Dimensional Regularization Gauss-Bonnet terms and Anomaly Actions

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based on work with
M.M. Maglio (GGI), D. Theofilopoulos (Salento U), (EPJ-C 2021,
and on ongling work with M. Maglio, D. Theofilopoulos,
M. Creti, R. Tommasi

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CFT's have been extensively studied in the last 50 years for a variety of reasons

1. string theory
2. critical behaviour of statistical systems
3. Possible applications to particle phenomenology (extensions of the Standard Model with a "possible" conformal phase
4. Early universe.

5. AdS/CFT correspondence. A theory in a conformal phase is dual – in a well defined sense- to a specific gravitational theory. Applications of this correspondence, from ordinary field theories, to cosmology (holography) as well as condensed matter physics have been overwhelming.

In d=2 spacetime dimensions the theory is particularly rich, but much less in higher dimensions. Neverthless, the power of the construction is significant, vene in the presence of only a finite number -rather than infinite- of symmetries.

Our discussions will be focused on theories with d> 2, where most of the activity, both in theory and phenomenology is.

The context in which we are going to investigate the impact of such symmetry is a phenomenological one, where **we assume that a conformal phase of matter exists** and may actually impact early cosmology.

One could also envision, at a more speculative level, a coupling of the Standard Model to conformal matter, with the inclusion of **extra degrees of freedom in the form of dilaton fields.**

We will briefly illustrate this case, just as example, though this specific point will be touched only briefly.

The methodology used in flat space, con be extended to a general spacetime and comes with specific issues which need to be clarified. In particular to Weyl-flat spacetimes, which is under investigation.

What conects flat **and** curved spacetime analysis is the **ANOMALY INDUCED ACTION and the two counterterms** which are necessary in order to make sense of such theories

Conformal symmetry induces significant constraints on multi-point functions which are controlled by hieriarchical equations, in the form of Conformal Ward Identities (CWIs)

These constraints can be formulated both in a flat and in a curved spacetime.

If we collect all the correlation functions into a single functional and study the impact of the quantum corrections, in the flat limit, starting from this functional, it is possible to investigate such constraints in momentum space in great generality, in any background, as constraints induced by the effective action.

Tensor correlators

TJJ and TTT correlators

Exact Correlators from Conformal Ward Identities in Momentum Space and the Perturbative TJJ Vertex — MM Maglio, CC 2017

The General 3-Graviton Vertex (TTT) of Conformal Field Theories in Momentum Space in $d = 4$ — MM Maglio, CC 2018

TTT in CFT: Trace Identities and the Conformal Anomaly Effective Action — MM Maglio, E Mottola, CC 2018

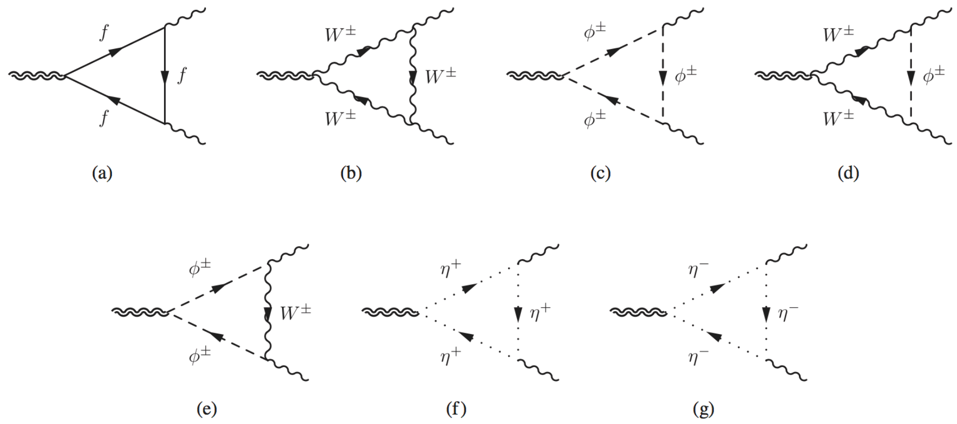
connection with the nonlocal anomaly action

Bzowski, McFadden Skenderis, 2013 BMS

The general reconstruction method is due to BMS

We have provided a simplified analysis of the TJJ and TTT by matching the general reconstruction to free field theory

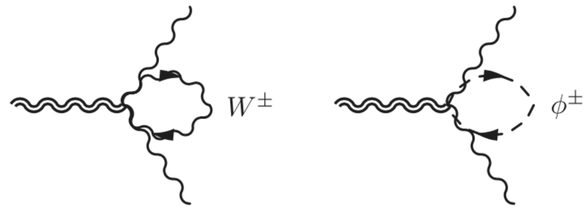
Suppose you want to couple the Standard Model to Gravity



$$
\begin{aligned}
S &= S_G + S_{\rm SM} + S_I \\
&= -\frac{1}{\kappa^2}\int d^4x\sqrt{-g}R + \int d^4x\sqrt{-g}\,\mathcal{L}_{\rm SM} \\
&\quad + \frac{1}{6}\int d^4x\sqrt{-g}R\mathcal{H}^\dagger\mathcal{H},
\end{aligned}
$$

Stress energy tensor, photon photon

An example



Einstein's equations take the form

$$\frac{\delta}{\delta g^{\mu\nu}(x)}S_G = -\frac{\delta}{\delta g^{\mu\nu}(x)}[S_{\rm SM} + S_I]$$

and the EMT in our conventions is defined as

$$T_{\mu\nu}(x) = \frac{2}{\sqrt{-g(x)}}\frac{\delta[S_{\rm SM} + S_I]}{\delta g^{\mu\nu}(x)},$$

Delle Rose, Serino, CC

which is classically covariantly conserved ($g^{\mu\rho}T_{\mu\nu;\rho} = 0$).

$$\mathcal{L}_{\text{grav}}(x) = -\frac{\kappa}{2} T^{\mu\nu}(x) h_{\mu\nu}(x).$$

$$g_{\mu\nu}(x) = \eta_{\mu\nu} + \kappa h_{\mu\nu}(x),$$

$$T_{\mu\nu} = T_{\mu\nu}^{\text{min}} + T_{\mu\nu}^{I},$$

$$T_{\mu\nu}^{\text{min}} = T_{\mu\nu}^{f.s.} + T_{\mu\nu}^{\text{ferm.}} + T_{\mu\nu}^{\text{Higgs}} + T_{\mu\nu}^{\text{Yukawa}} + T_{\mu\nu}^{g.\text{fix.}} + T_{\mu\nu}^{\text{ghost}}.$$

$$\mathcal{L}_{\mathcal{H}} = (D^{\mu}\mathcal{H})^{\dagger}(D_{\mu}\mathcal{H}) + \mu_{\mathcal{H}}^{2}\mathcal{H}^{\dagger}\mathcal{H} - \lambda(\mathcal{H}^{\dagger}\mathcal{H})^{2}\mu_{\mathcal{H}}^{2}, \qquad \lambda > 0,$$

Example : THE HIGGS SECTOR

$$\begin{aligned}
T_{\mu\nu}^{\text{Higgs}} = & -\eta_{\mu\nu}\mathcal{L}_{\text{Higgs}} + \partial_{\mu}H\partial_{\nu}H + \partial_{\mu}\phi\partial_{\nu}\phi + \partial_{\mu}\phi^{+}\partial_{\nu}\phi^{-} + \partial_{\nu}\phi^{+}\partial_{\mu}\phi^{-} + M_Z^2 Z_{\mu}Z_{\nu} + M_W^2(W_{\mu}^{+}W_{\nu}^{-} + W_{\nu}^{+}W_{\mu}^{-}) \\
& + M_W(W_{\mu}^{-}\partial_{\nu}\phi^{+} + W_{\nu}^{-}\partial_{\mu}\phi^{+} + W_{\mu}^{+}\partial_{\nu}\phi^{-} + W_{\nu}^{+}\partial_{\mu}\phi^{-}) + M_Z(\partial_{\mu}\phi Z_{\nu} + \partial_{\nu}\phi Z_{\mu}) + \frac{eM_W}{\sin\theta_W}H(W_{\mu}^{+}W_{\nu}^{-} + W_{\nu}^{+}W_{\mu}^{-}) \\
& + \frac{eM_Z}{\sin 2\theta_W}H(Z_{\mu}Z_{\nu}) - \frac{e}{2\sin\theta_W}[W_{\mu}^{+}(\phi^{-}\overleftrightarrow{\partial}_{\nu}(H + i\phi)) - W_{\mu}^{+}(\phi^{-}\overleftrightarrow{\partial}_{\nu}(H + i\phi))] - \frac{e}{2\sin\theta_W}[W_{\mu}^{-}(\phi^{+}\overleftrightarrow{\partial}_{\nu}(H - i\phi)) \\
& + W_{\nu}^{-}(\phi^{+}\overleftrightarrow{\partial}_{\mu}(H - i\phi))] + ie(A_{\mu} + \cot 2\theta_W Z_{\mu})(\phi^{-}\overleftrightarrow{\partial}_{\nu}\phi^{+}) + ie(A_{\nu} + \cot 2\theta_W Z_{\nu})(\phi^{-}\overleftrightarrow{\partial}_{\mu}\phi^{+}) \\
& - \frac{e}{\sin 2\theta_W}[Z_{\mu}(\phi\overleftrightarrow{\partial}_{\nu}H) + Z_{\nu}(\phi\overleftrightarrow{\partial}_{\mu}H)] - ieM_Z\sin\theta_W[Z_{\mu}(W_{\nu}^{+}\phi^{-} - W_{\nu}^{-}\phi^{+}) + Z_{\nu}(W_{\mu}^{+}\phi^{-} - W_{\mu}^{-}\phi^{+})] \\
& - ieM_W[A_{\mu}(W_{\nu}^{-} - W_{\nu}^{+}\phi^{-}) + A_{\nu}(W_{\mu}^{-} - W_{\mu}^{+}\phi^{-})] + \frac{e^2}{4\sin^2\theta_W}H^2[(W_{\mu}^{+}W_{\nu}^{-} + W_{\nu}^{+}W_{\mu}^{-} + 2Z_{\mu}Z_{\nu})] \\
& - \frac{ie^2}{2\cos\theta_W}H[Z_{\mu}(W_{\nu}^{+}\phi^{-} - W_{\nu}^{-}\phi^{+}) + Z_{\nu}(W_{\mu}^{+}\phi^{-} - W_{\mu}^{-}\phi^{+})] + \frac{e^2}{4\sin^2\theta_W}\phi^2(W_{\mu}^{+}W_{\nu}^{-} + W_{\nu}^{+}W_{\mu}^{-} + 2Z_{\mu}Z_{\nu}) \\
& + \frac{e^2}{\sin\theta_W^2}\phi^{+}\phi^{-}(W_{\mu}^{+}W_{\nu}^{-} + W_{\nu}^{+}W_{\mu}^{-}) - \frac{ie^2}{2\sin\theta_W}H[A_{\mu}(W_{\nu}^{-}\phi^{+} - W_{\nu}^{+}\phi^{-}) + A_{\nu}(W_{\mu}^{-}\phi^{+} - W_{\mu}^{+}\phi^{-})] \\
& + \frac{e^2}{2\cos\theta_W}\phi[Z_{\mu}(W_{\nu}^{+}\phi^{-} + W_{\nu}^{-}\phi^{+}) + Z_{\nu}(W_{\mu}^{+}\phi^{-} + W_{\mu}^{-}\phi^{+})] - \frac{e^2}{2\sin\theta_W}\phi[A_{\mu}(W_{\nu}^{-}\phi^{+} + W_{\nu}^{+}\phi^{-}) \\
& + A_{\nu}(W_{\mu}^{-}\phi^{+} + W_{\mu}^{+}\phi^{-})] + e^2\cot^2 2\theta_W\phi^{+}\phi^{-}Z_{\mu}Z_{\nu} + e^2\phi^{+}\phi^{-}A_{\mu}A_{\nu} + 2e^2\cot 2\theta_W\phi^{+}\phi^{-}(A_{\mu}Z_{\nu} + A_{\nu}Z_{\mu}). \qquad (19)
\end{aligned}$$

$$\Sigma_F^{\mu\nu\alpha\beta}(p,q) = \sum_{i=1}^{3} \Phi_{iF}(s,0,0,m_f^2)\phi_i^{\mu\nu\alpha\beta}(p,q),$$

$$\Sigma_B^{\mu\nu\alpha\beta}(p,q) = \sum_{i=1}^{3} \Phi_{iB}(s,0,0,M_W^2)\phi_i^{\mu\nu\alpha\beta}(p,q),$$

$$\Sigma_I^{\mu\nu\alpha\beta}(p,q) = \Phi_{1I}(s,0,0,M_W^2)\phi_1^{\mu\nu\alpha\beta}(p,q) + \Phi_{4I}(s,0,0,M_W^2)\phi_4^{\mu\nu\alpha\beta}(p,q).$$

$$\Phi_{1F}(s,0,0,m_f^2) = -i\frac{\kappa}{2}\frac{\alpha}{3\pi s}Q_f^2\left\{-\frac{2}{3}+\frac{4m_f^2}{s} - 2m_f^2 C_0(s,0,0,m_f^2,m_f^2,m_f^2)\left[1-\frac{4m_f^2}{s}\right]\right\},$$

$$\Phi_{1B}(s,0,0,M_W^2) = -i\frac{\kappa}{2}\frac{\alpha}{\pi s}\left\{\frac{5}{6}-\frac{2M_W^2}{s} + 2M_W^2 C_0(s,0,0,M_W^2,M_W^2,M_W^2)\left[1-\frac{2M_W^2}{s}\right]\right\},$$

Presence of a 1/s behavior
associated to an "effective dof"
Dilaton-like

$$\Phi^F_{1\,\mathrm{pole}} \equiv i\kappa\frac{\alpha}{9\pi s}Q_f^2,$$

$$\Phi_{1B,\mathrm{pole}} \equiv -i\frac{\kappa}{2}\frac{\alpha}{\pi s}\frac{5}{6}.$$

THINGS ARE CLEAR IN QCD

QCD TGG

$$
\begin{aligned}
T_{\mu\nu} &= -g_{\mu\nu}\mathcal{L}_{QCD} - F^a_{\mu\rho}F^{a\rho}_{\nu} - \frac{1}{\xi}g_{\mu\nu}\partial^\rho(A^a_\rho\partial^\sigma A^a_\sigma) + \frac{1}{\xi}(A^a_\nu\partial_\mu(\partial^\sigma A^a_\sigma) + A^a_\mu\partial_\nu(\partial^\sigma A^a_\sigma)) \\
&+ \frac{i}{4}\Big[\overline{\psi}\gamma_\mu(\overrightarrow{\partial}_\nu - igT^aA^a_\nu)\psi - \overline{\psi}(\overleftarrow{\partial}_\nu + igT^aA^a_\nu)\gamma_\mu\psi + \overline{\psi}\gamma_\nu(\overrightarrow{\partial}_\mu - igT^aA^a_\mu)\psi \\
&- \overline{\psi}(\overleftarrow{\partial}_\mu + igT^aA^a_\mu)\gamma_\nu\psi\Big] + \partial_\mu\overline{\omega}^a(\partial_\nu\omega^a - gf^{abc}A^c_\nu\omega^b) + \partial_\nu\overline{\omega}^a(\partial_\mu\omega^a - gf^{abc}A^c_\mu\omega^b),
\end{aligned}
$$

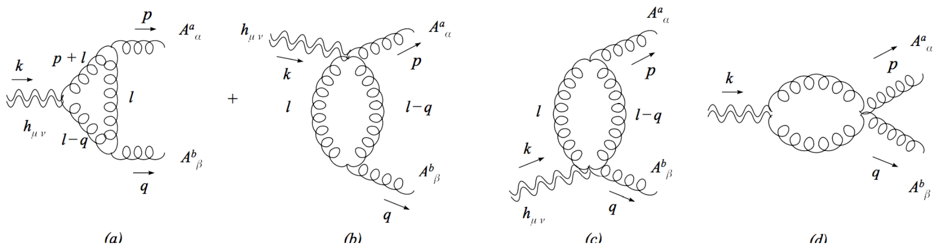
$$
\begin{aligned}
T^{g.f.}_{\mu\nu} &= \frac{1}{\xi}\left[A^a_\nu\partial_\mu(\partial\cdot A^a) + A^a_\mu\partial_\nu(\partial\cdot A^a)\right] - \frac{1}{\xi}g_{\mu\nu}\left[-\frac{1}{2}(\partial\cdot A)^2 + \partial^\rho(A^a_\rho\partial\cdot A^a)\right], \\
T^{gh}_{\mu\nu} &= \partial_\mu\bar{\omega}^aD^{ab}_\nu\omega^b + \partial_\nu\bar{\omega}^aD^{ab}_\mu\omega^b - g_{\mu\nu}\partial^\rho\bar{\omega}^aD^{ab}_\rho\omega^b.
\end{aligned}
$$

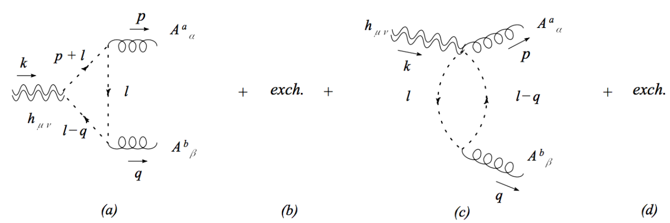
$$
\begin{aligned}
\partial^\mu T_{\mu\nu} &= -\frac{\delta S}{\delta\psi}\partial_\nu\psi - \partial_\nu\bar{\psi}\frac{\delta S}{\delta\bar{\psi}} + \frac{1}{2}\partial^\mu\left(\frac{\delta S}{\delta\psi}\sigma_{\mu\nu}\psi - \bar{\psi}\sigma_{\mu\nu}\frac{\delta S}{\delta\bar{\psi}}\right) - \partial_\nu A^a_\mu\frac{\delta S}{\delta A^a_\mu} \\
&+ \partial_\mu\left(A^a_\nu\frac{\delta S}{\delta A^a_\mu}\right) - \frac{\delta S}{\delta\omega^a}\partial_\nu\omega^a - \partial_\nu\bar{\omega}^a\frac{\delta S}{\delta\bar{\omega}^a}
\end{aligned}
$$

ORDINARY Ward IDENTITIES

$$
\begin{aligned}
\partial^\mu\langle T_{\mu\nu}(x)A^a_\alpha(x_1)A^b_\beta(x_2)\rangle_{trunc} &= -\partial_\nu\delta^4(x_1-x)D^{-1}_{\alpha\beta}(x_2,x) - \partial_\nu\delta^4(x_2-x)D^{-1}_{\alpha\beta}(x_1,x) \\
&+ \partial^\mu\left(g_{\alpha\nu}\delta^4(x_1-x)D^{-1}_{\beta\mu}(x_2,x) + g_{\beta\nu}\delta^4(x_2-x)D^{-1}_{\alpha\mu}(x_1,x)\right)
\end{aligned}
$$

Delle Rose, Armillis, CC





QCD TJJ: 1 form factor carries the entire anomaly

Delle Rose, Armillis, CC

$$\Phi_{1\,q}(s,\,0,\,0,\,m^2) \;=\; -\frac{g^2}{36\pi^2 s}+\frac{g^2 m^2}{6\pi^2 s^2}-\frac{g^2\, m^2}{6\pi^2 s}\mathcal{C}_0(s,0,0,m^2)\left[\frac{1}{2}-\frac{2m^2}{s}\right],$$

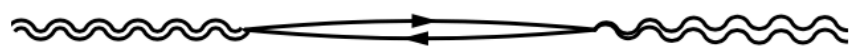
Quark contribution

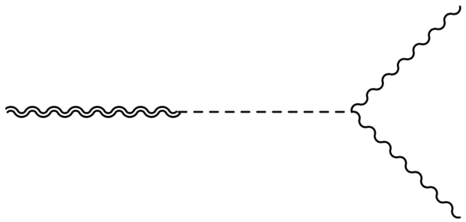
$$\Phi_1(s,0,0) \;=\; -\frac{g^2}{72\pi^2\, s}\,(2n_f-11C_A)+\frac{g^2}{6\pi^2}\sum_{i=1}^{n_f} m_i^2\left\{\frac{1}{s^2}-\frac{1}{2s}\mathcal{C}_0(s,0,0,m_i^2)\left[1-\frac{4m_i^2}{s}\right]\right\},$$

Entire TJJ vertex in QCD

It is reproduced by the effective action

$$\begin{aligned} S_{pole} &= -\frac{c}{6}\int d^4x\, d^4y\, R^{(1)}(x)\,\Box^{-1}(x,y)\, F^a_{\alpha\beta}\, F^{a\,\alpha\beta} \\ &= \frac{1}{3}\,\frac{g^3}{16\pi^2}\left(-\frac{11}{3}\,C_A+\frac{2}{3}\,n_f\right)\int d^4x\, d^4y\, R^{(1)}(x)\,\Box^{-1}(x,y)\, F_{\alpha\beta}F^{\alpha\beta} \end{aligned}$$





If we couple the SM to gravity
we would find at 1-loop
a general behavior.

It is reproduced by a
nonlocal effective action
where an intermediate
virtual state couples
to the anomaly

$$\mathcal{S}_A \sim \int d^4x\, d^4y R^{(1)}(x) \left(\frac{1}{\square}\right)(x,y) \left(b'\, E_4^{(2)}(y) + b\,(C^2)^{(2)}(y)\right),$$

This behavior is generic for conformal and chiral anomalies

$$\mathcal{S}_A \sim \beta(e) \int d^4x\, d^4y\, R^{(1)}(x) \left(\frac{1}{\square}\right)(x,y) F^{\mu\nu} F_{\mu\nu}(y),$$

CHIRAL AND CONFORMAl ANOMALIES

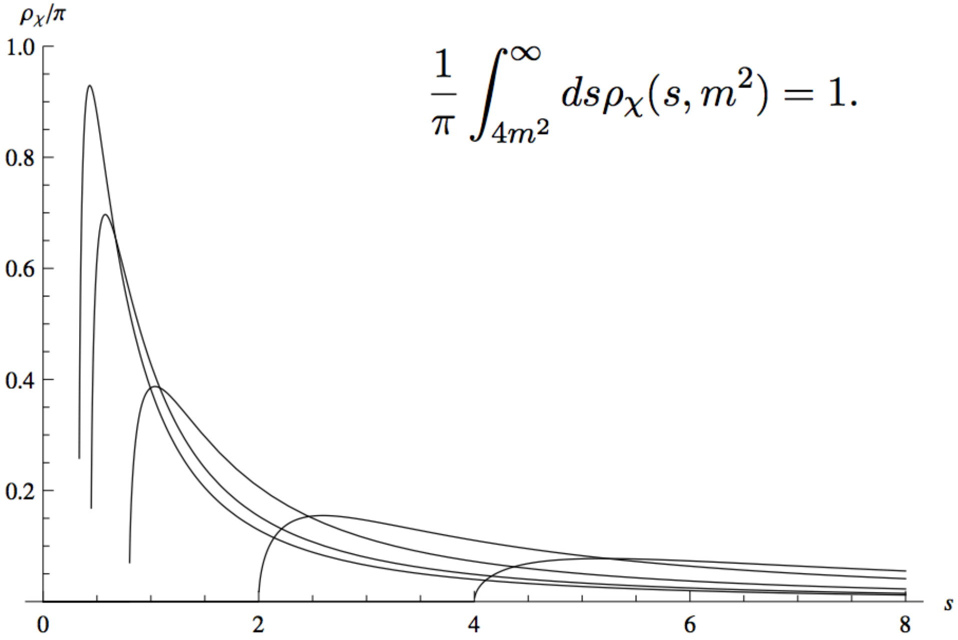
Superconformal Sum Rules and the Spectral Density Flow of the Composite Dilaton (ADD) Multiplet in $\mathcal{N}=1$ Theories

Costantini, Delle Rose, Serino, CC

Superconformal anomaly multiplet

$$
\begin{aligned}
\partial_\mu R^\mu &= \frac{g^2}{16\pi^2}\left(T(A)-\frac{1}{3}T(R)\right)F^{a\,\mu\nu}\tilde{F}^a_{\mu\nu}\,,\\
\bar\sigma_\mu S^\mu_A &= -i\frac{3\,g^2}{8\pi^2}\left(T(A)-\frac{1}{3}T(R)\right)\left(\bar\lambda^a\bar\sigma^{\mu\nu}\right)_A F^a_{\mu\nu}\,,\\
\eta_{\mu\nu}T^{\mu\nu} &= -\frac{3\,g^2}{32\pi^2}\left(T(A)-\frac{1}{3}T(R)\right)F^{a\,\mu\nu}F^a_{\mu\nu}\,.
\end{aligned}
$$

$$
\begin{aligned}
R^\mu &= \bar\lambda^a\bar\sigma^\mu\lambda^a+\frac{1}{3}\left(-\bar\chi_i\bar\sigma^\mu\chi_i+2i\phi_i^\dagger\mathcal{D}^\mu_{ij}\phi_j-2i(\mathcal{D}^\mu_{ij}\phi_j)^\dagger\phi_i\right)\,,\\
S^\mu_A &= i(\sigma^{\nu\rho}\sigma^\mu\bar\lambda^a)_A F^a_{\nu\rho}-\sqrt{2}(\sigma_\nu\bar\sigma^\mu\chi_i)_A(\mathcal{D}^\nu_{ij}\phi_j)^\dagger-i\sqrt{2}(\sigma^\mu\bar\chi_i)\mathcal{W}_i^\dagger(\phi^\dagger)\\
&- ig(\phi_i^\dagger T^a_{ij}\phi_j)(\sigma^\mu\bar\lambda^a)_A+S^\mu_{I\,A}\,,\\
T^{\mu\nu} &= -F^{a\,\mu\rho}F^{a\,\nu}{}_\rho+\frac{i}{4}\left[\bar\lambda^a\bar\sigma^\mu(\delta^{ac}\overrightarrow{\partial^\nu}-g\,t^{abc}A^{b\,\nu})\lambda^c+\bar\lambda^a\bar\sigma^\mu(-\delta^{ac}\overleftarrow{\partial^\nu}-g\,t^{abc}A^{b\,\nu})\lambda^c+(\mu\leftrightarrow\nu)\right]\\
&+ (\mathcal{D}^\mu_{ij}\phi_j)^\dagger(\mathcal{D}^\nu_{ik}\phi_k)+(\mathcal{D}^\nu_{ij}\phi_j)^\dagger(\mathcal{D}^\mu_{ik}\phi_k)+\frac{i}{4}\Big[\bar\chi_i\bar\sigma^\mu(\delta_{ij}\overrightarrow{\partial^\nu}+igT^a_{ij}A^{a\,\nu})\chi_j\\
&+ \bar\chi_i\bar\sigma^\mu(-\delta_{ij}\overleftarrow{\partial^\nu}+igT^a_{ij}A^{a\,\nu})\chi_j+(\mu\leftrightarrow\nu)\Big]-\eta^{\mu\nu}\mathcal{L}+T^{\mu\nu}_I\,,
\end{aligned}
$$



$$\frac{1}{\pi}\int_{4m^2}^{\infty} ds\,\rho_\chi(s,m^2) = 1.$$

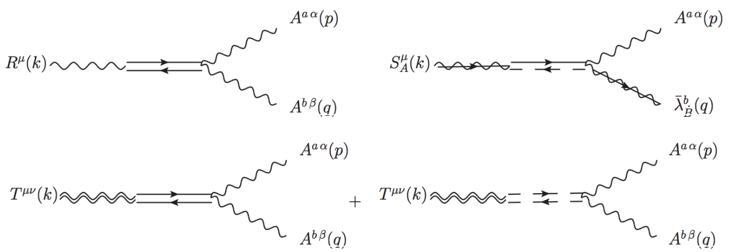
EXACT SUM RULE

The form factor that
carries the chiral
And conformal anomaly
away from the critical point c
shows a branch cut.

The spectral dentity exhibits a pole
As m→0

$$\lim_{m\to 0}\rho_\chi(s,m^2) = \lim_{m\to 0}\frac{2\pi m^2}{s^2}\log\left(\frac{1+\sqrt{\tau(s,m^2)}}{1-\sqrt{\tau(s,m^2)}}\right)\theta(s-4m^2) = \pi\delta(s)$$

Delle Rose, CC



THESE ANALYSIS
ARE PURELY
PERTURBATIVE.

RENORMALIZATION ASSOCIATED TO 2 COUNTERTERMS

$$C^2 = C_{\lambda\mu\nu\rho}C^{\lambda\mu\nu\rho} = R_{\lambda\mu\nu\rho}R^{\lambda\mu\nu\rho} - 2R_{\mu\nu}R^{\mu\nu} + \frac{R^2}{3}$$

WEYL TENSOR SQUARED

$$E = {}^*R_{\lambda\mu\nu\rho}{}^*R^{\lambda\mu\nu\rho} = R_{\lambda\mu\nu\rho}R^{\lambda\mu\nu\rho} - 4R_{\mu\nu}R^{\mu\nu} + R^2.$$

GAUSS BONNET TERM

The HIggs has to be conformally coupled (delle Rose, Serino, CC)

$$T^\mu_\mu = -\tfrac{1}{8}[2bC^2 + 2b'(E - \tfrac{2}{3}\Box R) + 2cF^2]$$

SCT

$$\begin{aligned}\mathcal{K}^\kappa T^{\mu\nu}(x) &\equiv \delta_\kappa T^{\mu\nu}(x) = \frac{\partial}{\partial b^\kappa}(\delta T^{\mu\nu}) \\ &= -(x^2\partial_\kappa - 2x_\kappa x\cdot\partial)T^{\mu\nu}(x) + 2\Delta_T x_\kappa T^{\mu\nu}(x) + 2(\delta_{\mu\kappa}x_\alpha - \delta_{\alpha\kappa}x_\mu)T^{\alpha\nu}(x) \\ &\quad +2(\delta_{\kappa\nu}x_\alpha - \delta_{\alpha\kappa}x_\nu)T^{\mu\alpha}.\end{aligned}$$

$$\begin{aligned}[K^\mu, D] &= -iK^\mu, \\ [P^\mu, K^\nu] &= 2i\delta^{\mu\nu}D + 2iJ^{\mu\nu}, \\ [K^\mu, K^\nu] &= 0, \\ [J^{\rho\sigma}, K^\mu] &= i\delta^{\mu\rho}K^\sigma - i\delta^{\mu\sigma}K^\rho.\end{aligned}$$

translations	$L_g = a^\mu\partial_\mu,$
rotations	$L_g = \frac{\omega^{\mu\nu}}{2}[x_\nu\partial_\mu - x_\mu\partial_\nu] - \Sigma_{\mu\nu},$
scale transformations	$L_g = \sigma\,[x\cdot\partial + \Delta],$
special conformal transformations	$L_g = b^\mu[x^2\partial_\mu - 2x_\mu\, x\cdot\partial - 2\Delta\, x_\mu - 2x_\nu\Sigma_\mu{}^\nu].$

FROM ORDINARY WARD IDENTITIES TO CONFORMAL WARD IDENTITIES

HIERARCHICAL SET OF EQUATIONS THAT NEED TO BE INVESTIGATED

RENORMALIZATION of the 3 T affects the 2 T and so on

To understand these states and the role of the counterterms
it is useful to move to momentum space

It is also useful to identify a mapping between general non perturbative solutions of the CWIs with the free field theory realizations

two and 3-point functions of primary scalar fields, in the scalar case, are easily fixed

$$\langle \mathcal{O}_1(x_1)\, \mathcal{O}_2(x_2)\rangle = \frac{C_{12}}{|x_1 - x_2|^{2\Delta_1}} \delta_{\Delta_1 \Delta_2}.$$

$$\langle \mathcal{O}_1(x_1)\, \mathcal{O}_2(x_2)\, \mathcal{O}_3(x_3)\rangle = \frac{C_{123}}{x_{12}^{\Delta_1+\Delta_2-\Delta_3}\; x_{23}^{\Delta_2+\Delta_3-\Delta-1}\; x_{13}^{\Delta_3+\Delta_1-\Delta_2}}.$$

$$\Gamma^{\mu\nu\alpha\beta}(x_1, x_2, x_3) = \langle T^{\mu\nu}(x_1) J^{\alpha}(x_2) J^{\beta}(x_3)\rangle$$ TJJ

$$\begin{aligned}
\mathcal{K}^{\kappa}\Gamma^{\mu\nu\alpha\beta}(x_1, x_2, x_3) \quad &= \quad \sum_{i=1}^{3} K^{\ \kappa}_{i\,scalar}(x_i)\Gamma^{\mu\nu\alpha\beta}(x_1, x_2, x_3) \\
&+2\left(\delta^{\mu\kappa} x_{1\rho} - \delta^{\kappa}_{\rho} x_1^{\mu}\right)\Gamma^{\rho\nu\alpha\beta} + 2\left(\delta^{\nu\kappa} x_{1\rho} - \delta^{\kappa}_{\rho} x_1^{\nu}\right)\Gamma^{\mu\rho\alpha\beta} \\
&2\left(\delta^{\alpha\kappa} x_{2\rho} - \delta^{\kappa}_{\rho} x_2^{\alpha}\right)\Gamma^{\mu\nu\rho\beta} + 2\left(\delta^{\beta\kappa} x_{3\rho} - \delta^{\kappa}_{\rho} x_3^{\beta}\right)\Gamma^{\mu\nu\alpha\rho} = 0,
\end{aligned}$$

analysis of the TT, TTT in coordinate space done long ago by Osborn and Petkou

Delle Rose, Mottola, Serino, C.C.
Bzowski, McFadden, Skenderis

2013

$$\Phi(x_1, x_2, \dots, x_n) = \int dp_1 dp_2 ... dp_{n-1} e^{i(p_1 x_1 + p_2 x_2 + \dots p_{n-1} x_{n-1} + \overline{p}_n x_n)} \Phi(p_1, p_2, \dots, \overline{p}_n).$$

$$\sum_{j=1}^{n} \left(x_j^\alpha \frac{\partial}{\partial x_j^\alpha} + \Delta_j \right) \Phi(x_1, x_2, \dots, x_n) = 0.$$

dilatation WI

$$\left[\sum_{j=1}^{n} \Delta_j - (n-1)d - \sum_{j=1}^{n-1} p_j^\alpha \frac{\partial}{\partial p_j^\alpha} \right] \Phi(p_1, p_2, \dots, \overline{p}_n) = 0.$$

momentum

SC WI

$$\sum_{j=1}^{n} \left(-x_j^2 \frac{\partial}{\partial x_j^\kappa} + 2 x_j^\kappa x_j^\alpha \frac{\partial}{\partial x_j^\alpha} + 2\Delta_j x_j^\kappa \right) \Phi(x_1, x_2, \dots, x_n) = 0$$

$$\sum_{j=1}^{n-1} \left(p_j^\kappa \frac{\partial^2}{\partial p_j^\alpha \partial p_j^\alpha} + 2(\Delta_j - d) \frac{\partial}{\partial p_j^\kappa} - 2 p_j^\alpha \frac{\partial^2}{\partial p_j^\kappa \partial p_j^\alpha} \right) \Phi(p_1, \dots p_{n-1}, \bar{p}_n) = 0.$$

momentum

We need to solve

n=3

$$\sum_{j=1}^{n-1}\left(p_j^\kappa\frac{\partial^2}{\partial p_j^\alpha\partial p_j^\alpha}+2(\Delta_j-d)\frac{\partial}{\partial p_j^\kappa}-2p_j^\alpha\frac{\partial^2}{\partial p_j^\kappa\partial p_j^\alpha}\right)\Phi(p_1,\dots p_{n-1},\bar{p}_n)=0.$$

$$\left[\sum_{j=1}^{n}\Delta_j-(n-1)d-\sum_{j=1}^{n-1}p_j^\alpha\frac{\partial}{\partial p_j^\alpha}\right]\Phi(p_1,p_2,\dots,\overline{p}_n)=0.$$

perform the change of variables (Delle Rose, Serino, Mottola, CC, 2013)

$$x=\frac{p_1^2}{p_3^2}\,,\qquad y=\frac{p_2^2}{p_3^2}\,,$$

$$\begin{aligned}\frac{\partial}{\partial p_1^\mu}&=2(p_{1\,\mu}+p_{2\,\mu})\frac{\partial}{\partial p_3^2}+\frac{2}{p_3^2}\left((1-x)p_{1\,\mu}-x\,p_{2\,\mu}\right)\frac{\partial}{\partial x}-2(p_{1\,\mu}+p_{2\,\mu})\frac{y}{p_3^2}\frac{\partial}{\partial y}\,,\\ \frac{\partial}{\partial p_2^\mu}&=2(p_{1\,\mu}+p_{2\,\mu})\frac{\partial}{\partial p_3^2}-2(p_{1\,\mu}+p_{2\,\mu})\frac{x}{p_3^2}\frac{\partial}{\partial x}+\frac{2}{p_3^2}\left((1-y)p_{2\,\mu}-y\,p_{1\,\mu}\right)\frac{\partial}{\partial y}.\end{aligned}$$

and assume an ansatz of the form

$$G_{123}(p_1^2,p_2^2,p_3^2)=(p_3^2)^{-d+\frac{1}{2}(\eta_1+\eta_2+\eta_3)}\,\Phi(x,y)$$

The equations are found to become a hypergeometric system of rank-4

$$\begin{cases} \Big[x(1-x)\frac{\partial^2}{\partial x^2} - y^2\frac{\partial^2}{\partial y^2} - 2\,x\,y\frac{\partial^2}{\partial x\partial y} + [\gamma - (\alpha+\beta+1)x]\,\frac{\partial}{\partial x} \\ \qquad\qquad -(\alpha+\beta+1)y\frac{\partial}{\partial y} - \alpha\,\beta\Big]\,\Phi(x,y) = 0\,, \\ \Big[y(1-y)\frac{\partial^2}{\partial y^2} - x^2\frac{\partial^2}{\partial x^2} - 2\,x\,y\frac{\partial^2}{\partial x\partial y} + [\gamma' - (\alpha+\beta+1)y]\,\frac{\partial}{\partial y} \\ \qquad\qquad -(\alpha+\beta+1)x\frac{\partial}{\partial x} - \alpha\,\beta\Big]\,\Phi(x,y) = 0\,, \end{cases}$$

Appell system of equations

(see **Campes de Feriet and Appell's** book)

$$\alpha = \frac{d}{2} - \frac{\eta_1+\eta_2-\eta_3}{2}\,, \qquad \gamma = \frac{d}{2} - \eta_1 + 1\,,$$

$$\beta = d - \frac{\eta_1+\eta_2+\eta_3}{2}\,, \qquad \gamma' = \frac{d}{2} - \eta_2 + 1\,.$$

$$F_4(\alpha,\beta;\gamma,\gamma';x,y) = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\frac{(\alpha)_{i+j}\,(\beta)_{i+j}}{(\gamma)_i\,(\gamma')_j}\,\frac{x^i}{i!}\,\frac{y^j}{j!}$$

$(\alpha)_i = \Gamma(\alpha+i)/\Gamma(\alpha)$ is the Pochhammer symbol.

GENERAL 3-POINT CORRELATOR

$$\langle O(p_1)\, O(p_2)\, O(p_3)\rangle = \left(p_3^2\right)^{-d+\frac{\Delta_t}{2}} C(\Delta_1, \Delta_2, \Delta_3, d)$$

$$\Bigg\{\Gamma\left(\Delta_1 - \frac{d}{2}\right)\Gamma\left(\Delta_2 - \frac{d}{2}\right)\Gamma\left(d - \frac{\Delta_1+\Delta_2+\Delta_3}{2}\right)\Gamma\left(d - \frac{\Delta_1+\Delta_2-\Delta_3}{2}\right)$$

$$\times\ F_4\left(\frac{d}{2} - \frac{\Delta_1+\Delta_2-\Delta_3}{2}, d - \frac{\Delta_t}{2}, \frac{d}{2} - \Delta_1 + 1, \frac{d}{2} - \Delta_2 + 1; x, y\right)$$

$$+\ \Gamma\left(\frac{d}{2} - \Delta_1\right)\Gamma\left(\Delta_2 - \frac{d}{2}\right)\Gamma\left(\frac{\Delta_1-\Delta_2+\Delta_3}{2}\right)\Gamma\left(\frac{d}{2} + \frac{\Delta_1-\Delta_2-\Delta_3}{2}\right)$$

$$\times\ x^{\Delta_1-\frac{d}{2}}\, F_4\left(\frac{\Delta_1-\Delta_2+\Delta_3}{2}, \frac{d}{2} - \frac{\Delta_2+\Delta_3-\Delta_1}{2}, \Delta_1 - \frac{d}{2} + 1, \frac{d}{2} - \Delta_2 + 1; x\right.$$

$$+\ \Gamma\left(\Delta_1 - \frac{d}{2}\right)\Gamma\left(\frac{d}{2} - \Delta_2\right)\Gamma\left(\frac{-\Delta_1+\Delta_2+\Delta_3}{2}\right)\Gamma\left(\frac{d}{2} + \frac{-\Delta_1+\Delta_2-\Delta_3}{2}\right)$$

$$\times\ y^{\Delta_2-\frac{d}{2}}\, F_4\left(\frac{\Delta_2-\Delta_1+\Delta_3}{2}, \frac{d}{2} - \frac{\Delta_1-\Delta_2+\Delta_3}{2}, \frac{d}{2} - \Delta_1 + 1, \Delta_2 - \frac{d}{2} + 1; x, y\right)$$

$$+\ \Gamma\left(\frac{d}{2} - \Delta_1\right)\Gamma\left(\frac{d}{2} - \Delta_2\right)\Gamma\left(\frac{\Delta_1+\Delta_2-\Delta_3}{2}\right)\Gamma\left(-\frac{d}{2} + \frac{\Delta_1+\Delta_2+\Delta_3}{2}\right)$$

$$\times\ x^{\Delta_1-\frac{d}{2}}\, y^{\Delta_2-\frac{d}{2}}\, F_4\left(-\frac{d}{2} + \frac{\Delta_t}{2}, \frac{\Delta_1+\Delta_2-\Delta_3}{2}, \Delta_1 - \frac{d}{2} + 1, \Delta_2 - \frac{d}{2} + 1; x, y\right)\Bigg\}.$$

linear combination of 4 fundamental solutions

It is important to verify that the symmetric solution above does not have any unphysical singularity in the

physical region, reproducing the expected behaviour in the large momentum limit $p_3 \gg p_1$

(MM Maglio, CC)

If we define

in terms of F4

$$B(\lambda,\mu) = \left(\frac{a}{c}\right)^{\lambda}\left(\frac{b}{c}\right)^{\mu}\Gamma\left(\frac{\alpha+\lambda+\mu-\nu}{2}\right)\Gamma\left(\frac{\alpha+\lambda+\mu+\nu}{2}\right)\Gamma(-\lambda)\Gamma(-\mu)\times$$

$$\times\ F_4\left(\frac{\alpha+\lambda+\mu-\nu}{2}, \frac{\alpha+\lambda+\mu+\nu}{2}; \lambda+1, \mu+1; \frac{a^2}{c^2}, \frac{b^2}{c^2}\right),$$

Then one obtains an explictly symmetric expression
(Bzowski, McFadden, Slkenderis)

$$\int_0^\infty ds\, s^{\alpha-1} K_\lambda(p_1 s) K_\mu(p_2 s) K_\nu(p_3 s) =$$

$$= \frac{2^{\alpha-4}}{c^\alpha} \left[B(\lambda,\mu) + B(\lambda,-\mu) + B(-\lambda,\mu) + B(-\lambda,-\mu)\right],$$

$$\Phi(p_1, p_2, p_3) = C_{123}\, p_1^{\Delta_1-\frac{d}{2}} p_2^{\Delta_2-\frac{d}{2}} p_3^{\Delta_3-\frac{d}{2}} \int_0^\infty dx\, x^{\frac{d}{2}-1} K_{\Delta_1-\frac{d}{2}}(p_1\, x)\, K_{\Delta_2-\frac{d}{2}}(p_2\, x)\, K_{\Delta_3-\frac{d}{2}}(p_3\, x)$$

The Bessel functions K_ν satisfy the equations

$$\frac{\partial}{\partial p}\left[p^\beta\, K_\beta(p\, x)\right] = -x\, p^\beta\, K_{\beta-1}(px)$$

$$K_{\beta+1}(x) = K_{\beta-1}(x) + \frac{2\beta}{x} K_\beta(x)$$

`The hypergeometric system of equations corrispondign to F4, can also be obtained by first rewriting the special CWI's which are four-vector equations to the scalar form (Bzowsky, McFadden, Skenderis, 2013)

$$K^\kappa(p_i) \equiv \sum_{j=1}^{2} \left(2(\Delta_j - d)\frac{\partial}{\partial p_j^\kappa} + p_j^\kappa \frac{\partial^2}{\partial p_j^\alpha \partial p_j^\alpha} - 2p_j^\alpha \frac{\partial^2}{\partial p_j^\kappa \partial p_j^\alpha} \right) \Phi(p_1, p_2, \bar{p}_3) = 0,$$

$$\frac{\partial \Phi}{\partial p_i^\mu} = \frac{p_i^\mu}{p_i}\frac{\partial \Phi}{\partial p_i} - \frac{\bar{p}_3^\mu}{p_3}\frac{\partial \Phi}{\partial p_3}.$$

chain rule

$${K_{scalar}}^\kappa \Phi = 0$$

$$K^\kappa_{scalar} = \sum_{i=1}^{3} p_i^\kappa\, K_i$$

$$K_i \equiv \frac{\partial^2}{\partial p_i \partial p_i} + \frac{d+1-2\Delta_i}{p_i}\frac{\partial}{\partial p_i}$$

$$\frac{\partial^2 \Phi}{\partial p_i \partial p_i} + \frac{1}{p_i}\frac{\partial \Phi}{\partial p_i}(d+1-2\Delta_1) - \frac{\partial^2 \Phi}{\partial p_3 \partial p_3} - \frac{1}{p_3}\frac{\partial \Phi}{\partial p_3}(d+1-2\Delta_3) = 0$$

$$K_{ij} \equiv K_i - K_j$$

$$K^\kappa_{13}\Phi = 0 \qquad \text{and} \qquad K^\kappa_{23}\Phi = 0.$$

The general (nonperturbative) result obtained for this and other correlators can be simplified by choosing 3 independent field theory solutions which are conformal at 1-loop (e.g. QED, QCD)

The simplification is drastic and allows to avoid all the complications related to the renomalization of the 3K integrals.

How to proceed

the TTT Case

$$\langle T^{\mu\nu}(x)\rangle = \frac{2}{\sqrt{g(x)}}\frac{\delta \mathcal{W}}{\delta g_{\mu\nu}(x)}$$

$$\mathcal{W} = \frac{1}{\mathcal{N}}\int \mathcal{D}\Phi\; e^{-S}$$

$$\langle T^{\mu_1\nu_1}(x_1)\ldots T^{\mu_n\nu_n}(x_n)\rangle \equiv \left[\frac{2}{\sqrt{g(x_1)}}\cdots\frac{2}{\sqrt{-g(x_n)}}\frac{\delta^n \mathcal{W}}{\delta g_{\mu_1\nu_1}(x_1)\ldots\delta g_{\mu_n\nu_n}(x_n)}\right]_{flat} = 2^n \frac{\delta^n \mathcal{W}}{\delta g_{\mu_1\nu_1}(x_1)\ldots\delta g_{\mu_n\nu_n}(x_n)}\bigg|_{flat}$$

$$\begin{aligned}\langle T^{\mu_1\nu_1}(x_1)T^{\mu_2\nu_2}(x_2)T^{\mu_3\nu_3}(x_3)\rangle = 8\bigg\{ &-\left\langle \frac{\delta S}{\delta g_{\mu_1\nu_1}(x_1)}\frac{\delta S}{\delta g_{\mu_2\nu_2}(x_2)}\frac{\delta S}{\delta g_{\mu_3\nu_3}(x_3)}\right\rangle \\ &+\left\langle \frac{\delta^2 S}{\delta g_{\mu_1\nu_1}(x_1)\delta g_{\mu_2\nu_2}(x_2)}\frac{\delta S}{\delta g_{\mu_3\nu_3}(x_3)}\right\rangle + \left\langle \frac{\delta^2 S}{\delta g_{\mu_1\nu_1}(x_1)\delta g_{\mu_3\nu_3}(x_3)}\frac{\delta S}{\delta g_{\mu_2\nu_2}(x_2)}\right\rangle \\ &+\left\langle \frac{\delta^2 S}{\delta g_{\mu_2\nu_2}(x_2)\delta g_{\mu_3\nu_3}(x_3)}\frac{\delta S}{\delta g_{\mu_1\nu_1}(x_1)}\right\rangle - \left\langle \frac{\delta^3 S}{\delta g_{\mu_1\nu_1}(x_1)\delta g_{\mu_2\nu_2}(x_2)\delta g_{\mu_3\nu_3}(x_3)}\right\rangle\bigg\}\end{aligned}$$

$$\sum_{j=1}^{2}\left[2(\Delta_j-d)\frac{\partial}{\partial p_j^\kappa}-2p_j^\alpha\frac{\partial}{\partial p_j^\alpha}\frac{\partial}{\partial p_j^\kappa}+(p_j)_\kappa\frac{\partial}{\partial p_j^\alpha}\frac{\partial}{\partial p_{j\alpha}}\right]\langle T^{\mu_1\nu_1}(p_1)\,T^{\mu_2\nu_2}(p_2)\,T^{\mu_3\nu_3}(\bar p_3)\rangle$$
$$+2\left(\delta^{\kappa(\mu_1}\frac{\partial}{\partial p_1^{\alpha_1}}-\delta^\kappa_{\alpha_1}\delta^{\lambda(\mu_1}\frac{\partial}{\partial p_1^\lambda}\right)\langle T^{\nu_1)\alpha_1}(p_1)\,T^{\mu_2\nu_2}(p_2)\,T^{\mu_3\nu_3}(\bar p_3)\rangle$$
$$+2\left(\delta^{\kappa(\mu_2}\frac{\partial}{\partial p_2^{\alpha_2}}-\delta^\kappa_{\alpha_2}\delta^{\lambda(\mu_2}\frac{\partial}{\partial p_2^\lambda}\right)\langle T^{\nu_2)\alpha_2}(p_2)\,T^{\mu_3\nu_3}(\bar p_3)\,T^{\mu_1\nu_1}(p_1)\rangle=0.$$

Reconstruction in the BMS approach

$$T^{\mu\nu}=t^{\mu\nu}+t^{\mu\nu}_{loc}$$

projectors

$$\pi^\mu_\alpha=\delta^\mu_\alpha-\frac{p^\mu p_\alpha}{p^2},\qquad \tilde\pi^\mu_\alpha=\frac{1}{d-1}\pi^\mu_\alpha$$
$$\Pi^{\mu\nu}_{\alpha\beta}=\frac{1}{2}\left(\pi^\mu_\alpha\pi^\nu_\beta+\pi^\mu_\beta\pi^\nu_\alpha\right)-\frac{1}{d-1}\pi^{\mu\nu}\pi_{\alpha\beta},$$
$$\mathcal{I}^{\mu\nu}_\alpha=\frac{1}{p^2}\left[2p^{(\mu}\delta^{\nu)}_\alpha-\frac{p_\alpha}{d-1}(\delta^{\mu\nu}+(d-2)\frac{p^\mu p^\nu}{p^2})\right]$$
$$\mathcal{I}^{\mu\nu}_{\alpha\beta}=\mathcal{I}^{\mu\nu}_\alpha p_\beta=\frac{p_\beta}{p^2}\left(p^\mu\delta^\nu_\alpha+p^\nu\delta^\mu_\alpha\right)-\frac{p_\alpha p_\beta}{p^2}\left(\delta^{\mu\nu}+(d-2)\frac{p^\mu p^\nu}{p^2}\right)$$
$$\mathcal{L}^{\mu\nu}_{\alpha\beta}=\frac{1}{2}\left(\mathcal{I}^{\mu\nu}_{\alpha\beta}+\mathcal{I}^{\mu\nu}_{\beta\alpha}\right)\qquad \tau^{\mu\nu}_{\alpha\beta}=\tilde\pi^{\mu\nu}\delta_{\alpha\beta}$$

transverse traceless sector

$$\langle t^{\mu_1\nu_1}(p_1)t^{\mu_2\nu_2}(p_2)t^{\mu_3\nu_3}(p_3)\rangle=\Pi_1{}^{\mu_1\nu_1}_{\alpha_1\beta_1}\Pi_2{}^{\mu_2\nu_2}_{\alpha_2\beta_2}\Pi_3{}^{\mu_3\nu_3}_{\alpha_3\beta_3}\langle T^{\alpha_1\beta_1}(p_1)T^{\alpha_2\beta_2}(p_2)T^{\alpha_3\beta_3}(p_3)\rangle$$

the tt sectors is parameterised in a specific way

BMS

$$
\begin{aligned}
&\langle t^{\mu_1\nu_1}(p_1)t^{\mu_2\nu_2}(p_2)t^{\mu_3\nu_3}(p_3)\rangle = \Pi^{\mu_1\nu_1}_{\alpha_1\beta_1}(p_1)\Pi^{\mu_2\nu_2}_{\alpha_2\beta_2}(p_2)\Pi^{\mu_3\nu_3}_{\alpha_3\beta_3}(p_3) \\
&\quad\times \Big[A_1\, p_2^{\alpha_1}p_2^{\beta_1}p_3^{\alpha_2}p_3^{\beta_2}p_1^{\alpha_3}p_1^{\beta_3} + A_2\,\delta^{\beta_1\beta_2}p_2^{\alpha_1}p_3^{\alpha_2}p_1^{\alpha_3}p_1^{\beta_3} + A_2\,(p_1\leftrightarrow p_3)\,\delta^{\beta_2\beta_3}p_3^{\alpha_2}p_1^{\alpha_3}p_2^{\alpha_1}p_2^{\beta_1} \\
&\qquad + A_2\,(p_2\leftrightarrow p_3)\,\delta^{\beta_3\beta_1}p_1^{\alpha_3}p_2^{\alpha_1}p_3^{\alpha_2}p_3^{\beta_2} + A_3\,\delta^{\alpha_1\alpha_2}\delta^{\beta_1\beta_2}p_1^{\alpha_3}p_1^{\beta_3} + A_3(p_1\leftrightarrow p_3)\,\delta^{\alpha_2\alpha_3}\delta^{\beta_2\beta_3}p_2^{\alpha_1}p_2^{\beta_1} \\
&\qquad\quad + A_3(p_2\leftrightarrow p_3)\,\delta^{\alpha_3\alpha_1}\delta^{\beta_3\beta_1}p_3^{\alpha_2}p_3^{\beta_2} + A_4\,\delta^{\alpha_1\alpha_3}\delta^{\alpha_2\beta_3}p_2^{\beta_1}p_3^{\beta_2} + A_4(p_1\leftrightarrow p_3)\,\delta^{\alpha_2\alpha_1}\delta^{\alpha_3\beta_1}p_3^{\beta_2}p_1^{\beta_3} \\
&\qquad\qquad + A_4(p_2\leftrightarrow p_3)\,\delta^{\alpha_3\alpha_2}\delta^{\alpha_1\beta_2}p_1^{\beta_3}p_2^{\beta_1} + A_5\delta^{\alpha_1\beta_2}\delta^{\alpha_2\beta_3}\delta^{\alpha_3\beta_1}\Big] \qquad (5.12)
\end{aligned}
$$

the entire correlator is reconstructed via

$$
\begin{aligned}
\langle T^{\mu_1\nu_1}\,T^{\mu_2\nu_2}\,T^{\mu_3\nu_3}\rangle &= \langle t^{\mu_1\nu_1}\,t^{\mu_2\nu_2}\,t^{\mu_3\nu_3}\rangle + \langle t_{loc}^{\mu_1\nu_1}\,t^{\mu_2\nu_2}\,t^{\mu_3\nu_3}\rangle + \langle t^{\mu_1\nu_1}\,t_{loc}^{\mu_2\nu_2}\,t^{\mu_3\nu_3}\rangle + \langle t^{\mu_1\nu_1}\,t^{\mu_2\nu_2}\,t_{loc}^{\mu_3\nu_3}\rangle \\
&\quad + \langle t_{loc}^{\mu_1\nu_1}\,t_{loc}^{\mu_2\nu_2}\,t^{\mu_3\nu_3}\rangle + \langle t_{loc}^{\mu_1\nu_1}\,t_{loc}^{\mu_2\nu_2}\,t^{\mu_3\nu_3}\rangle + \langle t^{\mu_1\nu_1}\,t_{loc}^{\mu_2\nu_2}\,t_{loc}^{\mu_3\nu_3}\rangle + \langle t_{loc}^{\mu_1\nu_1}\,t_{loc}^{\mu_2\nu_2}\,t_{loc}^{\mu_3\nu_3}\rangle
\end{aligned}
$$

at the same time one solves the diolatation WI

$$
\left(\sum_{j=1}^{3}\Delta_j - 2d - \sum_{j=1}^{2} p_j^{\alpha}\frac{\partial}{\partial p_j^{\alpha}}\right)\langle T^{\alpha_1\beta_1}T^{\alpha_2\beta_2}T^{\alpha_3\beta_3}\rangle
$$

the intermediate steps are rather technical

BMS

$K_{13}A_1 = 0$	$K_{23}A_1 = 0$
$K_{13}A_2 = 8A_1$	$K_{23}A_2 = 8A_1$
$K_{13}A_2(p_1 \leftrightarrow p_3) = -8A_1$	$K_{23}A_2(p_1 \leftrightarrow p_3) = 0$
$K_{13}A_2(p_2 \leftrightarrow p_3) = 0$	$K_{23}A_2(p_2 \leftrightarrow p_3) = -8A_1$
$K_{13}A_3 = 2A_2$	$K_{23}A_3 = 2A_2$
$K_{13}A_3(p_1 \leftrightarrow p_3) = -2A_2(p_1 \leftrightarrow p_3)$	$K_{23}A_3(p_1 \leftrightarrow p_3) = 0$
$K_{13}A_3(p_2 \leftrightarrow p_3) = 0$	$K_{23}A_3(p_2 \leftrightarrow p_3) = -2A_2(p_2 \leftrightarrow p_3)$
$K_{13}A_4 = -4A_2(p_2 \leftrightarrow p_3)$	$K_{23}A_4 = -4A_2(p_1 \leftrightarrow p_3)$
$K_{13}A_4(p_1 \leftrightarrow p_3) = 4A_2(p_2 \leftrightarrow p_3)$	$K_{23}A_4(p_1 \leftrightarrow p_3) = 4A_2(p_2 \leftrightarrow p_3) - 4A_2$
$K_{13}A_4(p_2 \leftrightarrow p_3) = 4A_2(p_1 \leftrightarrow p_3) - 4A_2$	$K_{23}A_4(p_2 \leftrightarrow p_3) = 4A_2(p_1 \leftrightarrow p_3)$
$K_{13}A_5 = 2\left[A_4 - A_4(p_1 \leftrightarrow p_3)\right]$	$K_{23}A_5 = 2\left[A_4 - A_4(p_2 \leftrightarrow p_3)\right]$

primary WI's

and some secondary WI's which connect 3- and 2-point functions

The primary can be solved in temrs of 3K integrals and deffine a generalised hypergeometric system of Appell type for F4.

Renormalization, anomalies and the anomaly action

Lagrangian realizations and reconstruction

MM Maglio, CC

$$S_{scalar} = \frac{1}{2}\int d^dx\,\sqrt{-g}\,\left[g^{\mu\nu}\nabla_\mu\phi\nabla_\nu\phi - \chi\,R\,\phi^2\right]$$

$$S_{fermion} = \frac{i}{2}\int d^dx\,e\,e^\mu_a\,\left[\bar\psi\gamma^a(D_\mu\psi) - (D_\mu\bar\psi)\gamma^a\psi\right],$$

in d=4 we need 3 sectors to perform the matching

$$S_M = -\frac{1}{4}\int d^4x\,\sqrt{-g}\,F^{\mu\nu}F_{\mu\nu},$$

$$S_{gf} = -\frac{1}{\xi}\int d^4x\,\sqrt{-g}\,(\nabla_\mu A^\mu)^2,$$

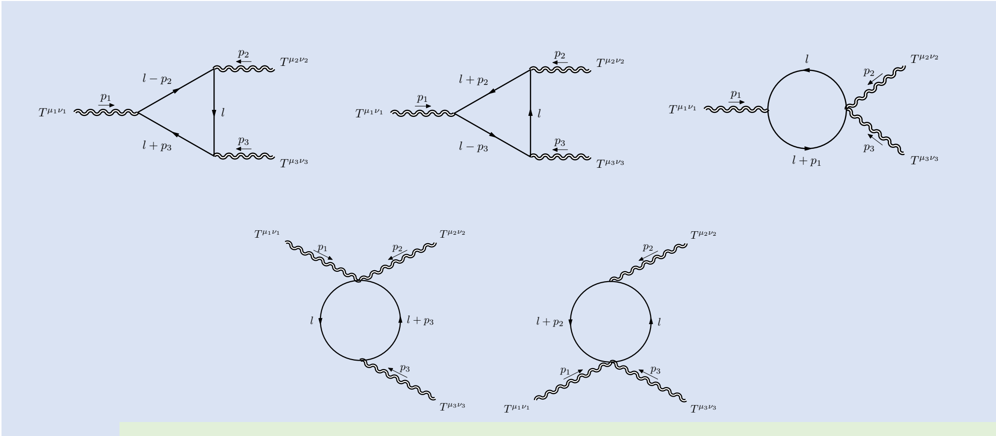
$$S_{gh} = \int d^4x\,\sqrt{-g}\,\partial^\mu\bar c\,\partial_\mu\,c.$$

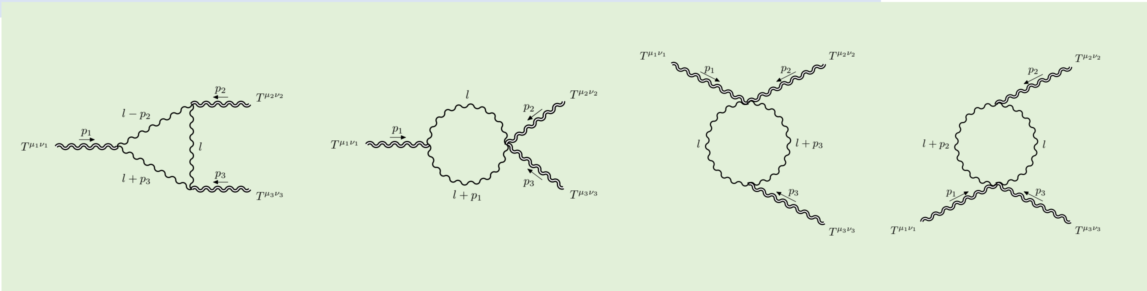
$$D_\mu = \partial_\mu + \Gamma_\mu = \partial_\mu + \frac{1}{2}\Sigma^{ab}\,e^\sigma_a\nabla_\mu\,e_{b\,\sigma}.$$

The Σ^{ab} are the generators of the Lorentz group in the spin 1/2 representation.

$$S_{abelian} = S_M + S_{gf} + S_{gh}$$

where $\chi = (d-2)/(4d-4)$ for a conformally coupled scalar in d dimensions, and R is the Ricci scalar. e^a_μ is the vielbein and e its determinant, with the covariant derivative D_μ given by





Can be matched to the complete solution of the CWI's

$$
\begin{aligned}
\partial_\nu \langle T^{\mu\nu}(x_1) T^{\rho\sigma}(x_2) T^{\alpha\beta}(x_3)\rangle = &\Big[\langle T^{\rho\sigma}(x_1) T^{\alpha\beta}(x_3)\rangle \partial^\mu \delta(x_1,x_2) + \langle T^{\alpha\beta}(x_1) T^{\rho\sigma}(x_2)\rangle \partial^\mu \delta(x_1,x_3)\Big] \\
&- \Big[\delta^{\mu\rho}\langle T^{\nu\sigma}(x_1) T^{\alpha\beta}(x_3)\rangle + \delta^{\mu\sigma}\langle T^{\nu\rho}(x_1) T^{\alpha\beta}(x_3)\rangle\Big]\partial_\nu \delta(x_1,x_2) \\
&- \Big[\delta^{\mu\alpha}\langle T^{\nu\beta}(x_1) T^{\rho\sigma}(x_2)\rangle + \delta^{\mu\beta}\langle T^{\nu\alpha}(x_1) T^{\rho\sigma}(x_2)\rangle\Big]\partial_\nu \delta(x_1,x_3)\,.
\end{aligned}
$$

$$
\begin{aligned}
p_{1\nu_1}\,\langle T^{\mu_1\nu_1}(p_1)\, T^{\mu_2\nu_2}(p_2)\, T^{\mu_3\nu_3}(p_3)\rangle = &-p_2^{\mu_1}\,\langle T^{\mu_2\nu_2}(p_1+p_2) T^{\mu_3\nu_3}(p_3)\rangle - p_3^{\mu_1}\,\langle T^{\mu_2\nu_2}(p_2) T^{\mu_3\nu_3}(p_1+p_3)\rangle \\
&+ p_{2\alpha}\left[\delta^{\mu_1\nu_2}\,\langle T^{\mu_2\alpha}(p_1+p_2) T^{\mu_3\nu_3}(p_3)\rangle + \delta^{\mu_1\mu_2}\,\langle T^{\nu_2\alpha}(p_1+p_2) T^{\mu_3\nu_3}(p_3)\rangle\right] \\
&+ p_{3\alpha}\left[\delta^{\mu_1\nu_3}\,\langle T^{\mu_3\alpha}(p_1+p_3) T^{\mu_2\nu_2}(p_2)\rangle + \delta^{\mu_1\mu_3}\,\langle T^{\nu_3\alpha}(p_1+p_3) T^{\mu_2\nu_2}(p_2)\rangle\right].
\end{aligned}
$$

while naive scale invariance gives the traceless condition

$$
g_{\mu\nu}\,\langle T^{\mu\nu}\rangle = 0.
$$

$$
\begin{aligned}
\beta_a(S) &= -\frac{3\pi^2}{720}\,, & \beta_b(S) &= \frac{\pi^2}{720}\,, \\
\beta_a(F) &= -\frac{9\pi^2}{360}\,, & \beta_b(F) &= \frac{11\pi^2}{720} \\
\beta_a(G) &= -\frac{18\pi^2}{360}\,, & \beta_b(G) &= \frac{31\pi^2}{360}
\end{aligned}
$$

After renormalization this equation is modified by the contribution of the conformal anomaly, by the general expression

$$
\begin{aligned}
g_{\mu\nu}(z)\langle T^{\mu\nu}(z)\rangle &= \sum_{I=F,S,G} n_I\left[\beta_a(I)\, C^2(z) + \beta_b(I)\, E(z)\right] + \frac{\kappa}{4} n_G F^{a\,\mu\nu}\, F^a_{\mu\nu}(z) \\
&\equiv \mathcal{A}(z,g)\,,
\end{aligned}
$$

in d=3 we need two sectors (scalar and fermion)

$$A_1^{d=3}(p_1,p_2,p_3) = \frac{\pi^3(n_S-4n_F)}{60(p_1+p_2+p_3)^6}\Big[p_1^3+6p_1^2(p_3+p_2)+(6p_1+p_2+p_3)\big((p_2+p_3)^2+3p_2p_3\big)\Big]$$

$$A_2^{d=3}(p_1,p_2,p_3) = \frac{\pi^3(n_S-4n_F)}{60(p_1+p_2+p_3)^6}\Big[4p_3^2\big(7(p_1+p_2)^2+6p_1p_2\big)+20p_3^3(p_1+p_2)+4p_3^4 + 3(5p_3+p_1+p_2)(p_1+p_2)\big((p_1+p_2)^2+p_1p_2\big)\Big] + \frac{\pi^3\,n_F}{3(p_1+p_2+p_3)^4}\Big[p_1^3+4p_1^2(p_2+p_3)+(4p_1+p_2+p_3)\big((p_2+p_3)^2+p_2p_3\big)\Big]$$

$$A_3^{d=3}(p_1,p_2,p_3) = \frac{\pi^3(n_S-4n_F)\,p_3^2}{240(p_1+p_2+p_3)^4}\Big[28p_3^2(p_1+p_2)+3p_3\big(11(p_1+p_2)^2+6p_1\,p_2\big)+7p_3^3 + 12(p_1+p_2)\big((p_1+p_2)^2+p_1p_2\big)\Big] + \frac{\pi^3 n_F\,p_3^2}{6(p_1+p_2+p_3)^3}\Big[3p_2(p_1+p_2)+2\big((p_1+p_2)^2+p_1p_2\big)+p_3^2\Big] - \frac{\pi^3(n_s+4n_F)}{16(p_1+p_2+p_3)^2}\Big[p_1^3+2p_1^2(p_2+p_3)+(2p_1+p_2+p_3)\big((p_2+p_3)^2-p_2p_3\big)\Big]$$

$$A_4^{d=3}(p_1,p_2,p_3) = \frac{\pi^3(n_S-4n_F)}{120(p_1+p_2+p_3)^4}\Big[(4p_3+p_1+p_2)\big(3(p_1+p_2)^4-3(p_1+p_2)^2p_1p_2+4p_1^2p_2^2\big) + 9p_3^2(p_1+p_2)\big((p_1+p_2)^2-3p_1p_2\big)-3p_3^5-12p_3^4(p_1+p_2)-9p_3^3\big((p_1+p_2)^2+2p_1p_2\big)\Big] + \frac{\pi^3\,n_F}{6(p_1+p_2+p_3)^3}\Big[(p_1+p_2)\big((p_1+p_2)^2-p_1p_2\big)(p_1+p_2+3p_3)-p_3^4-3p_3^3(p_1+p_2) - 6p_1p_2p_3^2\Big] - \frac{\pi^3(n_s+4n_F)}{8(p_1+p_2+p_3)^2}\Big[p_1^3+2p_1^2(p_2+p_3)+(2p_1+p_2+p_3)\big((p_2+p_3)^2-p_2p_3\big)\Big]$$

in d=4

The correlator in $d=4$ and the trace anomaly

we need 3 sectors and we need to renormalize because the gauge sector is not finite

$$\langle T^{\mu_1\nu_1}(p_1)T^{\mu_2\nu_2}(p_2)T^{\mu_3\nu_3}(p_3)\rangle_G = -V_G^{\mu_1\nu_1\mu_2\nu_2\mu_3\nu_3}(p_1,p_2,p_3) + \sum_{i=1}^{3} W_{G,i}^{\mu_1\nu_1\mu_2\nu_2\mu_3\nu_3}(p_1,p_2,p_3)$$

$$\langle t^{\mu_1\nu_1}(p_1)t^{\mu_2\nu_2}(p_2)t^{\mu_3\nu_3}(p_3)\rangle_G = \Pi^{\mu_1\nu_1}_{\alpha_1\beta_1}(p_1)\Pi^{\mu_2\nu_2}_{\alpha_2\beta_2}(p_2)\Pi^{\mu_3\nu_3}_{\alpha_3\beta_3}(p_3) \times \left[- V_G^{\alpha_1\beta_1\alpha_2\beta_2\alpha_3\beta_3}(p_1,p_2,p_3) + \sum_{i=1}^{3} W_{G,i}^{\alpha_1\beta_1\alpha_2\beta_2\alpha_3\beta_3}(p_1,p_2,p_3)\right]$$

$$\langle T^{\mu_1\nu_1}(p_1)T^{\mu_2\nu_2}(p_2)T^{\mu_3\nu_3}(p_3)\rangle = \sum_{I=F,G,S} n_I \, \langle T^{\mu_1\nu_1}(p_1)T^{\mu_2\nu_2}(p_2)T^{\mu_3\nu_3}(p_3)\rangle_I$$

A1 is finite

$$A_2^{Div} = \frac{\pi^2}{45\,\varepsilon}\left[26n_G - 7n_F - 2n_S\right]$$

$$A_3^{Div} = \frac{\pi^2}{90\,\varepsilon}\left[3(s+s_1)\big(6n_F + n_S + 12n_G\big) + s_2(11n_F + 62n_G + n_S)\right]$$

$$A_4^{Div} = \frac{\pi^2}{90\,\varepsilon}\left[(s+s_1)\big(29n_F + 98n_G + 4n_S\big) + s_2(43n_F + 46n_G + 8n_S)\right]$$

$$A_5^{Div} = \frac{\pi^2}{180\,\varepsilon}\Big\{n_F\left(43s^2 - 14s(s_1+s_2) + 43s_1^2 - 14s_1s_2 + 43s_2^2\right)$$

Anomalous CWI's in QED (MM Maglio,CC)

$$\mathrm{K}_{13} A_3^{Ren} = 2A_2^{Ren} - \frac{2\pi^2}{45}\,(7n_F - 26n_G + 2n_S)$$
$$\mathrm{K}_{23} A_3^{Ren} = 2A_2^{Ren} - \frac{2\pi^2}{45}\,(7n_F - 26n_G + 2n_S)$$

$$\mathrm{K}_{13} A_4^{Ren} = -4A_2^{Ren}(p_2 \leftrightarrow p_3) + \frac{4\pi^2}{45}\,(7n_F - 26n_G + 2n_S)$$
$$\mathrm{K}_{23} A_4^{Ren} = -4A_2^{Ren}(p_1 \leftrightarrow p_3) + \frac{4\pi^2}{45}\,(7n_F - 26n_G + 2n_S)$$

$$\mathrm{K}_{13} A_5^{Ren} = 2\left[A_4^{Ren} - A_4^{Ren}(p_1 \leftrightarrow p_3)\right] - \frac{4\pi^2}{9}(s - s_2)\,(5n_F + 2n_G + n_s)$$
$$\mathrm{K}_{23} A_5^{Ren} = 2\left[A_4^{Ren} - A_4^{Ren}(p_2 \leftrightarrow p_3)\right] - \frac{4\pi^2}{9}(s_1 - s_2)\,(5n_F + 2n_G + n_s)$$

one needs also to investigate the

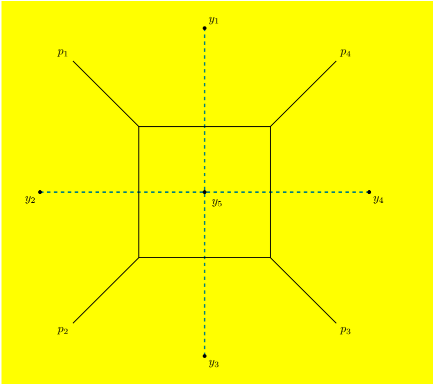
Secondary anomalous CWI's from free field theory

$$\langle T^{\mu_1\nu_1} T^{\mu_2\nu_2} T^{\mu_3\nu_3}\rangle_{Ren} = \langle t^{\mu_1\nu_1} t^{\mu_2\nu_2} t^{\mu_3\nu_3}\rangle_{Ren} + \langle T^{\mu_1\nu_1} T^{\mu_2\nu_2} T^{\mu_3\nu_3}\rangle_{Ren\,l\,t} + \langle T^{\mu_1\nu_1} T^{\mu_2\nu_2} T^{\mu_3\nu_3}\rangle_{anomaly}$$

4-point functions (MM Maglio, CC)

On Some Hypergeometric Solutions of the Conformal Ward Identities of Scalar 4-point Functions in Momentum Space

$$C_{13}=\left\{\frac{\partial^2}{\partial p_1^2}+\frac{(d-2\Delta_1+1)}{p_1}\frac{\partial}{\partial p_1}-\frac{\partial^2}{\partial p_3^2}-\frac{(d-2\Delta_3+1)}{p_3}\frac{\partial}{\partial p_3}\right.$$
$$+\frac{1}{s}\frac{\partial}{\partial s}\left(p_1\frac{\partial}{\partial p_1}+p_2\frac{\partial}{\partial p_2}-p_3\frac{\partial}{\partial p_3}-p_4\frac{\partial}{\partial p_4}\right)+\frac{(\Delta_3+\Delta_4-\Delta_1-\Delta_2)}{s}\frac{\partial}{\partial s}$$
$$+\frac{1}{t}\frac{\partial}{\partial t}\left(p_1\frac{\partial}{\partial p_1}+p_4\frac{\partial}{\partial p_4}-p_2\frac{\partial}{\partial p_2}-p_3\frac{\partial}{\partial p_3}\right)+\frac{(\Delta_2+\Delta_3-\Delta_1-\Delta_4)}{t}\frac{\partial}{\partial t}$$
$$\left.+\frac{(p_1^2-p_3^2)}{st}\frac{\partial^2}{\partial s\partial t}\right\}\Phi(p_1,p_2,p_3,p_4,s,t)=0.$$



+ ....

dual conformal symmetry

$$k=y_{51},\qquad p_1=y_{12},\qquad p_2=y_{23},\qquad p_3=y_{34}$$

$$\Phi_{Box}(p_1,p_2,p_3,p_4)=\int\frac{d^dk}{k^2(k+p_1)^2(k+p_1+p_2)^2(k+p_1+p_2+p_3)^2}$$

by requiring conformal invariance in momentum/coordinate space and in dual coordinate space

one btains a unique solution

For some vaues of the scaling dimensions exact solutions can be found

$$
\begin{aligned}
&\langle O(p_1)O(p_2)O(p_3)O(p_4)\rangle = \\
&\quad = \sum_{a,b} c(a,b)\Bigg[(s^2\, t^2)^{\Delta-\frac{3}{4}d} \left(\frac{p_1^2 p_3^2}{s^2 t^2}\right)^a \left(\frac{p_2^2 p_4^2}{s^2 t^2}\right)^b F_4\left(\alpha(a,b),\beta(a,b),\gamma(a),\gamma'(b),\frac{p_1^2 p_3^2}{s^2 t^2},\frac{p_2^2 p_4^2}{s^2 t^2}\right) \\
&\qquad + (s^2\, u^2)^{\Delta-\frac{3}{4}d} \left(\frac{p_2^2 p_3^2}{s^2 u^2}\right)^a \left(\frac{p_1^2 p_4^2}{s^2 u^2}\right)^b F_4\left(\alpha(a,b),\beta(a,b),\gamma(a),\gamma'(b),\frac{p_2^2 p_3^2}{s^2 u^2},\frac{p_1^2 p_4^2}{s^2 u^2}\right) \\
&\qquad + (t^2\, u^2)^{\Delta-\frac{3}{4}d} \left(\frac{p_1^2 p_2^2}{t^2 u^2}\right)^a \left(\frac{p_3^2 p_4^2}{t^2 u^2}\right)^b F_4\left(\alpha(a,b),\beta(a,b),\gamma(a),\gamma'(b),\frac{p_1^2 p_2^2}{t^2 u^2},\frac{p_3^2 p_4^2}{t^2 u^2}\right)\Bigg]
\end{aligned}
\tag{5.40}
$$

They are still hypergeometric functions F4 but quartic ratioa of momenta

Probably indication of a Yangian symmetry

DCC solutions (dual conformal/conformal) probably related ti a Yangian symmetry

$$\left\{\begin{array}{l}\left[\frac{\partial^2}{\partial p_1^2}+\frac{(d-2\Delta+1)}{p_1}\frac{\partial}{\partial p_1}-\frac{\partial^2}{\partial p_3^2}-\frac{(d-2\Delta+1)}{p_3}\frac{\partial}{\partial p_3}+\frac{(p_1^2-p_3^2)}{st}\frac{\partial^2}{\partial s\partial t}\right]I_{\tilde{\alpha}\{\beta_1,\beta_2,\beta_3\}}=0\\ \left[\frac{\partial^2}{\partial p_2^2}+\frac{(d-2\Delta+1)}{p_2}\frac{\partial}{\partial p_2}-\frac{\partial^2}{\partial p_4^2}-\frac{(d-2\Delta+1)}{p_4}\frac{\partial}{\partial p_4}+\frac{(p_2^2-p_4^2)}{st}\frac{\partial^2}{\partial s\partial t}\right]I_{\tilde{\alpha}\{\beta_1,\beta_2,\beta_3\}}=0\\ \left[\frac{\partial^2}{\partial p_3^2}+\frac{(d-2\Delta+1)}{p_3}\frac{\partial}{\partial p_3}-\frac{\partial^2}{\partial p_4^2}-\frac{(d-2\Delta+1)}{p_4}\frac{\partial}{\partial p_4}+\frac{(p_2^2-p_1^2)}{st}\frac{\partial^2}{\partial s\partial t}\right]I_{\tilde{\alpha}\{\beta_1,\beta_2,\beta_3\}}=0\end{array}\right.$$

new hypergeometric systems

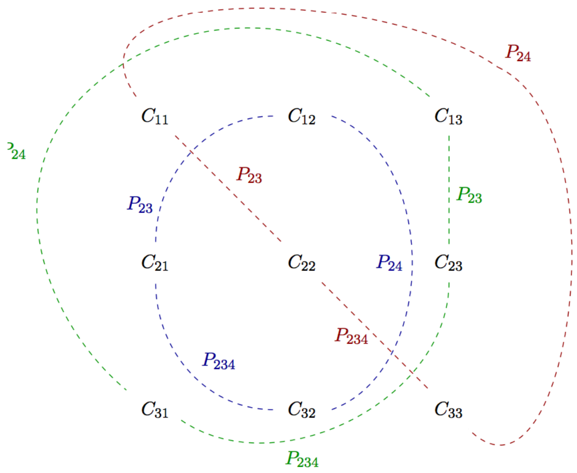
of variables

$$\begin{aligned}&\langle O(p_1)O(p_2)O(p_3)O(p_4)\rangle=\\&=\sum_{a,b}c(a,b)\Bigg[(s^2\,t^2)^{\Delta-\frac{3}{4}d}\left(\frac{p_1^2p_3^2}{s^2t^2}\right)^a\left(\frac{p_2^2p_4^2}{s^2t^2}\right)^b F_4\left(\alpha(a,b),\beta(a,b),\gamma(a),\gamma'(b),\frac{p_1^2p_3^2}{s^2t^2},\frac{p_2^2p_4^2}{s^2t^2}\right)\\&+\,(s^2\,u^2)^{\Delta-\frac{3}{4}d}\left(\frac{p_2^2p_3^2}{s^2u^2}\right)^a\left(\frac{p_1^2p_4^2}{s^2u^2}\right)^b F_4\left(\alpha(a,b),\beta(a,b),\gamma(a),\gamma'(b),\frac{p_2^2p_3^2}{s^2u^2},\frac{p_1^2p_4^2}{s^2u^2}\right)\\&+\,(t^2\,u^2)^{\Delta-\frac{3}{4}d}\left(\frac{p_1^2p_2^2}{t^2u^2}\right)^a\left(\frac{p_3^2p_4^2}{t^2u^2}\right)^b F_4\left(\alpha(a,b),\beta(a,b),\gamma(a),\gamma'(b),\frac{p_1^2p_2^2}{t^2u^2},\frac{p_3^2p_4^2}{t^2u^2}\right)\Bigg]\end{aligned}$$

Other analysis **TOOO** asymptotic limits of these equations
(for instance, fixed angle scattering) Lauricella functions are solutions

Maglio, Theofilopoulos, CC EPJ-C 2020

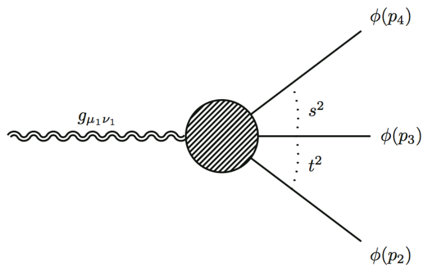
Four-point functions in momentum space: conformal ward identities in the scalar/tensor case



ig. 1 Orbits of the primary CWI's of the $TOOO$ under P_{23} and P_{24}

Orbits of form factors under permutations and their classification

$$C_{41} = \Bigg[\frac{2(p_3^2 - p_2^2 - t^2)}{t}\frac{\partial}{\partial t} + \frac{2(p_4^2 - p_2^2 - u^2)}{u}\frac{\partial}{\partial u} - 4\, p_2 \frac{\partial}{\partial p_2} + \frac{2}{\bar{p}_1^2}\left(\frac{d(d-2)(p_2^2 - s^2) - 2s^2}{(d-1)}\right) + \frac{4\Delta(d-1) - 2d^2 - 3(d-2)}{(d-1)} - \frac{(d-2)}{(d-1)}\frac{(p_2^2 - s^2)^2}{\bar{p}_1^4}\Bigg] \times A(p_2, p_3, p_4, s, t, u)$$
$$- \left[\frac{2}{\bar{p}_1^2}\left(\frac{d(p_3^2 - u^2) + 2u^2}{(d-1)}\right) + \frac{(d-2)}{(d-1)} + \frac{(d-2)}{(d-1)}\frac{(p_3^2 - u^2)^2}{\bar{p}_1^4}\right] \times A(p_3, p_2, p_4, u, t, s)$$
$$- \left[\frac{2}{\bar{p}_1^2}\left(\frac{d(p_4^2 - t^2) + 2t^2}{(d-1)}\right) + \frac{(d-2)}{(d-1)} + \frac{(d-2)}{(d-1)}\frac{(p_4^2 - t^2)^2}{\bar{p}_1^4}\right] \times A(p_4, p_3, p_2, t, s, u)\,,$$



Asymptotic SOLUTIONS of the TOOO higher hypergeopmetrics from the CWIs

$$\mathrm{K}_{ij} = \mathrm{K}_i - \mathrm{K}_j \ . \tag{D.4}$$

One can choose an arbitrary momentum as pivot in the ansatz for the solution of such system, for instance (x, y, z, p_4^2), where

$$x = \frac{p_1^2}{p_4^2}, \quad y = \frac{p_2^2}{p_4^2}, \quad z = \frac{p_3^2}{p_4^2} \tag{D.5}$$

$$\phi(p_1, p_2, p_3, p_4) = (p_4^2)^{n_s}\, x^a\, y^b\, z^c\, F(x, y, z),$$

$$\begin{aligned}\phi(p_1, p_2, p_3, p_4) &= C\, I_{d-1\left\{\Delta_1-\frac{d}{2},\Delta_2-\frac{d}{2},\Delta_3-\frac{d}{2},\Delta_4-\frac{d}{2}\right\}}(p_1, p_2, p_3, p_4) \\ &= \int_0^\infty dx\, x^{d-1} \prod_{i=1}^{4} (p_i)^{\Delta_i - \frac{d}{2}}\, K_{\Delta_i - \frac{d}{2}}(p_i\, x), \qquad \text{(1}\end{aligned}$$

$$F_C(\alpha, \beta, \gamma, \gamma', \gamma'', x, y, z) = \sum_{m_1, m_2, m_3}^{\infty} \times \frac{(\alpha)_{m_1+m_2+m_3} (\beta)_{m_1+m_2+m_3}}{(\gamma)_{m_1} (\gamma')_{m_2} (\gamma'')_{m_3} m_1!\, m_2!\, m_3!} x^{m_1} y^{m_2} z^{m_3}.$$

4K integrals

Maglio, Theofilopoulos, CC

first time that 4K appear in CWIs

particular solutons of these systems are Lauricella functions

$$\left\{ \begin{array}{c} x_j(1-x_j)\frac{\partial^2 F}{\partial x_j^2} + \sum_{s\neq j} x_r \text{ for } \sum_{r=j} x_s \frac{\partial^2 F}{\partial x_r \partial x_s} + [\gamma_j - (\alpha+\beta+1)x_j]\frac{\partial F}{\partial x_j} - (\alpha+\beta+1)\sum_{k\neq j} x_k \frac{\partial F}{\partial x_k} - \alpha\,\beta\, F = 0 \\ (j=1,2,3) \end{array} \right.$$

$$x = \frac{p_1^2}{p_4^2}, \quad y = \frac{p_2^2}{p_4^2}, \quad z = \frac{p_3^2}{p_4^2}$$

$$\begin{aligned}
S_1(\alpha,\beta,\gamma,\gamma',\gamma'',x,y,z) &= F_C\big(\alpha,\beta,\gamma,\gamma',\gamma'',x,y,z\big),\\
S_2(\alpha,\beta,\gamma,\gamma',\gamma'',x,y,z) &= x^{1-\gamma}\, F_C\big(\alpha-\gamma+1,\beta-\gamma+1,2-\gamma,\gamma',\gamma'',x,y,z\big)\,,\\
S_3(\alpha,\beta,\gamma,\gamma',\gamma'',x,y,z) &= y^{1-\gamma'}\, F_C\big(\alpha-\gamma'+1,\beta-\gamma'+1,\gamma,2-\gamma',\gamma'',x,y,z\big)\,,\\
S_4(\alpha,\beta,\gamma,\gamma',\gamma'',x,y,z) &= z^{1-\gamma''}\, F_C\big(\alpha-\gamma''+1,\beta-\gamma''+1,\gamma,\gamma',2-\gamma'',x,y,z\big),\\
S_5(\alpha,\beta,\gamma,\gamma',\gamma'',x,y,z) &= x^{1-\gamma}y^{1-\gamma'}\, F_C\big(\alpha-\gamma-\gamma'+2,\beta-\gamma-\gamma'+2,2-\gamma,2-\gamma',\gamma'',x,y,z\big)\,,\\
S_6(\alpha,\beta,\gamma,\gamma',\gamma'',x,y,z) &= x^{1-\gamma}z^{1-\gamma''}\, F_C\big(\alpha-\gamma-\gamma''+2,\beta-\gamma-\gamma''+2,2-\gamma,\gamma',2-\gamma'',x,y,z\big)\,,\\
S_7(\alpha,\beta,\gamma,\gamma',\gamma'',x,y,z) &= y^{1-\gamma'}z^{1-\gamma''}\, F_C\big(\alpha-\gamma'-\gamma''+2,\beta-\gamma'-\gamma''+2,\gamma,2-\gamma',2-\gamma'',x,y,z\big)\,,\\
S_8(\alpha,\beta,\gamma,\gamma',\gamma'',x,y,z) &= x^{1-\gamma}y^{1-\gamma'}z^{1-\gamma''}\\
&\quad\times\, F_C\big(\alpha-\gamma-\gamma'-\gamma''+2,\beta-\gamma-\gamma'-\gamma''+2,2-\gamma,2-\gamma',2-\gamma'',x,y,z\big)\,.
\end{aligned}$$

Giuseppe Lauricella (1867–1913)

TTTT implications for the anomaly action

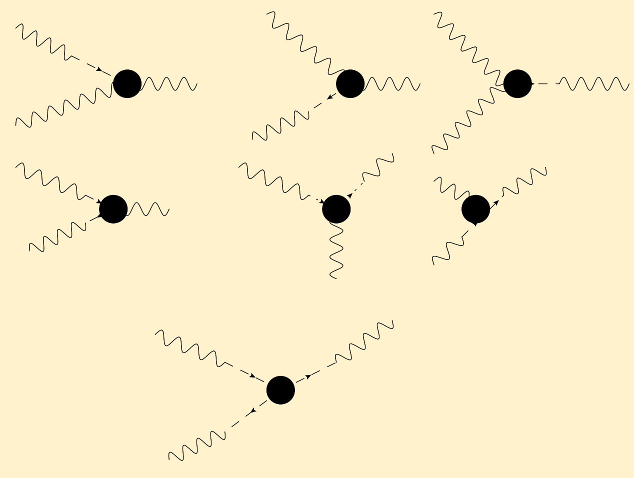
It can be derived from the analysis of the counterterms

$$V_{C^2}^{\mu_1\nu_1\mu_2\nu_2\mu_3\nu_3\mu_4\nu_4}(p_1,p_2,p_3,\bar{p}_4) \simeq \left[V_{C^2}^{\mu_1\nu_1\mu_2\nu_2\mu_3\nu_3\mu_4\nu_4}(p_1,p_2,p_3,\bar{p}_4)\right]_{d=4} + \varepsilon V'^{\mu_1\nu_1\mu_2\nu_2\mu_3\nu_3\mu_4\nu_4}_{C^2}(p_1,p_2,p_3,\bar{p}_4)$$

Decomposition into a transverse traceless sector + longitudinal sector

Maglio, Theofilopoulos, CC

Renormalization of the 4T affects the 3T and the 2T


(a)

the anomaly part

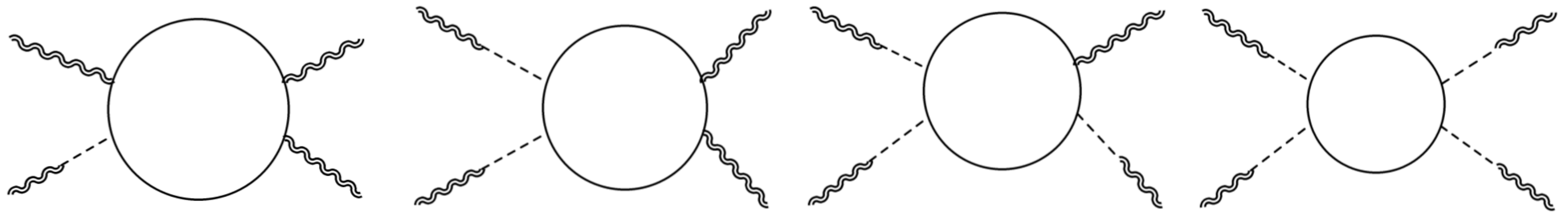
$$\langle T^{\mu_1\nu_1}(p_1)T^{\mu_2\nu_2}(p_2)T^{\mu_3\nu_3}(p_3)\rangle_{anomaly} = \frac{\hat{\pi}^{\mu_1\nu_1}(p_1)}{3\,p_1^2}\langle T(p_1)T^{\mu_2\nu_2}(p_2)T^{\mu_3\nu_3}(p_3)\rangle_{anomaly}$$

$$+ \frac{\hat{\pi}^{\mu_2\nu_2}(p_2)}{3\,p_2^2}\langle T^{\mu_1\nu_1}(p_1)T(p_2)T^{\mu_3\nu_3}(p_3)\rangle_{anomaly} + \frac{\hat{\pi}^{\mu_3\nu_3}(p_3)}{3\,p_3^2}\langle T^{\mu_1\nu_1}(p_1)T^{\mu_2\nu_2}(p_2)T(p_3)\rangle_{anomaly}$$

$$- \frac{\hat{\pi}^{\mu_1\nu_1}(p_1)\hat{\pi}^{\mu_2\nu_2}(p_2)}{9\,p_1^2p_2^2}\langle T(p_1)T(p_2)T^{\mu_3\nu_3}(p_3)\rangle_{anomaly} - \frac{\hat{\pi}^{\mu_2\nu_2}(p_2)\hat{\pi}^{\mu_3\nu_3}(p_2)}{9p_2^2p_3^2}\langle T^{\mu_1\nu_1}(p_1)T(p_2)T(p_3)\rangle_{anomaly}$$

$$- \frac{\hat{\pi}^{\mu_1\nu_1}(p_1)\hat{\pi}^{\mu_3\nu_3}(\bar{p}_3)}{9p_1^2p_3^2}\langle T(p_1)T^{\mu_2\nu_2}(p_2)T(p_3)\rangle_{anomaly} + \frac{\hat{\pi}^{\mu_1\nu_1}(p_1)\hat{\pi}^{\mu_2\nu_2}(p_2)\hat{\pi}^{\mu_3\nu_3}(\bar{p}_3)}{27p_1^2p_2^2p_3^2}\langle T(p_1)T(p_2)T(\bar{p}_3)\rangle_{anomaly}.$$

The 4rth order anomaly action: organization



$$\mathcal{S}_A = \int d^4x_1 d^4x_2 \langle T\cdot h(x_1) T\cdot h(x_2)\rangle + \int d^4x_1 d^4x_2 d^4x_3 \langle T\cdot h(x_1) T\cdot h(x_2) T\cdot h(x_3)\rangle_{pole}$$

$$+ \int d^4x_1 d^4x_2 d^4x_3 d^4x_4 \left(\langle T\cdot h(x_1) T\cdot h(x_2) T\cdot h(x_3) T\cdot h(x_4)\rangle_{pole} + \right.$$

$$\left. + \langle T\cdot h(x_1) T\cdot h(x_2) T\cdot h(x_3) T\cdot h(x_4)\rangle_{0T}\right),$$

$$
\begin{aligned}
&\langle T^{\mu_1\nu_1}(p_1)T^{\mu_2\nu_2}(p_2)T^{\mu_3\nu_3}(p_3)T^{\mu_4\nu_4}(\bar{p}_4)\rangle_{0-residue} = \\
&= \mathcal{I}^{\mu_1\nu_1}_{\alpha_1}(p_1)\, p_{1\beta_1}\ \langle T^{\alpha_1\beta_1}(p_1)T^{\mu_2\nu_2}(p_2)T^{\mu_3\nu_3}(p_3)T^{\mu_4\nu_4}(\bar{p}_4)\rangle_{anomaly} + (perm.) \\
&- \Bigg\{ \Bigg[\mathcal{I}^{\mu_2\nu_2}_{\alpha_2}(p_2)\, \mathcal{I}^{\mu_1\nu_1}_{\alpha_1}(p_1)\, p_{2\beta_2}\, p_{1\beta_1}\ \langle T^{\alpha_1\beta_1}(p_1)T^{\alpha_2\beta_2}(p_2)T^{\mu_3\nu_3}(p_3)T^{\mu_4\nu_4}(p_4)\rangle_{anom} \\
&\quad + \mathcal{I}^{\mu_1\nu_1}_{\alpha_1}(p_1)\, \frac{\pi^{\mu_2\nu_2}(p_2)}{3}\ p_{1\beta_1}\ \langle T^{\alpha_1\beta_1}(p_1)T(p_2)T^{\mu_3\nu_3}(p_3)T^{\mu_4\nu_4}(\bar{p}_4)\rangle_{anomaly} \\
&\quad + \frac{\pi^{\mu_1\nu_1}(p_1)}{3}\, \mathcal{I}^{\mu_2\nu_2}_{\alpha_2}(p_2)\ p_{2\beta_2}\ \langle T(p_1)T^{\alpha_2\beta_2}(p_2)T^{\mu_3\nu_3}(p_3)T^{\mu_4\nu_4}(\bar{p}_4)\rangle_{anomaly} \Bigg] + (perm.) \Bigg\} \\
&+ \Bigg\{ \Bigg[\mathcal{I}^{\mu_1\nu_1}_{\alpha_1}(p_1)\, \mathcal{I}^{\mu_2\nu_2}_{\alpha_2}(p_2) \frac{\pi^{\mu_3\nu_3}(p_3)}{3}\ \langle T^{\alpha_1\beta_1}(p_1)T^{\alpha_2\beta_2}(p_2)T(p_3)T^{\mu_4\nu_4}(\bar{p}_4)\rangle_{anomaly} + (13) + (23) \Bigg] + (perm.) \\
&+ \Bigg[\mathcal{I}^{\mu_1\nu_1}_{\alpha_1}(p_1) \frac{\pi^{\mu_2\nu_2}(p_2)}{3}\ \frac{\pi^{\mu_3\nu_3}(p_3)}{3}\ p_{1\beta_1}\ \langle T^{\alpha_1\beta_1}(p_1)T(p_2)T(p_3)T^{\mu_4\nu_4}(\bar{p}_4)\rangle_{anomaly} + (12) + (13) \Bigg] + (perm.) \Bigg\} \\
&\quad - \Bigg\{ \mathcal{I}^{\mu_1\nu_1}_{\alpha_1}(p_1)\, \mathcal{I}^{\mu_2\nu_2}_{\alpha_2}(p_2)\ \frac{\pi^{\mu_3\nu_3}(p_3)}{3}\ \frac{\pi^{\mu_4\nu_4}(p_4)}{3}\, p_{1\beta_1}\, p_{2\beta_2}\ \langle T^{\alpha_1\beta_1}(p_1)T^{\alpha_2\beta_2}(p_2)T(p_3)T(p_4)\rangle_{anomaly} \\
&\qquad + (13) + (23) + (14) + (24) + (13)(24) \Bigg\} \\
&\quad - \Bigg\{ \mathcal{I}^{\mu_1\nu_1}_{\alpha_1}(p_1)\ \frac{\pi^{\mu_2\nu_2}(p_2)}{3}\ \frac{\pi^{\mu_3\nu_3}(p_3)}{3}\ \frac{\pi^{\mu_4\nu_4}(p_4)}{3} p_{1\beta_1}\ \langle T^{\alpha_1\beta_1}(p_1)T(p_2)T(p_3)T(\bar{p}_4)\rangle_{anomaly} \\
&\qquad + (12) + (13) + (14) \Bigg\}. \qquad (9.65
\end{aligned}
$$

A new Weyl invarianrt term

The anomaly contribution at 4T

$$
\begin{aligned}
&\langle T^{\mu_1\nu_1}(p_1)T^{\mu_2\nu_2}(p_2)T^{\mu_3\nu_3}(p_3)T^{\mu_4\nu_4}(\bar{p}_4)\rangle_{poles} = \\
&= \frac{\pi^{\mu_1\nu_1}(p_1)}{3}\,\langle T(p_1)T^{\mu_2\nu_2}(p_2)T^{\mu_3\nu_3}(p_3)T^{\mu_4\nu_4}(\bar{p}_4)\rangle_{anomaly} + (perm.) \\
&- \frac{\pi^{\mu_1\nu_1}(p_1)}{3}\,\frac{\pi^{\mu_2\nu_2}(p_2)}{3}\,\langle T(p_1)T(p_2)T^{\mu_3\nu_3}(p_3)T^{\mu_4\nu_4}(\bar{p}_4)\rangle_{anomaly} + (perm.) \\
&+ \frac{\pi^{\mu_1\nu_1}(p_1)}{3}\,\frac{\pi^{\mu_2\nu_2}(p_2)}{3}\,\frac{\pi^{\mu_3\nu_3}(p_2)}{3}\,\langle T(p_1)T(p_2)T(p_3)T^{\mu_4\nu_4}(\bar{p}_4)\rangle_{anomaly} + (perm.) \\
&- \frac{\pi^{\mu_1\nu_1}(p_1)}{3}\,\frac{\pi^{\mu_2\nu_2}(p_2)}{3}\,\frac{\pi^{\mu_3\nu_3}(p_3)}{3}\,\frac{\pi^{\mu_4\nu_4}(p_4)}{3}\,\langle T(p_1)T(p_2)T(p_3)T(\bar{p}_4)\rangle_{anomaly}\,.
\end{aligned}
$$

It is possible to generalize this to all orders

WHAT DO WE LEARN FOR GRAVITY from the TTTT in flat and curves spacetime

M.M. Maglio, E. Mottola, CC

RIEGERT ACTION works for the TTT

$$g_{\mu\nu} = e^{2\sigma}\bar{g}_{\mu\nu}. \qquad \sqrt{-g}\,\Delta_4 = \sqrt{-\bar{g}}\,\bar{\Delta}_4$$

$$\Delta_4 \equiv \nabla_\mu\left(\nabla^\mu\nabla^\nu + 2R^{\mu\nu} - \tfrac{2}{3}Rg^{\mu\nu}\right)\nabla_\nu = \Box^2 + 2R^{\mu\nu}\nabla_\mu\nabla_\nu - \tfrac{2}{3}R\Box + \tfrac{1}{3}(\nabla^\mu R)\nabla_\mu$$

$$\mathcal{S}^{NL}_{\rm anom}[g] = \frac{1}{4}\int d^4x\sqrt{-g_x}\left(E - \tfrac{2}{3}\Box R\right)_x \int d^4x'\sqrt{-g_{x'}}\,D_4(x,x')\left[\frac{b'}{2}\left(E - \tfrac{2}{3}\Box R\right) + b\,C^2\right]_{x'}$$

Weyl invariant terms are missing, **as for any anomaly induced action**

The action is b uilt working in d=4 using

$$\sqrt{-g}\,C^2 = \sqrt{-\overline{g}}\,\overline{C}^2$$
$$\sqrt{-g}\,\left(E - \tfrac{2}{3}\Box R\right) = \sqrt{-\overline{g}}\,\left(\overline{E} - \tfrac{2}{3}\overline{\Box}\bar{R}\right) + 4\,\sqrt{-\overline{g}}\,\bar{\Delta}_4\,\sigma$$

Open issue

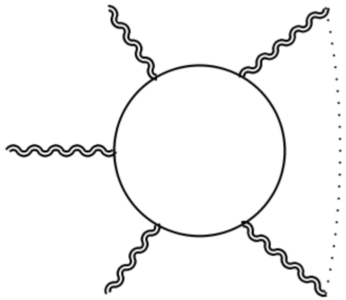
`iMPLICATION FOR THE EARLY UNIVERSE

Einstein-Gauss BOnnet Gravity oce we extend our analysis fo the Weyl flat case (work in progress)

$$\mathcal{S}(g) = \mathcal{S}(\bar{g}) + \sum_{n=1}^{\infty} \frac{1}{2^n n!} \int d^d x_1 \ldots d^d x_n \sqrt{g_1} \ldots \sqrt{g_n} \, \langle T^{\mu_1 \nu_1} \ldots T^{\mu_n \nu_n} \rangle_{\bar{g}} \delta g_{\mu_1 \nu_1}(x_1) \ldots \delta g_{\mu_n \nu_n}(x_n).$$

For anomaly actions

$$\mathcal{S}(g) = \sum_n$$



which is constrained by the Wess-Zumino consistency condition

$$[\delta_{\sigma_1}, \delta_{\sigma_2}] \, \mathcal{S} = 0$$

$$\mathcal{S}_{eff} \sim \int d^4x \sqrt{g} \left(\Lambda + c_1(g) R + c_2 \text{“}R^2\text{”} \right),$$

SAKHAROV induced gravity

$$\delta_\sigma \mathcal{S} = \frac{1}{(4\pi)^2} \int d^4x \sqrt{g} \, \sigma \, \left(c_1 R_{\mu\nu\rho\sigma} R^{\mu\nu\rho\sigma} + c_2 R_{\mu\nu} R^{\mu\nu} + c_3 R^2 + c_4 \Box R \right)$$

Anomaly action

TOPOLOGICAL TERMS IN THE EARLY UNIVERSE FROM THE CONFORMAL ANOMALY

$$E_d = \frac{1}{2^{d/2}} \delta^{\nu_1 \cdots \nu_d}_{\mu_1 \cdots \mu_d} R^{\mu_1 \mu_2}{}_{\nu_1 \nu_2} \cdots R^{\mu_{d-1} \mu_d}{}_{\nu_{d-1} \nu_d} \,,$$

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R + \Lambda g_{\mu\nu} + \frac{\alpha}{d-4} \mathcal{H}_{\mu\nu} = 8\pi G_N T_{\mu\nu},$$

some inconsistencies
in the Einstein Gauss-Bonnet term

$$\begin{aligned}
\mathcal{L}_0 &= 1 \\
\mathcal{L}_1 &= R \\
\mathcal{L}_2 &= E_4 \\
\mathcal{L}_3 &= R^3 - 12 R R_{\mu\nu} R^{\mu\nu} + 16 R_{\mu\nu} R^{\mu}{}_{\rho} R^{\nu\rho} + 24 R_{\mu\nu} R_{\rho\sigma} R^{\mu\rho\nu\sigma} + 3 R R_{\mu\nu\rho\sigma} R^{\mu\nu\rho\sigma} \\
&\quad - 24 R_{\mu\nu} R^{\mu}{}_{\rho\sigma\kappa} R^{\nu\rho\sigma\kappa} + 4 R_{\mu\nu\rho\sigma} R^{\mu\nu\eta\zeta} R^{\rho\sigma}{}_{\eta\zeta} - 8 R_{\mu\rho\nu\sigma} R^{\mu}{}_{\eta}{}^{\nu}{}_{\zeta} R^{\rho\eta\sigma\zeta} \,.
\end{aligned}$$

LOVELOCK Theorem can be violated?

Maglio, Theofilopoulos,
M. Creti', R Tommasi

$$\mathcal{S}_{EGB} = \int d^d x \sqrt{g} \left(R + \frac{\alpha}{d-4} E \right)$$

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R + \Lambda g_{\mu\nu} + \frac{\alpha}{d-4} \mathcal{H}_{\mu\nu} = 8\pi G_N T_{\mu\nu},$$

singular limits of EGB gravity
are related to anomaly actions

taken face value this equation is inconsistent

one needs to include back-reaction of matter on gravity

CONCLUSIONS AND PERSPECTIVES

The Conformal Ward Identities allow to define in a new way the anomaly induced action.
This can be studied both in flat and in curved spacetime, by a careful analysis of such constraints.

So far, we have covered the flat spacetime limit, up to the 4T. We can easily extend this analysis to the nT.

We have also shown how the inclusion of dual conformal/conformal symmetry allows to extend the power of the CWI, probably because of some underlying Yangian symmetry.

This requires a complete understanding of topological terms in curved spacetime, which in CFTs appear as counterterms in the anomaly action