Conformal Blocks and Light-Ray Operators in Celestial CFT

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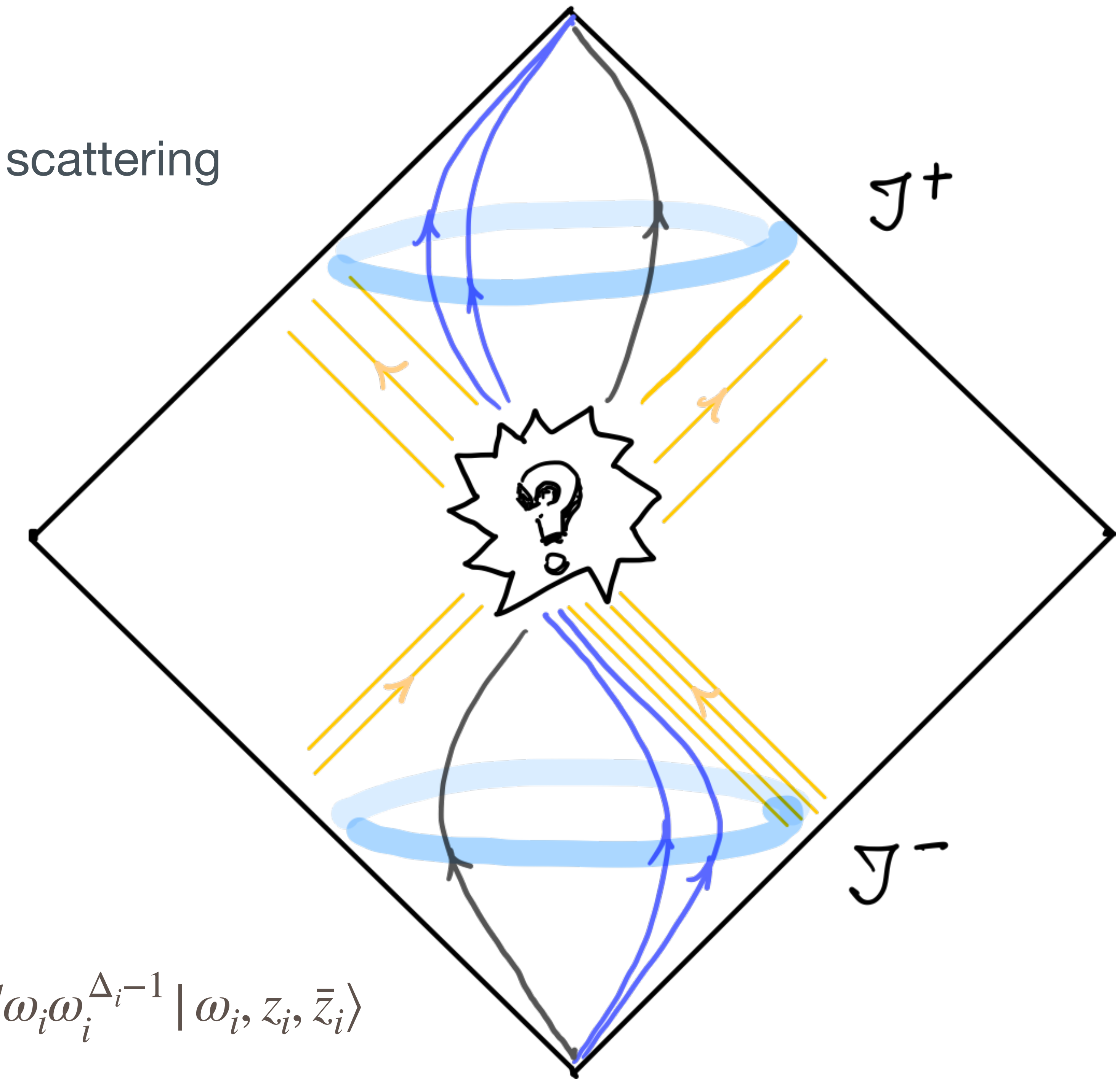
Motivation

- Celestial amplitudes: new toolkit for scattering in asymptotically flat spacetimes

- Study scattering in basis of boost eigenstates

- Map from momentum space to celestial sphere for massless external states:

$$|p_i\rangle \rightarrow |\omega_i, z_i, \bar{z}_i\rangle \rightarrow |\Delta_i, z_i, \bar{z}_i\rangle = \int_0^\infty d\omega_i \omega_i^{\Delta_i - 1} |\omega_i, z_i, \bar{z}_i\rangle$$

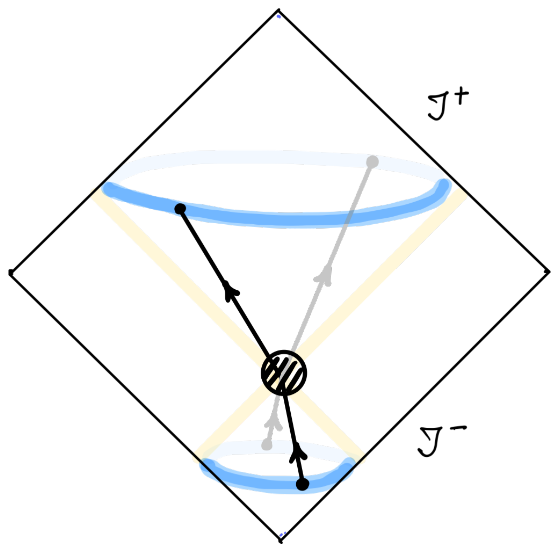


Celestial holography

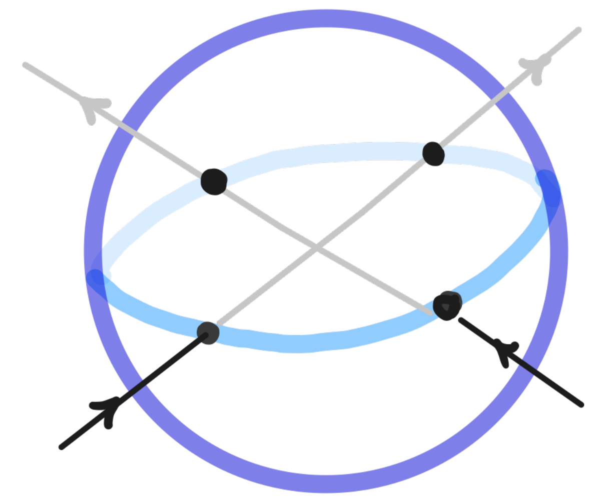
Outline

- Conformal blocks for scalar scattering in CCFT
  - → massive scalar exchanges of positive integral dimension
  - → spinning massive exchanges: light-ray transforms of massive scalars on the principal series
- Comments on light-transforms and shadows
- Open problems

Massless scalar 4-point scattering



Momentum space



Celestial sphere

$$\mathbb{A}(p_i) = \mathcal{M}(s,t)\delta^{(4)}\left(\sum_{i=1}^{4} p_i\right)$$

$$s = -(p_1+p_2)^2 \equiv \omega^2$$

$$t = -(p_1+p_3)^2 \equiv -z\omega^2$$

$\rightarrow$

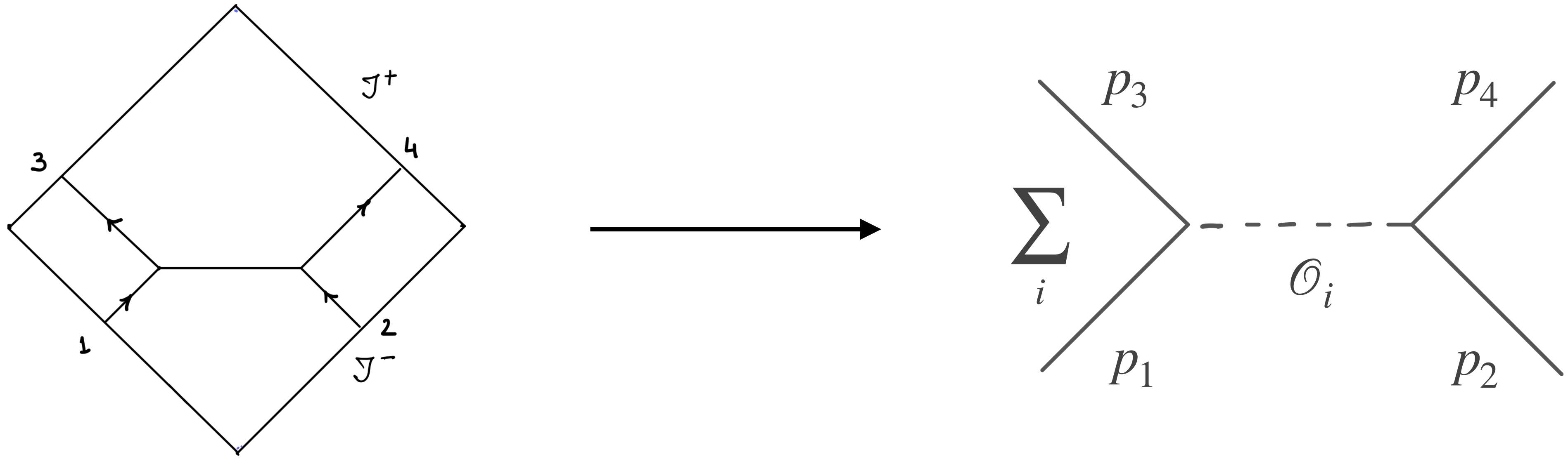
$$\widetilde{\mathcal{A}}(z_i,\bar{z}_i;\beta) = K(z_i,\bar{z}_i)X(z,\beta)\int_0^\infty d\omega\,\omega^{\beta-1}\mathcal{M}(\omega^2,-z\omega^2)$$

$$\beta = \sum_{i=1}^{4}\Delta_i - 4, \qquad z = -\frac{t}{s} = \frac{z_{13}z_{24}}{z_{12}z_{34}} \in [0,1]$$

$$K(z_i,\bar{z}_i) = \prod_{i<j} z_{ij}^{\frac{h}{3}-h_i-h_j}\bar{z}_{ij}^{\frac{\bar{h}}{3}-\bar{h}_i-\bar{h}_j}$$

$$X(z,\beta) = \delta(z-\bar{z})\,|z(1-z)|^{\frac{1}{6}(\beta+4)}$$

Tree-level example



$$\mathscr{M}(s,t) = -\,g^2\frac{1}{t-m^2} \quad\longrightarrow\quad \widetilde{\mathscr{A}}(z_i,\bar{z}_i;\beta) = g^2K(z_i,\bar{z}_i)X(z,\beta)\int_0^\infty d\omega\,\omega^{\beta-1}\frac{1}{z\omega^2+m^2}$$

$$= I_{13-24}(z_i,\bar{z}_i)\underbrace{N_{gm}(\beta)\,|\,z\,|^2\,|\,1-z\,|^{h_{13}-h_{24}}\,\delta(z-\bar{z})}_{f_t(z,\bar{z})}$$

Goal: Decompose $f_t(z,\bar{z})$ into conformal blocks.

Roadmap: Treat $z,\bar{z}$ as real independent variables and expand $f_t(z,\bar{z})$ on a complete set of orthogonal solutions to the two particle conformal Casimir over the Lorentzian square, $z,\bar{z}\in[0,1]$.

Conformal partial waves

- 1-3 conformal Casimir equation $\left(\mathcal{D}_z + \mathcal{D}_{\bar{z}}\right) \Psi_{h,\bar{h}}(z,\bar{z}) = \left[h(h-1) + \bar{h}(\bar{h}-1)\right] \Psi_{h,\bar{h}}(z,\bar{z})$

$$\mathcal{D}_z = z^2(1-z)\frac{\partial^2}{\partial z^2} - (1 - h_{13} + h_{24})z^2\frac{\partial}{\partial z} + h_{13}h_{24}z$$

admits solutions of the form $\Psi_{h,\bar{h}}(z,\bar{z}) = \Psi_h(z)\Psi_{\bar{h}}(\bar{z})\,.$

- Conformal partial waves $\Psi_h(z) = \frac{1}{2}\left(Q(h)k_h(z) + Q(1-h)k_{1-h}(z)\right), \quad h = \frac{1}{2} + \alpha, \quad \alpha \in i\mathbb{R},$

where $k_{h,\bar{h}}(z,\bar{z}) = k_h(z)k_{\bar{h}}(\bar{z}), \quad k_h(z) = z^h {}_2F_1(h - h_{12}, h + h_{34}; 2h; z)$ are $SL(2,\mathbb{C})$ blocks.

- $Q(h)$ fixed by **orthogonality** $\langle \Psi_h, \Psi_{h'} \rangle = \int_0^1 \frac{dz}{\mu(z)} \Psi_h(z)\Psi_{h'}(z) = \frac{N(h)}{2}\left[\delta(h+h') + \delta(h-h')\right]$

- **Completeness** $\int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty} \frac{dh}{N(h)} \Psi_h(z)\Psi_h(z') = \mu(z)\delta(z-z'), \quad \mu(z) = z^2(1-z)^{h_{12}-h_{34}}$

[Hogervorst, Rutter, van Rees '17, '20]

Conformal blocks for celestial scattering

$$f_t(z,\bar{z}) = N_{gm}(\beta)\,|z|^2\,|1-z|^{h_{13}-h_{24}}\delta(z-\bar{z}) = \int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty}\frac{dh}{N(h)}\int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty}\frac{d\bar{h}}{N(\bar{h})}g(h,\bar{h})\Psi_{h,\bar{h}}(z,\bar{z})$$

- Upon deforming the contour

$$f_t(z,\bar{z}) = n(\beta)\Big[\sum_{n=1}^{\infty} D^{nn}C_{13n}C_{24n}\cos\pi\left(\frac{n}{2}+h_{13}\right)\cos\pi\left(\frac{n}{2}+h_{24}\right)k_{\frac{1+n}{2}}(z)k_{\frac{1+n}{2}}(\bar{z})$$

$$+\frac{1}{2}\int_{-i\infty}^{i\infty}d\alpha C^L_{13\alpha}C^L_{24\alpha}D^{L,\alpha\alpha}k_{\frac{1}{2}+\alpha}(z)k_{\frac{1}{2}-\alpha}(\bar{z})\Big].$$

- Massive scalar exchanges $C_{ijn} = \frac{g}{m^4}\left(\frac{m}{2}\right)^{2h_i+2h_j}\mathrm{B}\left(\frac{1+n}{2}+h_{ij},\frac{1+n}{2}-h_{ij}\right),\quad D^{nn} = \frac{nm^2}{2\pi}.$

[Atanasov, Melton, A.R., Strominger ’21]

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$$g(h,\bar{h}) \rightarrow \langle f_t(z,\bar{z}), \Psi_{h,\bar{h}}(z,\bar{z})\rangle$$

- Upon deforming the contour

$$f_t(z,\bar{z}) = n(\beta)\Big[\sum_{n=1}^{\infty} D^{nn} C_{13n} C_{24n} \cos\pi\left(\frac{n}{2}+h_{13}\right)\cos\pi\left(\frac{n}{2}+h_{24}\right) k_{\frac{1+n}{2}}(z) k_{\frac{1+n}{2}}(\bar{z})$$

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$$\langle f_t(z,\bar z),\Psi_{h,\bar h}(z,\bar z)\rangle$$

- Upon deforming the contour

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matches 3-point of 2 massless and 1 massive scalars

massive 2-point

[Atanasov, Melton, A.R., Strominger '21]

Conformal blocks for celestial scattering

$$f_t(z,\bar z) = N_{gm}(\beta)\,|z|^2\,|1-z|^{h_{13}-h_{24}}\delta(z-\bar z) = \int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty}\frac{dh}{N(h)}\int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty}\frac{d\bar h}{N(\bar h)}\,g(h,\bar h)\Psi_{h,\bar h}(z,\bar z)$$

$$\langle f_t(z,\bar z), \Psi_{h,\bar h}(z,\bar z)\rangle$$

- Upon deforming the contour

$$f_t(z,\bar z) = n(\beta)\Big[\sum_{n=1}^{\infty} D^{nn}C_{13n}C_{24n}\cos\pi\left(\frac{n}{2}+h_{13}\right)\cos\pi\left(\frac{n}{2}+h_{24}\right)k_{\frac{1+n}{2}}(z)k_{\frac{1+n}{2}}(\bar z)$$

$$+\frac{1}{2}\int_{-i\infty}^{i\infty} d\alpha\, C^L_{13\alpha}C^L_{24\alpha}D^{L,\alpha\alpha}k_{\frac{1}{2}+\alpha}(z)k_{\frac{1}{2}-\alpha}(\bar z)\Big].$$

- Light-ray exchanges

$$C^L_{ij\alpha} = -\pi i\frac{g}{m^4}\left(\frac{m}{2}\right)^{2h_i+2h_j}\frac{1}{\alpha},\qquad D^{L,\alpha\alpha} = -\frac{m^2\alpha^2}{\pi^2 i}.$$

matches 3-point of 2 massless and
1 light-transformed massive scalars

light-transformed massive 2-point

[Atanasov, Melton, A.R., Strominger '21]

Light-transformed conformal primary wavefunctions

- Light transforms relate the bulk conformal primary wavefunctions $\phi_\Delta(z,\bar{z}) = \frac{i^\Delta \Gamma(\Delta)}{(-q\cdot X)^\Delta}$ of $\Delta = 1 + i\lambda,\ J = 0$ to conformal primary wavefunctions

$$\Phi(z,\bar{z};X) \propto \int_{-\infty}^{\infty} \frac{dz'}{(z'-z)^{2-\Delta}} \frac{1}{(-q(z',\bar{z})\cdot X)^\Delta} \propto \frac{(\partial_z q\cdot X)^{1-\Delta}}{-q\cdot X}, \quad \Delta_L = 1,\ J_L = -i\lambda\,.$$

- Two, three point functions of massless particles are singular in conformal primary basis, eg. tree-level gluons

$$\widetilde{\mathcal{A}}_3 \equiv \langle \mathcal{O}_1^-(z_1,\bar{z}_1)\mathcal{O}_2^-(z_2,\bar{z}_2)\mathcal{O}_3^+(z_3,\bar{z}_3)\rangle \propto \delta(\lambda_1+\lambda_2+\lambda_3) z_{12}^{1-i(\lambda_1+\lambda_2)} z_{23}^{i\lambda_1-1} z_{13}^{i\lambda_2-1}\delta(\bar{z}_{13})\delta(\bar{z}_{23})\,.$$

[Pasterski, Shao, Strominger '18]

- Light-transform leads to finite 3-point, eg. ambidextrous light transform of $\mathcal{O}^-$ and $\mathcal{O}^+$.

[Sharma '21]

New basis for celestial amplitudes?

- Shadows also turn singular 3-point functions into non-singular ones (prescription less clear).

- Evidence that a ``good" basis for CCFT will involve light-transforms and/or shadows.

- Study symmetry generators and associated constraints in the light/shadow bases.

- Massless momenta $P = \hat{q} e^{\partial_\Delta}$ transform to

$$\widetilde{P} = \frac{(\Delta - 1)^2}{2} \left(\partial_z \partial_{\bar{z}} \hat{q} + \frac{1}{\Delta - 1} \partial_z \hat{q} \partial_{\bar{z}} + \frac{1}{\Delta - 1} \partial_{\bar{z}} \hat{q} \partial_z + \frac{\hat{q} \partial_z \partial_{\bar{z}}}{(\Delta - 1)^2} \right) e^{-\partial_\Delta} \quad \text{upon shadow.}$$

- Can be used to derive constraints on celestial amplitudes involving shadows!

[work in progress w. Laurent Freidel]

Open questions

- Principle to dictate choice of basis?

- Transform amplitudes to this basis, figure out how symmetries act: Poincare, soft symmetries, etc.

- 4-point functions, conformal blocks in the new basis $\Longrightarrow$ simplifications?

  [Fan, Fotopoulos, Stieberger, Taylor, Zhu '21]

- Use constraints to implement a bootstrap program $\Longrightarrow$ non-perturbative aspects of gravity in AFS?