Generalization of AdS/CFT Correspondence

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Workshop on
Quantum Geometry, Field Theory and Gravity

Contents

Brief review on AdS/CFT correspondence

$\mathcal{N}=4$ supersymmetric $U(N)$ Yang-Mills

A. Tsuchiya's talk

Conformal invariance: $SO(4,2)$

Gauge multiplet: $(A_\mu, \Phi_a,\ \lambda^i)$, $\quad a=1.\cdots,6,\quad i=1,\cdots,4$

Global R-symmetry: $SO(6)\cong SU(4)$

Global symmetry of $\mathcal{N}=4$: $SO(4,2)\times SO(6)$

Isometry of vacuum geometry: $AdS_5\times\mathbb{S}^5$

$SO(4,2)$: 15 generators

Poincaré = $(P_\mu, L_{\mu\nu})$: 10=4+6

Special conformal = K_μ: 4

Dilatation = D: 1

$SO(6)$: 15 generators

Rotations = M_{ab}: 15

$\mathcal{N} = 4$ Super-Yang-Mills for Asymptotically Flat Spacetimes

We want: $AdS_5 \times \mathbb{S}^5$ ⟹ Asymptotically flat spacetimes,
e.g., $\mathbb{R}^{3,1} \times \mathbb{R}^6$ or $\mathbb{R}^{3,1} \times CY_3$

Isometry of vacuum geometry: $\mathbb{R}^{3,1} \times \mathbb{R}^6$

⇕

Global symmetry of $\mathcal{N} = 4$: $ISO(3,1) \times ISO(6)$

$SO(4,2)$: 15 generators

Poincaré = $(P_\mu, L_{\mu\nu})$: 10=4+6

~~Special conformal = K_μ: 4~~

~~Dilatation = D: 1~~

$SO(6)$: 15 generators

Rotations = M_{ab}: 15

Find a vacuum to break the special conformal and dilatation symmetries only, but to preserve the Poincaré and $ISO(6)$ symmetries.

Coulomb branch of $\mathcal{N} = 4$ super Yang-Mills

Consider the limit $N \to \infty$: (Aoki, et al, '99)

Coulomb branch $\begin{cases} \text{Commutative vacuum: } [\Phi_a, \Phi_b]|_{vac} = 0 \quad \Longrightarrow \quad \langle \Phi_a \rangle_{vac} = \mathrm{diag}(\alpha_{a1}, \cdots, \alpha_{aN}) \\ \text{Noncommutative vacuum: } [\Phi_a, \Phi_b]|_{vac} = -iB_{ab}\, I_{N\times N} \;\Longrightarrow\; \langle \Phi_a \rangle_{vac} = p_a = B_{ab} y^b \end{cases}$

$[B_{\mathrm{ab}}] = \alpha'^{-1}\, \mathrm{I}_{3\times 3} \times i\sigma^2$: 6×6 symplectic matrix

y^a satisfy the Heisenberg-Moyal algebra: $[y^a, y^b] = i\, \theta^{ab} I_{N\times N}$ where $\theta \equiv B^{-1}$

We assign the mass dimension: $[\Phi_a] = M$, $[y^a] = M^{-1}$, $[B_{ab}] = M^2$, $[\theta^{ab}] = M^{-2}$

$U(N)$ gauge symmetry is broken to a subgroup H or NC $U(1)_\star$.

The second vacuum will be called the NC Coulomb branch.
Note that the Moyal-Heisenberg vacuum saves the NC nature of matrices while the conventional vacuum dismisses the property.

D=10 $\mathcal{N}=1$ NC $U(1)$ Gauge Theory from $\mathcal{N}=4$ Super-Yang-Mills

Suppose that fluctuations around the NC Coulomb branch take the form

$$D_\mu = \partial_\mu - iA_\mu(x,y), \qquad \Phi_a = p_a + A_a\,(x,y) \in \mathcal{A}_\theta\ , \qquad \Psi(x,y) = \begin{pmatrix} P_+\lambda^i \\ P_-\tilde{\lambda}_i \end{pmatrix}(x,y)$$

The above adjoint scalar fields now obey the deformed algebra given by

$$[\Phi_a, \Phi_b] = [p_a + A_a\,(x,y),\ p_b + A_b\,(x,y)] = -i\,(B_{ab} - F_{ab}(x,y)),\ \ F_{ab} = \partial_a A_b - \partial_b A_a - i[A_a, A_b],$$

$$D_\mu \Phi_a = \left[\partial_\mu - iA_\mu(x,y),\ p_a + A_a\,(x,y)\right] = \partial_\mu A_a - \partial_a A_\mu - i\left[A_\mu, A_a\right] = F_{\mu a}(x,y),$$

$$F_{\mu\nu}(x,y) = i\left[D_\mu,\ D_\nu\right] = i[\partial_\mu - iA_\mu(x,y), \partial_\nu - iA_\nu(x,y)],$$

with the definition $\partial_a \equiv -i\ ad_{p_a} = -i[p_a,\ \cdot\ ]$.

Similarly, for $\left[D_\mu, \lambda^i\right]$, $\left[\Phi_a, \lambda^i\right]$ and $\left[\Phi_a, \bar{\lambda}_i\right]$.

D=10 $\mathcal{N}=1$ NC $U(1)$ Gauge Theory from $\mathcal{N}=4$ Super-Yang-Mills

Plugging the fluctuations into the four-dimensional $U(N\to\infty)$ super Yang-Mills theory, we get the ten-dimensional supersymmetric NC $U(1)$ gauge theory with the action

$$S=\int d^4x\,\mathrm{Tr}\Big\{-\frac{1}{4}F_{\mu\nu}F^{\mu\nu}-\frac{1}{2}D_\mu\Phi_a D^\mu\Phi_a+\frac{g^2}{4}[\Phi_a,\Phi_b]^2$$
$$+i\bar\lambda_i\bar\sigma^\mu D_\mu\lambda^i+\frac{i}{2}g\bar\Sigma^a_{ij}\lambda^i[\Phi_a,\lambda^j]-\frac{i}{2}g\Sigma^{a,ij}\bar\lambda_i[\Phi^a,\bar\lambda_j]\Big\}$$

$$\Updownarrow$$

$$S=\int d^{10}X\Big\{-\frac{1}{4G^2_{\mathrm{YM}}}(F_{MN}-B_{MN})^2+\frac{i}{2}\bar\Psi\Gamma^M D_M\Psi\Big\}$$

where $B_{MN}=\begin{pmatrix}0&0\\0&B_{ab}\end{pmatrix}$ and $G^2_{YM}=(2\pi)^3|\mathrm{Pf}\,\theta|g^2$ (HSY, '09, '13, '14)

D=10 $\mathcal{N}=1$ NC $U(1)$ Gauge Theory from $\mathcal{N}=4$ Super-Yang-Mills

The action is invariant under $\mathcal{N}=1$ supersymmetry transformations given by

$$\delta A_M = i\bar{\alpha}\Gamma_M \Psi\,, \quad \delta\Psi = \frac{1}{2}(F_{MN} - B_{MN})\Gamma^{MN}\alpha\,.$$

We want to emphasize that the relationship between the four-dimensional $U(N)$ super-Yang–Mills theory and ten-dimensional NC $U(1)$ super-Yang–Mills theory in the NC Coulomb branch is not a dimensional reduction but they are exactly equivalent to each other.
Therefore any quantity in lower-dimensional $U(N)$ gauge theory can be transformed into an object in higher-dimensional NC $U(1)$ gauge theory using the compatible ordering.

For example, a Wilson loop in $U(N)$ gauge theory (Ishibashi, et al, '99, Ambjorn, et. al, '99, …..)

$$W_N = \frac{1}{N}\,\mathrm{Tr}\,P\exp\left(i\oint (A_\mu \dot{x}^\mu + \Phi_a \dot{y}^a)ds\right)$$

can be translated into a corresponding NC $U(1)$ Wilson "line" defined by

$$\hat{W} = \frac{1}{V_6}\int d^6y\, P_\star \exp\left(i\int_\Gamma (B_{ab}\dot{y}^a y^b + A_M \dot{x}^M)ds\right)$$

where V_6 is a volume of extra six-dimensional space.

D=10 $\mathcal{N}=1$ NC $U(1)$ Gauge Theory from $\mathcal{N}=4$ Super-Yang-Mills

This can be confirmed by using the fact well-known from quantum mechanics:

The map $\mathcal{A}_\theta^4 \to \mathcal{A}_N^4 \equiv \mathcal{A}_N\left(C^\infty(\mathbb{R}^{3,1})\right) = C^\infty(\mathbb{R}^{3,1}) \otimes \mathcal{A}_N$ is a Lie algebra homomorphism where N =dim $(\mathcal{H}) \to \infty$:

$$\sum_{n,m=0}^{\infty} |n\rangle\langle n| f(x,y)\, |m\rangle\langle m| = \sum_{n,m=0}^{\infty} f_{nm}(x) |n\rangle\, \langle m|$$

for $f(x,y) \in \mathcal{A}_\theta^4$ and $[f(x)]_{nm} \in \mathcal{A}_N^4$.

Via the matrix representation, one can recover the $\mathcal{N}=4$ supersymmetric $U(N)$ Yang-Mills theory from D=10 $\mathcal{N}=1$ NC $U(1)$ gauge theory.

Now we will show that $\mathbb{R}^{3,1} \times CY_3$ is a ten-dimensional geometry emergent from D=10 $\mathcal{N}=1$ NC $U(1)$ gauge theory. (HSY, ‘14)

Asymptotically Flat Spacetimes from D=10 $\mathcal{N}=1$ NC $U(1)$ Gauge Theory

Consider a Kähler manifold $(\boldsymbol{M}, \boldsymbol{g})$ where $ds^2 = g_{i\bar{j}}(z,\bar{z})dz^i d\bar{z}^{\bar{j}}, \quad i,\bar{j} = 1,\cdots,n$

and $g_{i\bar{j}}(z,\bar{z}) = \frac{\partial^2 K(z,\bar{z})}{\partial z^i \partial \bar{z}^{\bar{j}}}$. The real function $K(z,\bar{z})$ is callled Kähler potential.

Given a Kähler metric, one can introduce a fundamental two-form defined by

$$\Omega = \sqrt{-1}\, g_{i\bar{j}}(z,\bar{z}) dz^i \wedge d\bar{z}^{\bar{j}},$$

which is a nondegenerate, closed 2-form, $\mathrm{d}\Omega = 0$. So the Kähler form is a symplectic 2-form. That means the Kähler manifold $(\boldsymbol{M}, \boldsymbol{g})$ is a symplectic manifold $(\boldsymbol{M}, \boldsymbol{\Omega})$ too although the reverse is not necessarily true.

The Kähler potential is not unique but admits a Kähler transformation

$$K(z,\bar{z}) \longrightarrow K(z,\bar{z}) + f(z) + \bar{f}(\bar{z}).$$

Note that the Kähler form can be written as $\Omega = \mathrm{d}\mathcal{A}$ where $\mathcal{A} = \frac{\sqrt{-1}}{2}(\partial - \bar{\partial})K(z,\bar{z})$

and $\partial = dz^i \frac{\partial}{\partial z^i}$, $\bar{\partial} = d\bar{z}^{\bar{i}} \frac{\partial}{\partial \bar{z}^{\bar{i}}}$, $d = \partial + \bar{\partial}$. Then the Kähler transformation corresponds to a gauge transformation for the 1-form $\mathcal{A}$ given by

$$\mathcal{A} \longrightarrow \mathcal{A} + d\lambda, \quad \text{where} \quad \lambda = \frac{\sqrt{-1}}{2}(\bar{f}(\bar{z}) - f(z)).$$

This implies that the 1-form $\mathcal{A}$ corresponds to $U(1)$ gauge fields.

Kähler Geometry As A $U(1)$ Gauge Theory

Kähler geometry corresponds to a dynamical symplectic geometry and is locally described by

$$\mathfrak{B} = (N = \cup_\alpha U_\alpha\, ,\ \mathcal{F}_\alpha = B + F_\alpha). \qquad \text{(Griffiths \& Harris, 107 pp)}$$

In this picture, the dynamical $U(1)$ gauge fields defined on a symplectic manifold (N, B) manifest themselves as local deformations of the symplectic or Kähler structure.
~~This is the analog of the picture $\mathfrak{B}$ for the Lorentz force.~~

What is the gauge theory description for gravity?

Find a local coordinate transformation $\varphi_\alpha \in \mathrm{Diff}(U_\alpha)$: $y^\mu \mapsto x^a(y)$, such that

$$\varphi_\alpha^*(B + F_\alpha) = B \quad \Leftrightarrow \quad \varphi_\alpha^*(\delta + h_\alpha) = \delta$$

Thus the picture essentially states the equivalence principle in general relativity.
In terms of local coordinates, $x^a(y) \equiv \theta^{ab}\phi_b(y) = \theta^{ab}\,(p_b + a_b(y))$

$$\left(B_{ab} + F_{ab}(x)\right)\frac{\partial x^a}{\partial y^\mu}\frac{\partial x^b}{\partial y^\nu} = B_{\mu\nu} \quad \Leftrightarrow \quad \Theta^{ab}(x) \equiv \left(\frac{1}{B+F(x)}\right)^{ab} = \{x^a(y), x^b(y)\} = \left(\theta\left(-B + f(y)\right)\theta\right)^{ab}$$

where $\theta \equiv B^{-1} \in \Gamma(\Lambda^2 TM)$ is a Poisson bivector and
$f_{ab}(y) = \partial_a a_b - \partial_b a_a + \{a_a, a_b\}$ is the field strength of *symplectic* $U(1)$ gauge fields $a_b(y)$.

Duality Between K$\ddot{a}$hler Geometry and $U(1)$ Gauge Theory

What is the relation between gauge theory and gravity ? (J. Lee & HSY, ‘18)

(A. Iqbal, C. Vafa, N. Nekrasov and A. Okounkov, hep-th/0312022,
D. Maulik, N. Nekrasov, A. Okounkov and R. Pandharipande, math.AG/0312059)

$$\begin{tikzcd}
\text{Kähler gravity} \arrow[r, "\mathfrak{J}_\varepsilon^{-1}"] \arrow[d, "\mathcal{Q}"'] & \text{Symplectic } U(1) \text{ gauge theory} \arrow[d, "\mathcal{Q}"] \\
\text{Quantized Kähler gravity} & \text{NC } U(1) \text{ gauge theory} \arrow[l, "\mathfrak{J}_\theta"']
\end{tikzcd}$$

$$\begin{tikzcd}
\text{Calabi -- Yau manifold} \arrow[r, "\mathfrak{J}_\varepsilon^{-1}"] \arrow[d, "\mathcal{Q}"'] & \text{Symplectic } U(1) \text{ instanton} \arrow[d, "\mathcal{Q}"] \\
\text{Quantized Calabi -- Yau manifold} & \text{NC } U(1) \text{ instanton} \arrow[l, "\mathfrak{J}_\theta"']
\end{tikzcd}$$

Here $\mathcal{J}$ means an isomorphism between two theories.
In some sense $\mathcal{J}$ corresponds to the gauge-gravity duality.
It turns out that it can be interpreted as the large N duality too.

Gauge-Gravity Duality for Asymptotically Flat Spacetimes

For any dynamical variable, e.g. $\Phi_a = p_a + \hat{A}_a(y) \in \mathcal{A}_\theta$, we can associate a differential operator, the so-called polyvector fields in $\mathcal{D}_\theta$, by the adjoint map

$$\mathcal{A}_\theta \ \to \mathcal{D}_\theta : \Phi_a(y) \mapsto ad_{\Phi_a} = [\Phi_a(y), \ \cdot \ ] \equiv \hat{V}_a.$$

The adjoint map $\mathcal{A}_\theta \ \to \mathcal{D}_\theta$ is also a Lie algebra homomorphism. For example, using the commutation relation

$$[\Phi_a, \Phi_b] = -i\left(B_{ab} - \hat{F}_{ab}\right), \qquad \hat{F}_{ab} = \partial_a \hat{A}_b - \partial_b \hat{A}_a - i\left[\hat{A}_a, \hat{A}_b\right],$$

we get the relation

$$\hat{V}_{\hat{F}_{ab}} = [\hat{V}_a, \hat{V}_b] \in \mathcal{D}_\theta \ .$$

The generalized vector fields $\hat{V}_a$ take the following form

$$\hat{V}_a = V_a^\mu \frac{\partial}{\partial y^\mu} + \sum_{p=2}^{\infty} V_a^{\mu_1 \cdots \mu_p} \frac{\partial}{\partial y^{\mu_1}} \cdots \frac{\partial}{\partial y^{\mu_p}}.$$

Let us truncate the above polyvector fields to ordinary vector fields given by

$$\mathcal{X}(M) = \left\{V_a = V_a^\mu(y) \frac{\partial}{\partial y^\mu} | a, \mu = 1, \cdots, 6 \right\}.$$

Gauge-Gravity Duality for Asymptotically Flat Spacetimes

The orthonormal vielbeins on TM are obtained by the prescription

$$V_A = \lambda\, E_A \in \Gamma(TM) \quad \text{or} \quad e^A = \lambda\, v^A \in \Gamma(T^*M).$$

The conformal factor $\lambda \in C^\infty(M)$ is determined by the volume-preserving condition

$$\mathcal{L}_{V_A} \mathcal{V}_t = (\nabla \cdot V_A - 8\, V_A \ln \lambda) = 0 \;\; \text{with} \;\; \mathcal{V}_t = \lambda^2 d^4x \wedge v^1 \wedge \cdots \wedge v^6.$$

If the structure equation of vector fields $V_A \in \Gamma(TM)$ is defined by

$$[V_A, V_B] = -g_{AB}{}^C\, V_C\,,$$

the volume-preserving condition can be written as

$$g_{BA}{}^B = V_A \ln \lambda^2.$$

In the end, the Riemannian metric on a 10-dimensional emergent spacetime manifold M is given by

$$\begin{aligned} ds^2 &= \mathcal{G}_{MN}(X) dX^M \otimes dX^N = e^A \otimes e^A \\ &= \lambda^2 v^A \otimes v^A = \lambda^2 \big(\eta_{\mu\nu} dx^\mu dx^\nu + v^a_b v^a_c (dy^b - \mathbf{A}^b)(dy^c - \mathbf{A}^c) \big) \end{aligned}$$

Vacua of $\mathcal{N} = 4$ super Yang-Mills theory in the NC Coulomb branch

The vacua of $\mathcal{N} = 4$ super Yang-Mills theory in the NC Coulomb branch are characterized by the BPS equation given by

$$\delta\Psi = \frac{1}{2} F_{MN} \Gamma^{MN} \alpha \, .$$

For simplicity, we set $A_\mu(x, y) = 0$ and assume that NC U(1) gauge fields $A_a(x, y)$, $a = 1, \cdots, 6$, along extra dimensions depend only on NC coordinates y_a, i.e. $A_a(y)$.

The solution of the BPS equation (3) is known as Hermitian Yang-Mills instantons obeying

$$\hat{F}_{ab} = -\frac{1}{2} T_{ab}{}^{cd} \hat{F}_{cd}$$

where $T_{ab}{}^{cd} = \frac{1}{2} \varepsilon_{ab}{}^{cdef} I_{ef}$. The self-duality equations (known as Donaldson-Uhlenbeck-Yau equations) on $\mathbb{C}^3$ are given by (HSY, '14)

$$\hat{F}_{ij} = \hat{F}_{\bar{\imath}\bar{\jmath}} = 0, \qquad \qquad \sum_{i=1}^{3} \hat{F}_{i\bar{\imath}} = 0.$$

$$\Big\downarrow \qquad \qquad \qquad \qquad \Big\downarrow$$

Kähler condition: $\omega_{ij} = \omega_{\bar{\imath}\bar{\jmath}}$ Ricci-flat condition: $\sum_{i=1}^{3} \omega_{i\bar{\imath}} = 0$

Hermitian $U(1)$ instantons $\cong$ Calabi-Yau manifolds

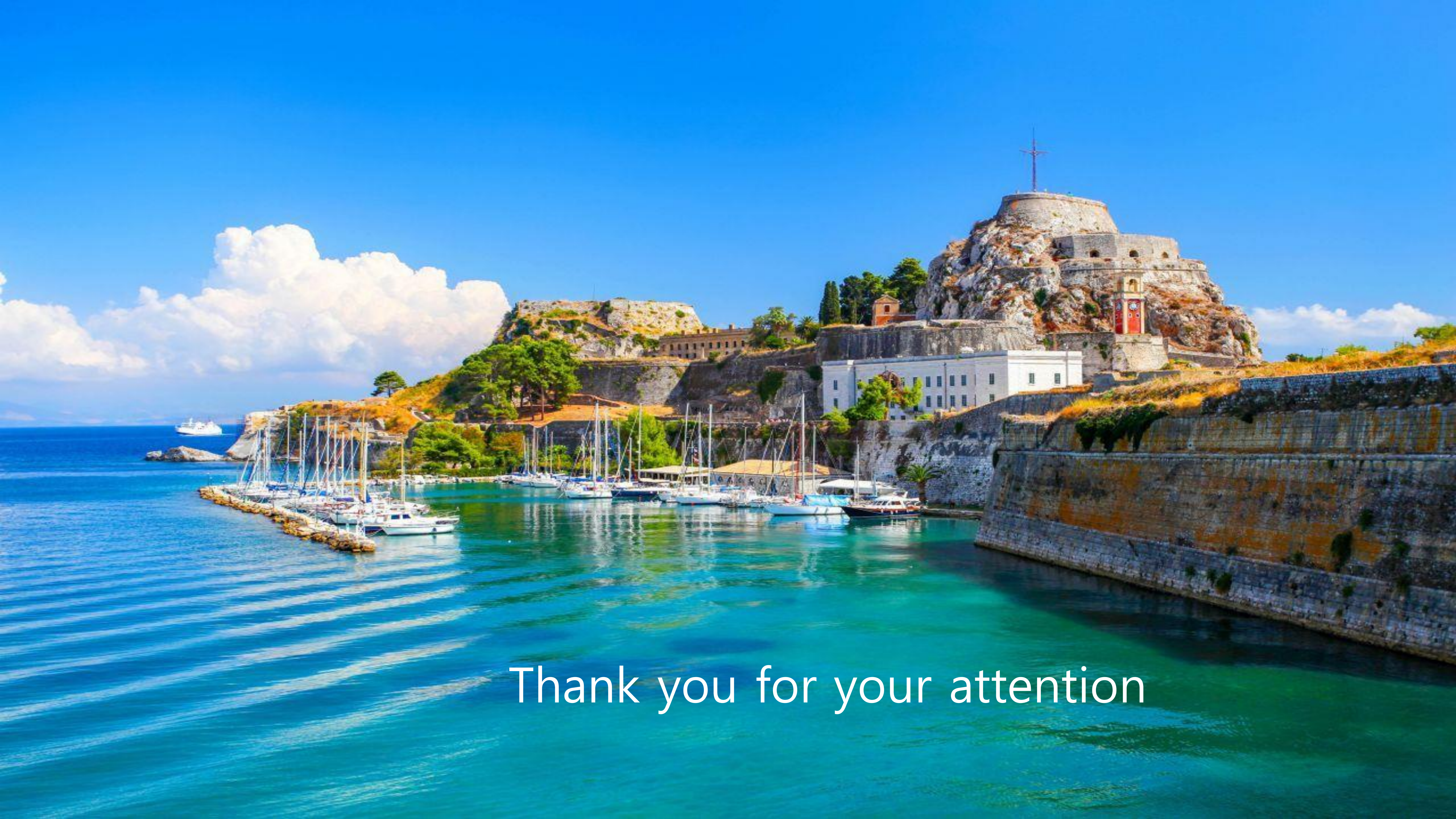
Summary and Speculation

☞ I emphasize that the AdS/CFT correspondence is a particular example of emergent gravity from a large N gauge theory in the noncommutative (NC) Coulomb branch.

☞ We showed that the four-dimensional $\mathcal{N} = 4$ supersymmetric large N gauge theory in the NC Coulomb branch is isomorphically mapped to the ten-dimensional $\mathcal{N} = 1$ supersymmetric NC $U(1)$ gauge theory.

☞ Wild speculation 1: The NC Coulomb branch admits infinitely degenerate vacua whose dual geometries interpolate from $\mathbb{R}^{9,1}$ to $AdS_5 \times \mathbb{S}^5$.

☞ Wild speculation 2: Emergent gravity from NC gauge theory generalizes the AdS/CFT correspondence to asymptotically flat spacetimes as well as AdS spacetimes.


Thank you for your attention