$O(d,d)$ transformations preserve classical integrability

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Based on 1907.03759 with
Domenico Orlando (INFN, Turin), Susanne Reffert (University of Bern), and
Kentaroh Yoshida (Kyoto University)

The plan of my talk

1. Motivation
2. Classical integrability of WZW models
3. Apply doubled formalism
4. Example (, time permitting)
5. Conclusions and outlook

1. Motivation

1.1 Integrability of string theory

- AdS_d/CFT_{d-1}: attractive examples of gauge/gravity(string) duality

d=5: [Maldacena-1998]

type IIB string on $AdS_5 \times S_5$ '$\Longleftrightarrow$' 4d $\mathcal{N}=4$ $SU(N)$ SYM ($N \to \infty$)

- <u>Intriguing</u>: integrable structures

allows us to determine physical quantities exactly,
even at finite coupling, without relying on supersymmetries.

e.g. scattering amplitudes, conformal dims. of composite ops.
spectrum of strings etc…

→ Many directions of applications of integrability techniques!

A comprehensive review:
[Beisert et al-2010]

An ongoing series of
winter schools
of integrability (=YRISW)

1.1 Integrability of string theory

- AdS$_d$/CFT$_{d-1}$: attractive examples of gauge/gravity(string) duality

d=5: [various examples]

integrable deformed type IIB string on AdS$_5$xS$_5$ '⇔' integrable deformed 4d $\mathcal{N}$=4 $SU(N)$ SYM ($N \to \infty$)

- Intriguing: integrable structures

allows us to determine physical quantities exactly,
even at finite coupling, without relying on supersymmetries.

e.g. scattering amplitudes, conformal dims. of composite ops.
spectrum of strings etc…

- On string theory side: integrable deformations

construct a variety of examples of ↑dualities keeping integrability

→ Want to follow a systematic approach for such deformations.

→ **Yang-Baxter (YB) deformation**

1.2 YB deformation [Klimcik-2002, 2014] [Delduc,Magro,Vicedo-2013] [Kawaguchi,Matsumoto,Yoshida-2014]

- The YB deformed sigma model is characterized by a classical r-matrix, linear operator solving the (modified) YB equation

$$\boxed{[R(X),R(Y)]+R([R(X),Y]+[X,R(Y)])=[X,Y]} \qquad X,Y\in\mathfrak{g}$$

- The YB sigma model reads for $g\in G$

$$\boxed{S=-\frac{T}{2}\int d^2\xi\,\left(\eta^{\alpha\beta}-\epsilon^{\alpha\beta}\right)\mathrm{Tr}\left[\left(g^{-1}\partial_\alpha g\right)\frac{1}{1-\eta R}\left(g^{-1}\partial_\beta g\right)\right]}$$

with a linear R-operator: $R(X)=\mathrm{Tr}_2[r_{12}(1\otimes X)]\equiv\sum_i\left(a_i\mathrm{Tr}\,(b_iX)-b_i\mathrm{Tr}\,(a_iX)\right)$

$$r_{12}=\sum_i a_i\wedge b_i\equiv\sum_i(a_i\otimes b_i-b_i\otimes a_i)$$

and a const. deformation parameter: η

$$X,a_i,b_i\in\mathfrak{g}$$

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and a const. deformation parameter: η

- Given some r-matrix, read off the corresp. deformed background via comparison with the canonical form of string sigma model.

→ So systematic that lots of integrable deformed backgrounds produced.

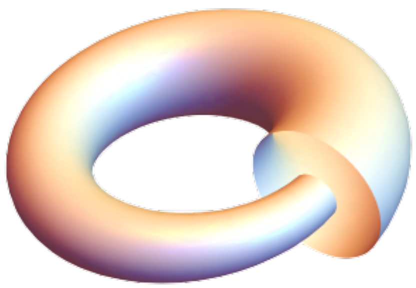
1.3 YB deformation → $O(d,d)$

☆ Some YB deformed backgrounds closely related to $O(d,d)$ **transformations**

- TsT transformations e.g. [Matsumoto,Yoshida-2014] [Osten,van Tongeren-2016]

  T-duality - angular shift - T-duality on the $U(1) \times U(1)$ directions [Maldacena,Russo] [Hashimoto,Itzhaki] [Maldacena,Lunin] [Alday,Aryutunov,Frolov,] etc...

- T-fold backgrounds [Melgarejo-2017]

  Non-geometric backgrounds, solutions in the generalized supergravity [Arutyunov,Frolov,Hooare,Roiban,Tseytlin]

- Non-Abelian T-dual backgrounds [Borsato,Wulff-2018] [Lust,Osten-2018]

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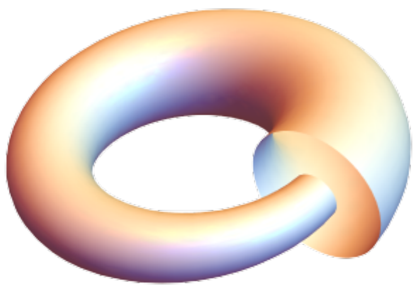

★ Current-current deformations (as YB-def. [Borsato,Wulff-2018])

- marginal deformations of 2d CFTs
- generated by global $O(d,d;\mathbb{R})$ transf.

$$\frac{\partial L}{\partial \eta} \sim J_\eta(z)\bar{J}_\eta(\bar{z})$$

- traditional but related to the ``recent" $T\bar{T}$-deformation

e.g. [Hassan,Sen-1992]
[Giveon,Kiritsis-1994]
[Forste-1994]
[Forste,Roggenkamp-2003]
[Israel,Kounnas,Petropoulos,-2005]…

e.g. [Cavaglia et al-2016]
[Giveon,Itzhaki,Kutasov,2017]…

1.4 In my work, $O(d,d)$ → Integrability

☆ Without YB, extract integrable structures of ANY global $O(d,d;\mathbb{R})$ transf.
(bottom-up approach = complimentary to YB, or my collaborator)

☆ **Construct the Lax pairs of $O(d,d)$ deformed models (see later→)**

(→whose existence define the classical integrability)

1.4 In my work, $O(d,d)$ → Integrability

☆ Without YB, extract integrable structures of ANY global $O(d,d;\mathbb{R})$ transf.
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☆ Construct the Lax pairs of $O(d,d)$ deformed models (see later→)

✤ $O(d,d)$ transformations ``rotate" the generalized metric: e.g. [Hassan,Sen-1992]
[Blumenhagen, Deser,Plauschinn,Deser,Schmid,2013]

Given $\mathcal{H} = \begin{pmatrix} G - BG^{-1}B & BG^{-1} \\ -G^{-1}B & G^{-1} \end{pmatrix}$, the deformed bgr. data via field redefinition

undeformed	$h \in O(d,d)$	transformed		deformed
$\mathcal{H}(G,B)$	$\longrightarrow$	$h^t\mathcal{H}(G,B)h$	$=$	$\mathcal{H}(G',B')$

Nowadays, the gen.metric is a crucial object in the $O(d,d)$-inv. formalisms

→ $O(d,d)$ transf. well-controlled in the doubled formalism, why not use it!?

✦ Integrable deformations and doubled formalism [Demulder,Hassler,Thompson-2018]...
✦ also, developments in the doubled sigma model [Marotta,Pezzella,VItale-2018,2019]...

1.5 Upshot

☆ I studied how to construct Lax pairs (=def. of classical integrability) in the $O(d,d)$ deformed models using the doubled formalism.
[Hull-2004]

☆ The resulting Lax pairs form the algebra of symmetries hidden by deformations.

✦ ~ ***Revisit the traditional using a modern language.***

→ Return to the basics

2. Classical integrability of WZW models

2.1 Terminology

cf: conventions in [Ricci,Tseytlin,Wolf-2007]

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- Given a J satisfying $\mathrm{d} \star J = 0$ and $\mathrm{d}J + J \wedge J = 0$,

  Lax pairs given by

$$\boxed{\mathcal{L} = a_\lambda J + b_\lambda \star J}$$

$$a_\lambda = \frac{1}{2}(1 - \cosh\lambda)\,, \qquad b_\lambda = \frac{1}{2}\sinh\lambda$$

spectral parameter

☆ Lax pairs are flat $\mathrm{d}\mathcal{L} + \mathcal{L} \wedge \mathcal{L} = 0$ **on-shell**.

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- An infinite number of charges generated by the monodromy matrix $\mathcal{T}(t;\lambda)$

$$\begin{aligned}\mathcal{T}(t;\lambda)&=\mathcal{P}\mathrm{exp}\left[-\int \mathrm{d}x'\mathcal{L}_x(x')\right]\\&=1+\sum_{n=0}\lambda^{n+1}\mathcal{Q}^{(n)}(t)\end{aligned}\quad\longrightarrow\quad\boxed{\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{Q}^{(n)}(t)=0}$$

$\forall n\in\mathbb{Z}_{\geq 0}$

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$$\mathcal{Q}^{(0)} = \frac{1}{2}\int \mathrm{d}x\, J_t(t,x)$$

$$\mathcal{Q}^{(1)} = -\frac{1}{4}\int \mathrm{d}x\, J_x(t,x) + \frac{1}{2}\int \mathrm{d}x \int^x \mathrm{d}x' J_t(t,x) J_t(t,x')$$

... infinitely many non-local charges

- An infinite number of charges generated by the monodromy matrix $\mathcal{T}(t;\lambda)$

$$\mathcal{T}(t;\lambda) = \mathcal{P}\exp\left[-\int \mathrm{d}x' \mathcal{L}_x(x')\right]$$

$$= 1 + \sum_{n=0} \lambda^{n+1} \mathcal{Q}^{(n)}(t) \longrightarrow \boxed{\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{Q}^{(n)}(t) = 0}$$

$$\forall n \in \mathbb{Z}_{\geq 0}$$

2.2 Basics of $SU(2)$ WZNW on S^3 w/ H-flux

- The WZNW action

$$S[g] = -\frac{1}{4}\int_{\Sigma_2} \mathrm{Tr}\left[g^{-1}\mathrm{d}g \wedge \star g^{-1}\mathrm{d}g\right] + \frac{i}{3!}\int_{\mathcal{V}_3} \mathrm{Tr}\left(g^{-1}\mathrm{d}g\right)^{\wedge 3}$$

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has conserved and flat currents

$$J_{\mathrm{L}} = (1 - i\star)\, g^{-1}\mathrm{d}g, \quad J_{\mathrm{R}} = (1 + i\star)\left(-\mathrm{d}g g^{-1}\right)$$

- For $g \in SU(2)$, six Lax pairs can be constructed.

- Monodromy matrix, and then non-local charges…..

2.3 Gauged Lax pairs

- Lax connections are equivalent up to gauge transformations:

$$\mathcal{L} \to \hat{\mathcal{L}} = h^{-1}\mathcal{L}h + h^{-1}\mathrm{d}h$$

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- For a specific $g = e^{-Z_+T_2}e^{YT_1}e^{+Z_-T_2}$ with $[T_\alpha, T_\beta] = \epsilon_{\alpha\beta\gamma}T_\gamma$,

$$= e^{-(Z_1+Z_2)T_2}e^{YT_1}e^{+(Z_1-Z_2)T_2}$$

using the gaugings $\begin{cases} h_\mathrm{L} = e^{-(Z_1-Z_2)T_2} \\ h_\mathrm{R} = e^{-(Z_1+Z_2)T_2} \end{cases}$, our proper starting point is

$$\hat{\mathcal{L}}^1_\mathrm{L} = +F_1(\lambda)\mathrm{d}Y,$$

$$\hat{\mathcal{L}}^2_\mathrm{L} = -F_1(\lambda)\left[\mathrm{d}Z_1 - \mathrm{d}Z_2 - \cos Y\left(\mathrm{d}Z_1 + \mathrm{d}Z_2\right)\right] - \left(\mathrm{d}Z_1 - \mathrm{d}Z_2\right),$$

$$\hat{\mathcal{L}}^3_\mathrm{L} = +F_1(\lambda)\sin Y\left(\mathrm{d}Z_1 + \mathrm{d}Z_2\right),$$

$$F_1(\lambda) = [(ib_\lambda - a_\lambda) + (ia_\lambda - b_\lambda)\star]$$
$$F_2(\lambda) = [(ib_\lambda + a_\lambda) + (ia_\lambda + b_\lambda)\star]$$

$$\hat{\mathcal{L}}^1_\mathrm{R} = +F_2(\lambda)\mathrm{d}Y,$$

$$\hat{\mathcal{L}}^2_\mathrm{R} = +F_2(\lambda)\left[\mathrm{d}Z_1 + \mathrm{d}Z_2 - \cos Y\left(\mathrm{d}Z_1 - \mathrm{d}Z_2\right)\right] - \left(\mathrm{d}Z_1 + \mathrm{d}Z_2\right)$$

$$\hat{\mathcal{L}}^3_\mathrm{R} = +F_2(\lambda)\sin Y\left(\mathrm{d}Z_1 - \mathrm{d}Z_2\right)$$

No explicit dep. on (Z_1, Z_2)

2.3 Gauged Lax pairs and $O(2,2)$ deformed Lax pairs

- Lax pairs should depend on th
  only through their derivatives (

$$G_{\mu\nu}\mathrm{d}X^\mu\mathrm{d}X^\nu = \frac{1}{4}\mathrm{d}Y^2 + \sin^2\left(\frac{Y}{2}\right)\mathrm{d}Z_1^2 + \cos^2\left(\frac{Y}{2}\right)\mathrm{d}Z_2^2$$

$$B_{\mu\nu}\mathrm{d}X^\mu \wedge \mathrm{d}X^\nu = \cos^2\left(\frac{Y}{2}\right)\mathrm{d}Z_1 \wedge \mathrm{d}Z_2.$$

$$\hat{\mathcal{L}}^1_{\mathrm{L}} = +F_1(\lambda)\mathrm{d}Y,$$

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$$\hat{\mathcal{L}}^3_{\mathrm{R}} = +F_2(\lambda)\sin Y\left(\mathrm{d}Z_1 - \mathrm{d}Z_2\right)$$

☆ The strategy at $O(2,2)$ (in the basis of (Z_1, Z_2)) deformed Lax pairs:

1. Build a map ($O(2,2)$ transf. rule) for $(\mathrm{d}Z_1, \mathrm{d}Z_2)$: $\boxed{\mathcal{L}' = \hat{\mathcal{L}}(\mathrm{d}Z \to \mathfrak{D}(\mathrm{d}Z'))}$
2. Check the flatness (**on-shell** = EoMs in the deformed model).

2.3 Gauged Lax pairs and $O(2,2)$ deformed Lax pairs

cf: [Ricci,Tseytlin,Wolf-2007]

- Lax pairs should depend on the U(1) isometric directions **only** through their derivatives (cf: T-duality, Buscher rule).

$$
\begin{aligned}
\hat{\mathcal{L}}^1_{\mathrm{L}} &= +F_1(\lambda)\mathrm{d}Y, \\
\hat{\mathcal{L}}^2_{\mathrm{L}} &= -F_1(\lambda)\left[\mathrm{d}Z_1 - \mathrm{d}Z_2 - \cos Y\left(\mathrm{d}Z_1 + \mathrm{d}Z_2\right)\right] - \left(\mathrm{d}Z_1 - \mathrm{d}Z_2\right), \\
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\end{aligned}
$$

$$
\begin{aligned}
\hat{\mathcal{L}}^1_{\mathrm{R}} &= +F_2(\lambda)\mathrm{d}Y, \\
\hat{\mathcal{L}}^2_{\mathrm{R}} &= +F_2(\lambda)\left[\mathrm{d}Z_1 + \mathrm{d}Z_2 - \cos Y\left(\mathrm{d}Z_1 - \mathrm{d}Z_2\right)\right] - \left(\mathrm{d}Z_1 + \mathrm{d}Z_2\right) \\
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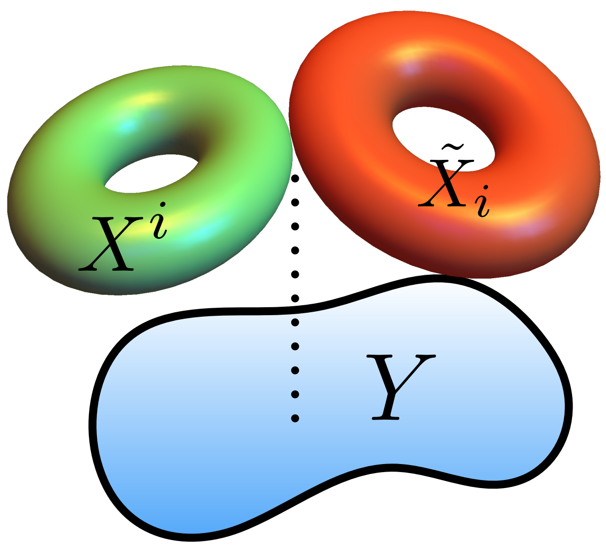
3. Apply doubled formalism

3.1 Action and constraint of the doubled sigma model

[Hull-2004]

- **Doubled formalism** based on the doubled torus fibration over a base manifold. To T^d fiber of the physical sigma model description, we add a dual torus $\tilde{T}^d$ so the fiber becomes the doubled torus T^{2d}.



- For the doubled coords. $\mathbb{X}^I = (X^i, \tilde{X}_i)$, the doubled sigma model reads

$$S = \int \frac{1}{2}\mathcal{H}_{IJ}\mathrm{d}\mathbb{X}^I \wedge \star\mathrm{d}\mathbb{X}^J + \mathcal{L}(Y)...$$

$O(d,d)$ invariance: $\mathcal{H} \to h^t\mathcal{H}h, \quad \mathrm{d}\mathbb{X} \to h^{-1}\mathrm{d}\mathbb{X}$

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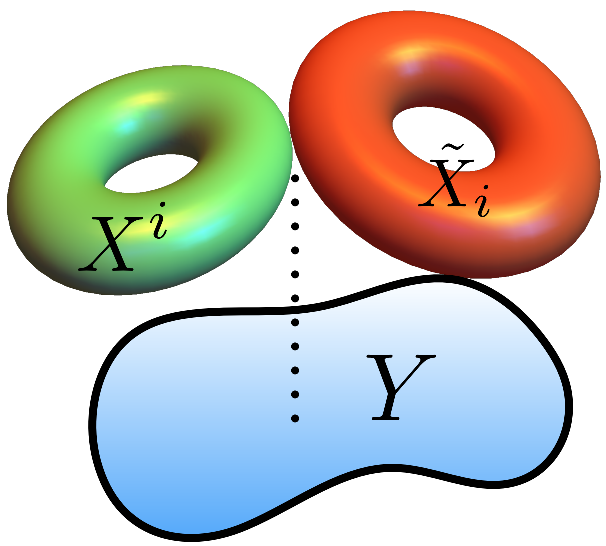

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$O(d,d)$ invariance: $\mathcal{H} \to h^t \mathcal{H} h, \quad \mathrm{d}\mathbb{X} \to h^{-1}\mathrm{d}\mathbb{X}$

- It is equipped with the self-duality constraint:

$$\boxed{\mathrm{d}\mathbb{X}^I = L^{IJ}\mathcal{H}_{JK} \star \mathrm{d}\mathbb{X}^K} \qquad L = \begin{pmatrix} 0 & \mathbf{1}_d \\ \mathbf{1}_d & 0 \end{pmatrix}$$

truncates the doubled formalism to the standard sigma model.

3.2 Winding coords and $O(d,d)$ map

[Rennecke-2014]
[Orlando,Reffert,Y.S.,Yoshida-2019]

- Unpackaging the self-duality constraint, the winding coords.

$$\mathrm{d}\tilde{X}_i = \star\left(G_{ij} + B_{ij}\star\right)\mathrm{d}X^j = \star J_i$$

turn into (dual of) Noether currents for $U(1)$-isometry: $\mathrm{d}^2\tilde{X}_i = 0 = \mathrm{d}\star J_i$

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- Therefore, the $O(d,d)$ transformation by $g = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$ leads to

$O(d,d)$ map: $$\boxed{\mathrm{d}X^i = \alpha^i{}_j \mathrm{d}X'^j + \beta^{ik} \star J'_k} = \mathfrak{D}(\mathrm{d}X')$$

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- This map results in the $O(d,d)$ deformed Lax pairs as

$$\boxed{\mathcal{L}'(\mathrm{d}X') = \hat{\mathcal{L}}(\mathrm{d}X \to \mathfrak{D}(\mathrm{d}X'))}$$

→ Flatness ON-SHELL (=EoMs in the deformed model) to be checked →

3.4 Maps for on-shell conditions [Orlando,Reffert,Y.S.,Yoshida-2019]

- Start from the undeformed (gauged) Lax pairs

adapted coords.

$$k_X = \partial_X$$

$$\mathrm{d}\hat{\mathcal{L}} + \hat{\mathcal{L}} \wedge \hat{\mathcal{L}} \propto \sum_i (\text{EoMs for } \mathrm{d}X, Y)_i$$

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1. The $O(d,d)$ transformation for the (diff. of) winding coords. by $g = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$ is

$$\mathrm{d}\tilde{X}_i = \gamma_{ij}\mathrm{d}X'^j + \delta_i{}^k \mathrm{d}\tilde{X}'_k$$

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$$\star J_i = \gamma_{ij} \mathrm{d}X'^j + \delta_i{}^k \star J'_k$$

3.4 Maps for on-shell conditions [Orlando,Reffert,Y.S.,Yoshida-2019]

- Start from the undeformed (gauged) Lax pairs

adapted coords.

$$k_X = \partial_X$$

$$\mathrm{d}\hat{\mathcal{L}} + \hat{\mathcal{L}} \wedge \hat{\mathcal{L}} \propto \sum_i (\text{EoMs for } \mathrm{d}X, Y)_i$$

$$\mathcal{L}'(\mathrm{d}X') = \hat{\mathcal{L}}(\mathrm{d}X \to \mathfrak{D}(\mathrm{d}X'))$$

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2. Using the $O(d,d)$ invariance of the doubled action and the self-duality constraint,

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→ Flat on-shell after $O(d,d)$

3.5 Recipe summarized

[Orlando,Reffert,Y.S.,Yoshida-2019]
extended from [Ricci,Tseytlin,Wolf-2007]

Recipe for $O(d,d)$ deformed Lax pairs

1. Find flat Noether currents J to construct Lax pairs $\mathcal{L}$.

2. Find gauged Lax pairs $\hat{\mathcal{L}}$ explicitly indep. of adapted coords.

3. Apply the $O(d,d)$ map from the doubled formalism to $\hat{\mathcal{L}}$ to obtain $\mathcal{L}'$.

★ Flatness guaranteed .

4. Example
(,time permitting)

4.1 Examples of $O(2,2)$ deformations

- Imagine the S^3 background

$$G_{\mu\nu}\mathrm{d}X^\mu\mathrm{d}X^\nu = \frac{1}{4}\mathrm{d}Y^2 + \sin^2\left(\frac{Y}{2}\right)\mathrm{d}Z_1^2 + \cos^2\left(\frac{Y}{2}\right)\mathrm{d}Z_2^2$$

$$B_{\mu\nu}\mathrm{d}X^\mu \wedge \mathrm{d}X^\nu = \cos^2\left(\frac{Y}{2}\right)\mathrm{d}Z_1 \wedge \mathrm{d}Z_2.$$

- Gauged Lax pairs are given by

$$\begin{aligned}
\hat{\mathcal{L}}_{\mathrm{L}}^1 &= +F_1(\lambda)\mathrm{d}Y, \\
\hat{\mathcal{L}}_{\mathrm{L}}^2 &= -F_1(\lambda)\left[\mathrm{d}Z_1 - \mathrm{d}Z_2 - \cos Y\left(\mathrm{d}Z_1 + \mathrm{d}Z_2\right)\right] - \left(\mathrm{d}Z_1 - \mathrm{d}Z_2\right), \\
\hat{\mathcal{L}}_{\mathrm{L}}^3 &= +F_1(\lambda)\sin Y\left(\mathrm{d}Z_1 + \mathrm{d}Z_2\right),
\end{aligned}$$

$$\begin{aligned}
\hat{\mathcal{L}}_{\mathrm{R}}^1 &= +F_2(\lambda)\mathrm{d}Y, \\
\hat{\mathcal{L}}_{\mathrm{R}}^2 &= +F_2(\lambda)\left[\mathrm{d}Z_1 + \mathrm{d}Z_2 - \cos Y\left(\mathrm{d}Z_1 - \mathrm{d}Z_2\right)\right] - \left(\mathrm{d}Z_1 + \mathrm{d}Z_2\right) \\
\hat{\mathcal{L}}_{\mathrm{R}}^3 &= +F_2(\lambda)\sin Y\left(\mathrm{d}Z_1 - \mathrm{d}Z_2\right)
\end{aligned}$$

- Use the doubled formalism based on $\mathbb{Z} = (Z_1, Z_2, \tilde{Z}_1, \tilde{Z}_2)$

4.2 Examples of $O(2,2)$ deformations

- Whatever $O(2,2)$ element you choose, e.g. ,

$$g = \begin{pmatrix} \boxed{\begin{matrix} 1 & 0 \\ 0 & \frac{1}{1+\tan\alpha} \end{matrix}} & \boxed{\begin{matrix} 0 & \tan\alpha \\ -\frac{\tan\alpha}{1+\tan\alpha} & 0 \end{matrix}} \\ \begin{matrix} 0 & \frac{1}{1+\tan\alpha} \\ -1 & 0 \end{matrix} & \begin{matrix} \frac{1}{1+\tan\alpha} & 0 \\ 0 & 1 \end{matrix} \end{pmatrix}$$

(current-current deformation)

- Write down the $O(2,2)$ map

$$\mathrm{d}Z_1 = \mathfrak{D}_1(\mathrm{d}Z') = 1 \cdot \mathrm{d}Z_1' + \tan\alpha \star J_2'$$

$$\mathrm{d}Z_2 = \mathfrak{D}_2(\mathrm{d}Z') = \frac{1}{1+\tan\alpha}\mathrm{d}Z_2' - \frac{\tan\alpha}{1+\tan\alpha} \star J_1'$$

- Then,

$$\mathcal{L}_{\mathrm{L}}^{\prime 1} = +F_1(\lambda)\mathrm{d}Y,$$

$$\mathcal{L}_{\mathrm{L}}^{\prime 2} = -F_1(\lambda)\left[\mathfrak{D}_1(\mathrm{d}Z') - \mathfrak{D}_2(\mathrm{d}Z') - \cos Y\left(\mathfrak{D}_1(\mathrm{d}Z') + \mathfrak{D}_2(\mathrm{d}Z')\right)\right] - \left(\mathfrak{D}_1(\mathrm{d}Z') - \mathfrak{D}_2(\mathrm{d}Z')\right),$$

$$\mathcal{L}_{\mathrm{L}}^{\prime 3} = +F_1(\lambda)\sin Y\left(\mathfrak{D}_1(\mathrm{d}Z') + \mathfrak{D}_2(\mathrm{d}Z')\right),$$

$$\mathcal{L}_{\mathrm{R}}^{\prime 1} = +F_2(\lambda)\mathrm{d}Y,$$

$$\mathcal{L}_{\mathrm{R}}^{\prime 2} = +F_2(\lambda)\left[\mathfrak{D}_1(\mathrm{d}Z') + \mathfrak{D}_2(\mathrm{d}Z') - \cos Y\left(\mathfrak{D}_1(\mathrm{d}Z') - \mathfrak{D}_2(\mathrm{d}Z')\right)\right] - \left(\mathfrak{D}_1(\mathrm{d}Z') + \mathfrak{D}_2(\mathrm{d}Z')\right),$$

$$\mathcal{L}_{\mathrm{R}}^{\prime 3} = +F_2(\lambda)\sin Y\left(\mathfrak{D}_1(\mathrm{d}Z') - \mathfrak{D}_2(\mathrm{d}Z')\right).$$

$\mathrm{d}\mathcal{L}' + \mathcal{L}' \wedge \mathcal{L}' = 0$ **still holds.**

5. Conclusions and Outlook

6. Conclusions and outlook: O(d,d) is not odd

☆ This work completed the classical integrability of ANY global $O(d,d)$ transformation using the doubled formalism from the bottom.

→ $O(d,d)$ deformed Lax pairs involve windings = $U(1)$ currents

→ All the current-current deformations are integrable.

→ Using our recipe, we can extract all the Lax pairs for the symmetry hidden by the $O(d,d)$ deformations. cf: [Beisert-2009]

☆ Some (technical) things to do:

✦ Algebra of non-local charges… [work in progress] [Kawaguchi,Yoshida-2010]…

✦ $O(d,d)$ map interpreted as deformations on the spectral parameter?

✦ Local $O(d,d)$ deformations…

☆ Doubled formalism based on the Abelian isometries

→ Any extension?

[I'm just saying] (What would be a collective T-duality invariant framework…(if any) ?)

Thank you!

Ευχαριστώ