\kappa\text{-approx-sol-HERM}}\,, \quad \text{for} \quad \mu = 1,\,\ldots\,,10\,. \tag{B.4}$$

Considering the absolute values of the matrix entries, there is not yet a strong signal for a diagonal/band-diagonal structure.

But perhaps we see a trend: look, for example, at the pattern of the $\mu = 2$ matrix as it evolves with improving approximations (decreasing penalty function).

# B. Approximate solution

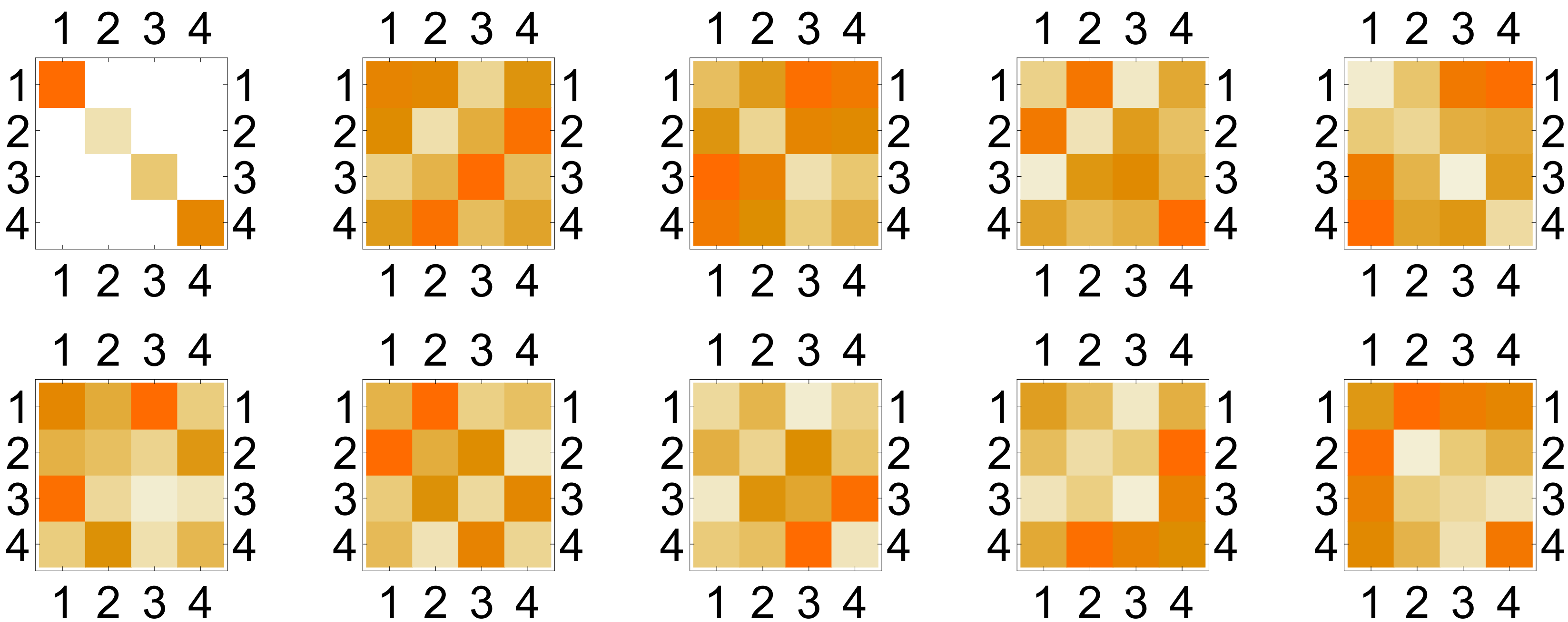

Figure 4: $(D,\, N) = (10,\, 4)$ approximate solution Abs $\left[\widehat{a}\,'^{\,\mu}_{\kappa\text{-approx-sol-HERM}}\right]$ with $f_{\text{penalty}} = 422.468$. Shown are $\mu = 1,\, \ldots\, , 5$ on the top row and $\mu = 6,\, \ldots\, , 10$ on the bottom row.

# B. Approximate solution

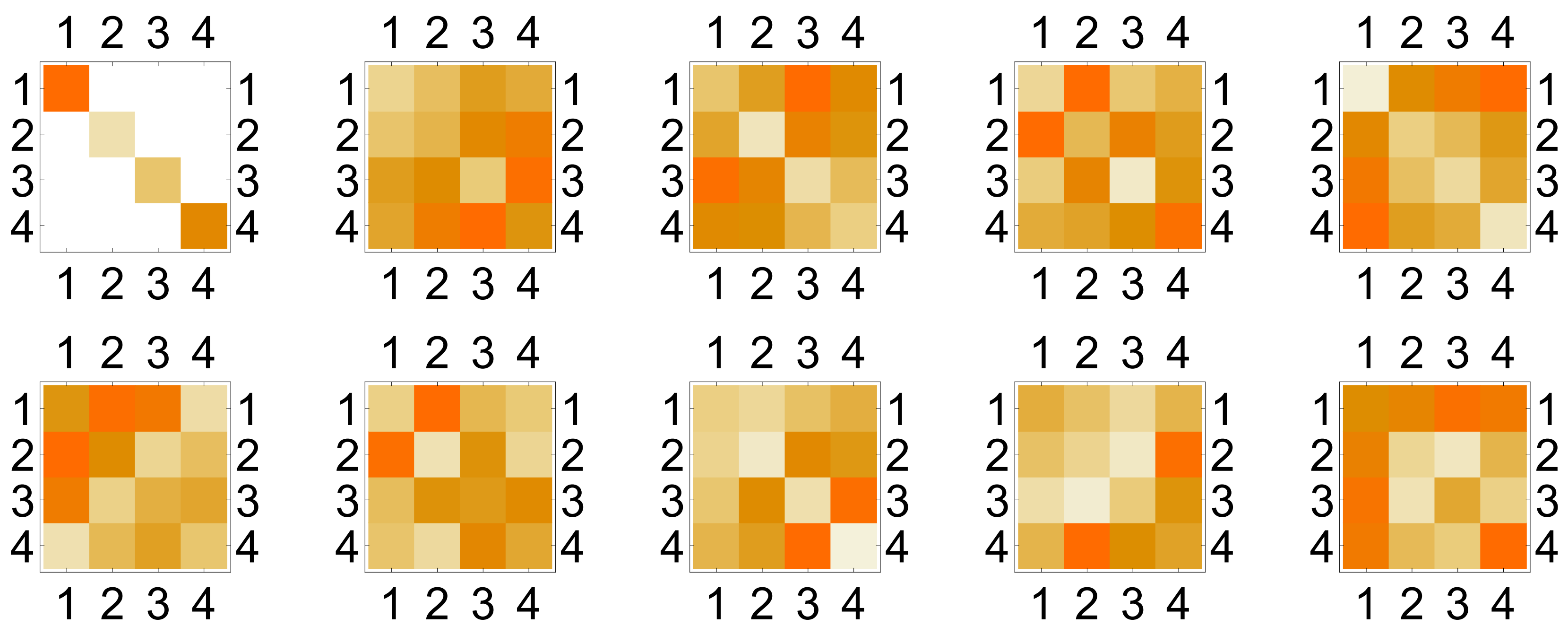


Figure 5: $(D,\,N)\;=\;(10,\,4)$ approximate solution Abs $\left[\widehat{a}\,{}'^{\,\mu}_{\kappa\text{-approx-sol-HERM}}\right]$ with $f_{\text{penalty}} = 310.932$. Shown are $\mu = 1, \ldots, 5$ on the top row and $\mu = 6, \ldots, 10$ on the bottom row.

# B. Approximate solution

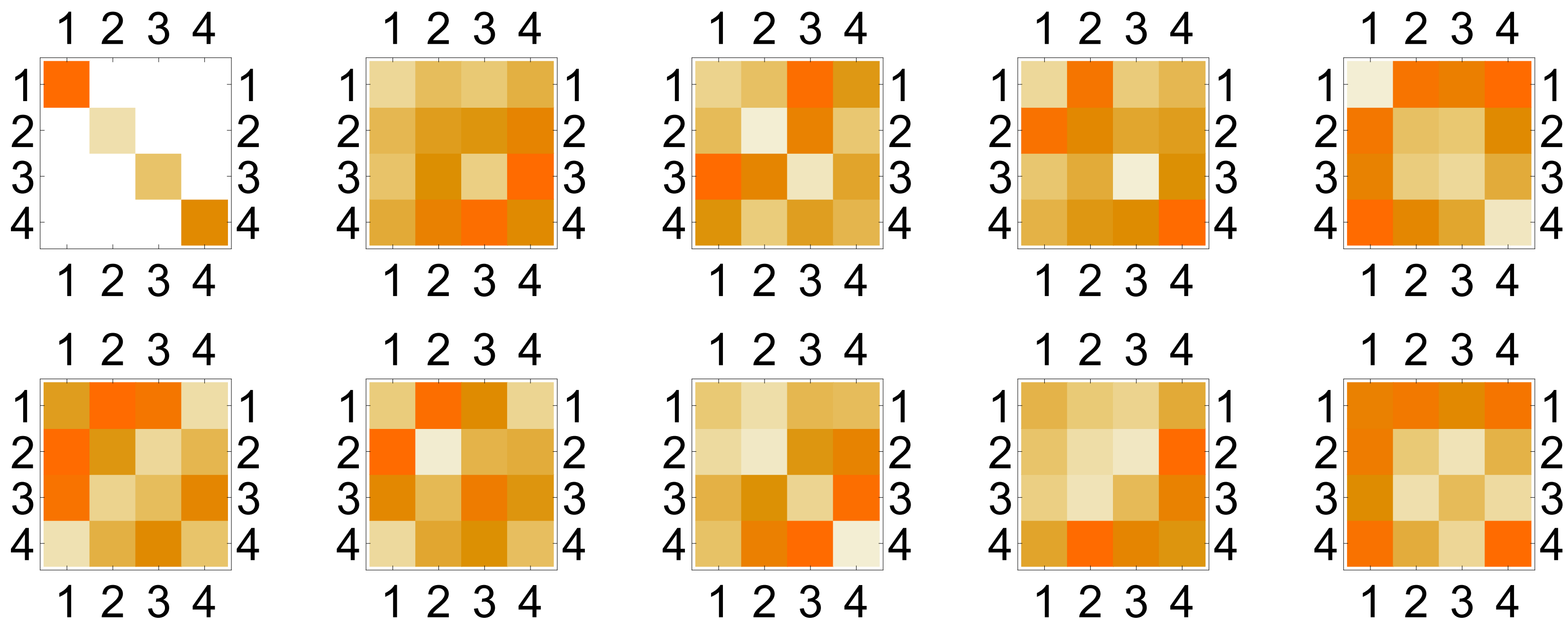


Figure 6: $(D,\, N) = (10,\, 4)$ approximate solution Abs $\left[\widehat{a}^{\,\prime\,\mu}_{\kappa\text{-approx-sol-HERM}}\right]$ with $f_{\text{penalty}} = 209.33$. Shown are $\mu = 1, \ldots, 5$ on the top row and $\mu = 6, \ldots, 10$ on the bottom row.

# C. Pseudorandom numbers

In this appendix, we give the explicit realization (the "$\kappa$-realization") of the pseudorandom numbers used for the approximate solution of App. B.

Specifically, we take the following 4 pseudorandom numbers for the master momenta:

$$\widehat{p}_{\kappa\text{-realization}} = \left\{ -\frac{111}{250},\, \frac{19}{200},\, -\frac{63}{200},\, \frac{189}{1000} \right\}, \tag{C.1}$$

and the following 150 pseudorandom numbers entering the Hermitian master-noise matrices:

# C. Pseudorandom numbers

$$\widehat{\eta}^{\,1}_{\kappa}\text{-realization} =$$

$$\begin{pmatrix} \frac{593}{2000\sqrt{2}} - \frac{151}{500} & \frac{1}{2000} - \frac{9i}{40} & \frac{353}{2000} + \frac{6i}{25} & -\frac{987}{2000} - \frac{51i}{400} \\ \frac{1}{2000} + \frac{9i}{40} & \frac{151}{500} + \frac{593}{2000\sqrt{2}} & \frac{1}{8} - \frac{63i}{2000} & -\frac{131}{400} - \frac{367i}{2000} \\ \frac{353}{2000} - \frac{6i}{25} & \frac{1}{8} + \frac{63i}{2000} & -\frac{369}{1000} - \frac{593}{2000\sqrt{2}} & -\frac{169}{400} + \frac{171i}{500} \\ -\frac{987}{2000} + \frac{51i}{400} & -\frac{131}{400} + \frac{367i}{2000} & -\frac{169}{400} - \frac{171i}{500} & \frac{369}{1000} - \frac{593}{2000\sqrt{2}} \end{pmatrix}, \tag{C.2}$$

$$\widehat{\eta}^{\,2}_{\kappa}\text{-realization} =$$

$$\begin{pmatrix} \frac{69}{2000} - \frac{17}{50\sqrt{2}} & -\frac{153}{500} - \frac{47i}{500} & \frac{897}{2000} - \frac{61i}{1000} & \frac{269}{2000} - \frac{237i}{2000} \\ -\frac{153}{500} + \frac{47i}{500} & -\frac{69}{2000} - \frac{17}{50\sqrt{2}} & \frac{1}{250} - \frac{13i}{200} & -\frac{103}{400} + \frac{367i}{1000} \\ \frac{897}{2000} + \frac{61i}{1000} & \frac{1}{250} + \frac{13i}{200} & \frac{17}{50\sqrt{2}} - \frac{123}{250} & \frac{1}{20} - \frac{71i}{2000} \\ \frac{269}{2000} + \frac{237i}{2000} & -\frac{103}{400} - \frac{367i}{1000} & \frac{1}{20} + \frac{71i}{2000} & \frac{123}{250} + \frac{17}{50\sqrt{2}} \end{pmatrix}, \tag{C.3}$$

# C. Pseudorandom numbers

$$\widehat{\eta}^{\,3}_{\kappa\text{-realization}} =$$

$$\begin{pmatrix} \frac{313}{1000}+\frac{7}{250\sqrt{2}} & \frac{7}{200}+\frac{431i}{2000} & \frac{17}{40}-\frac{59i}{250} & -\frac{437}{2000}+\frac{69i}{1000} \\ \frac{7}{200}-\frac{431i}{2000} & \frac{7}{250\sqrt{2}}-\frac{313}{1000} & -\frac{991}{2000}-\frac{49i}{250} & \frac{83}{1000}-\frac{279i}{2000} \\ \frac{17}{40}+\frac{59i}{250} & -\frac{991}{2000}+\frac{49i}{250} & \frac{871}{2000}-\frac{7}{250\sqrt{2}} & \frac{49}{250}+\frac{121i}{500} \\ -\frac{437}{2000}-\frac{69i}{1000} & \frac{83}{1000}+\frac{279i}{2000} & \frac{49}{250}-\frac{121i}{500} & -\frac{871}{2000}-\frac{7}{250\sqrt{2}} \end{pmatrix}, \tag{C.4}$$

$$\widehat{\eta}^{\,4}_{\kappa\text{-realization}} =$$

$$\begin{pmatrix} \frac{293}{1000}+\frac{429}{2000\sqrt{2}} & -\frac{11}{250}-\frac{469i}{2000} & -\frac{439}{1000}-\frac{483i}{2000} & -\frac{11}{50}+\frac{41i}{400} \\ -\frac{11}{250}+\frac{469i}{2000} & \frac{429}{2000\sqrt{2}}-\frac{293}{1000} & \frac{409}{1000}-\frac{343i}{2000} & \frac{3}{16}-\frac{407i}{1000} \\ -\frac{439}{1000}+\frac{483i}{2000} & \frac{409}{1000}+\frac{343i}{2000} & -\frac{6}{125}-\frac{429}{2000\sqrt{2}} & -\frac{141}{500}-\frac{31i}{2000} \\ -\frac{11}{50}-\frac{41i}{400} & \frac{3}{16}+\frac{407i}{1000} & -\frac{141}{500}+\frac{31i}{2000} & \frac{6}{125}-\frac{429}{2000\sqrt{2}} \end{pmatrix}, \tag{C.5}$$

# C. Pseudorandom numbers

$$\widehat{\eta}^{\,5}_{\kappa}\text{-realization} =$$

$$\begin{pmatrix} \frac{183}{500\sqrt{2}}-\frac{48}{125} & \frac{37}{125}+\frac{249i}{1000} & -\frac{511}{2000}-\frac{43i}{500} & -\frac{129}{2000}+\frac{19i}{200} \\ \frac{37}{125}-\frac{249i}{1000} & \frac{48}{125}+\frac{183}{500\sqrt{2}} & \frac{49}{400}-\frac{463i}{1000} & \frac{23}{200}+\frac{433i}{2000} \\ -\frac{511}{2000}+\frac{43i}{500} & \frac{49}{400}+\frac{463i}{1000} & -\frac{46}{125}-\frac{183}{500\sqrt{2}} & \frac{821}{2000}+\frac{199i}{2000} \\ -\frac{129}{2000}-\frac{19i}{200} & \frac{23}{200}-\frac{433i}{2000} & \frac{821}{2000}-\frac{199i}{2000} & \frac{46}{125}-\frac{183}{500\sqrt{2}} \end{pmatrix}, \tag{C.6}$$

$$\widehat{\eta}^{\,6}_{\kappa}\text{-realization} =$$

$$\begin{pmatrix} \frac{181}{1000}-\frac{181}{2000\sqrt{2}} & -\frac{143}{500}+\frac{i}{5} & \frac{921}{2000}-\frac{313i}{1000} & -\frac{603}{2000}-\frac{449i}{2000} \\ -\frac{143}{500}-\frac{i}{5} & -\frac{181}{1000}-\frac{181}{2000\sqrt{2}} & \frac{609}{2000}-\frac{39i}{2000} & \frac{443}{2000}+\frac{383i}{1000} \\ \frac{921}{2000}+\frac{313i}{1000} & \frac{609}{2000}+\frac{39i}{2000} & \frac{941}{2000}+\frac{181}{2000\sqrt{2}} & -\frac{17}{1000}+\frac{27i}{2000} \\ -\frac{603}{2000}+\frac{449i}{2000} & \frac{443}{2000}-\frac{383i}{1000} & -\frac{17}{1000}-\frac{27i}{2000} & \frac{181}{2000\sqrt{2}}-\frac{941}{2000} \end{pmatrix}, \tag{C.7}$$

# C. Pseudorandom numbers

$$\widehat{\eta}^{\,7}_{\kappa}\text{-realization} =$$

$$\begin{pmatrix} \frac{57}{2000}+\frac{83}{200\sqrt{2}} & -\frac{129}{500}-\frac{147i}{2000} & \frac{387}{2000}-\frac{611i}{2000} & \frac{313}{2000}+\frac{191i}{400} \\ -\frac{129}{500}+\frac{147i}{2000} & \frac{83}{200\sqrt{2}}-\frac{57}{2000} & -\frac{61}{500}-\frac{11i}{50} & -\frac{969}{2000}+\frac{927i}{2000} \\ \frac{387}{2000}+\frac{611i}{2000} & -\frac{61}{500}+\frac{11i}{50} & -\frac{2}{125}-\frac{83}{200\sqrt{2}} & \frac{489}{1000}+\frac{361i}{1000} \\ \frac{313}{2000}-\frac{191i}{400} & -\frac{969}{2000}-\frac{927i}{2000} & \frac{489}{1000}-\frac{361i}{1000} & \frac{2}{125}-\frac{83}{200\sqrt{2}} \end{pmatrix}, \qquad \text{(C.8)}$$

$$\widehat{\eta}^{\,8}_{\kappa}\text{-realization} =$$

$$\begin{pmatrix} \frac{321}{2000}-\frac{193}{2000\sqrt{2}} & -\frac{273}{2000}-\frac{441i}{2000} & \frac{427}{1000}-\frac{317i}{2000} & -\frac{407}{2000}-\frac{57i}{400} \\ -\frac{273}{2000}+\frac{441i}{2000} & -\frac{321}{2000}-\frac{193}{2000\sqrt{2}} & \frac{23}{2000}+\frac{93i}{400} & \frac{163}{500}+\frac{18i}{125} \\ \frac{427}{1000}+\frac{317i}{2000} & \frac{23}{2000}-\frac{93i}{400} & \frac{307}{1000}+\frac{193}{2000\sqrt{2}} & \frac{333}{2000}-\frac{9i}{25} \\ -\frac{407}{2000}+\frac{57i}{400} & \frac{163}{500}-\frac{18i}{125} & \frac{333}{2000}+\frac{9i}{25} & \frac{193}{2000\sqrt{2}}-\frac{307}{1000} \end{pmatrix}, \qquad \text{(C.9)}$$

# C. Pseudorandom numbers

$$\widehat{\eta}^{\,9}_{\kappa\text{-realization}} =$$

$$\begin{pmatrix} -\frac{39}{400}-\frac{7}{200\sqrt{2}} & \frac{1}{500}-\frac{i}{125} & -\frac{981}{2000}+\frac{56i}{125} & -\frac{137}{1000}+\frac{547i}{2000} \\ \frac{1}{500}+\frac{i}{125} & \frac{39}{400}-\frac{7}{200\sqrt{2}} & -\frac{201}{1000}+\frac{101i}{1000} & -\frac{63}{500}+\frac{31i}{125} \\ -\frac{981}{2000}-\frac{56i}{125} & -\frac{201}{1000}-\frac{101i}{1000} & \frac{197}{400}+\frac{7}{200\sqrt{2}} & \frac{597}{2000}-\frac{57i}{125} \\ -\frac{137}{1000}-\frac{547i}{2000} & -\frac{63}{500}-\frac{31i}{125} & \frac{597}{2000}+\frac{57i}{125} & \frac{7}{200\sqrt{2}}-\frac{197}{400} \end{pmatrix}, \qquad \text{(C.10)}$$

$$\widehat{\eta}^{\,10}_{\kappa\text{-realization}} =$$

$$\begin{pmatrix} \frac{577}{2000\sqrt{2}}-\frac{319}{2000} & \frac{969}{2000}+\frac{2i}{5} & -\frac{243}{2000}+\frac{29i}{500} & -\frac{19}{125}-\frac{151i}{1000} \\ \frac{969}{2000}-\frac{2i}{5} & \frac{319}{2000}+\frac{577}{2000\sqrt{2}} & \frac{17}{50}+\frac{119i}{2000} & \frac{463}{2000}-\frac{33i}{400} \\ -\frac{243}{2000}-\frac{29i}{500} & \frac{17}{50}-\frac{119i}{2000} & \frac{119}{1000}-\frac{577}{2000\sqrt{2}} & \frac{257}{1000}+\frac{219i}{500} \\ -\frac{19}{125}+\frac{151i}{1000} & \frac{463}{2000}+\frac{33i}{400} & \frac{257}{1000}-\frac{219i}{500} & -\frac{119}{1000}-\frac{577}{2000\sqrt{2}} \end{pmatrix}, \qquad \text{(C.11)}$$