Non-commutative deformations of gauge theories and L_∞ algebras

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Based on arXiv: 1803.00732, 1806.10314, 1903.02867, 1905.08753
in collaboration with Munich String Theory Group

September 17, 2019

Outline of the talk:

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2. L_∞ algebras
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   - Relation to deformations
3. L_∞ bootstrap
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4. NC deformation of the abelian Chern-Simons
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   - NC field strength; dynamics and the action principle
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NC gauge theories with $\Theta^{ab}(x)$

The star product is given by

$$f \star g = f \cdot g + \frac{i}{2}\Theta^{ij}\,\partial_i f\,\partial_j g - \frac{1}{8}\Theta^{ij}\Theta^{kl}\,\partial_i\partial_k f\,\partial_j\partial_l g$$
$$-\frac{1}{12}\left(\Theta^{im}\partial_m\Theta^{jk}\right)\left(\partial_i\partial_j f\,\partial_k g + \partial_i\partial_j g\,\partial_k f\right) + \mathcal{O}(\Theta^3)\,.$$

The standard Leibniz rule is violated,

$$\partial_a(f \star g) = \partial_a f \star g + f \star \partial_a g + \frac{i}{2}(\partial_a\Theta^{ij})\partial_i f \partial_j g + \mathcal{O}(\Theta^2)\,.$$

So, if to substitute all point-wise products with a star products in the action, the theory will not be gauge invariant.

Possible solution: [Madore et al' 00], taking, $D_a = c[x_a, \cdot]_\star$, one gets $D_a[f,g]_\star = [D_a f, g]_\star + [f, D_a g]_\star$, as a consequence of, $[x_a,[f,g]_\star]_\star + [f,[g,x_a]_\star]_\star + [g,[x_a,f]_\star]_\star \equiv 0$.
However, the commutative limit is not well defined.

The statement of the problem

We are looking for the non-commutative theory satisfying the following two properties:

1. Gauge invariance
2. Correct commutative limit

In fact, non-commutative gauge theory should be a deformation in Θ, $\partial\Theta$, $\partial\partial\Theta$, etc., of well defined commutative theory.

To this end we employ the framework of L_∞ algebras which are good for both description of gauge theories and deformations.

For simplicity in this talk we discuss only the associative deformations, when the star commutator satisfies the Jacobi identity.

Definition of L_∞ in ℓ-picture

- is a graded vector space:

$$\mathbf{L} = \bigoplus_n L_n = \cdots \oplus \underbrace{L_{-3}}_{\text{Noether}} \oplus \underbrace{L_{-2}}_{\text{eom}} \oplus \underbrace{L_{-1}}_{\text{fields}} \oplus \underbrace{L_0}_{\text{gauge}} \oplus \underbrace{L_1}_{\text{ghosts}} \oplus \ldots$$

- endowed with multi-linear maps: $\ell_n(v_1, \ldots, v_n)$, such that,

$$\ell_n(v_1, \ldots, v_n) \in L_{n-2+\sum_{i=1}^n \deg(v_i)}\,,$$

- which are graded anti-symmetric,

$$\ell_n(\ldots, v_1, v_2, \ldots) = (-1)^{1+\deg(v_1)\deg(v_2)}\, \ell_n(\ldots, v_2, v_1, \ldots)\,,$$

- and satisfy the relations (generalized Jacobi identities):

$$\mathcal{J}_n(v_1, \ldots, v_n) := \sum_{i+j=n+1} (-1)^{i(j-1)} \sum_\sigma (-1)^\sigma\, \chi(\sigma; v)\, \ell_j\big(\, \ell_i(v_{\sigma(1)}\,, \ldots, v_{\sigma(i)})\,, v_{\sigma(i+1)}, \ldots, v_{\sigma(n)}\,\big) = 0\,.$$

Strong homotopy algebras [Lada, Stasheff' 92]

The first L_∞ relations read

$$\mathcal{J}_1 := \ell_1\big(\,\ell_1(v)\,\big) = 0\,,$$
$$\mathcal{J}_2 := \ell_1\big(\,\ell_2(v_1, v_2)\,\big) - \ell_2\big(\,\ell_1(v_1), v_2\,\big) - (-1)^{v_1}\ell_2\big(\,v_1, \ell_1(v_2)\,\big) = 0\,,$$

meaning that ℓ_1 is a nilpotent derivation with respect to ℓ_2.

$$\begin{aligned}\mathcal{J}_3 := {} & \ell_1\big(\ell_3(v_1, v_2, v_3)\big) + \ell_3\big(\ell_1(v_1), v_2, v_3\,\big) \\ & +(-1)^{v_1}\ell_3\big(v_1, \ell_1(v_2), v_3\,\big) + (-1)^{v_1+v_2}\ell_3\big(v_1, v_2, \ell_1(v_3)\big) \\ & +\ell_2\big(\ell_2(v_1, v_2), v_3\,\big) + (-1)^{(v_2+v_3)v_1}\ell_2\big(\ell_2(v_2, v_3), v_1\,\big) \\ & +(-1)^{(v_1+v_2)v_3}\ell_2\big(\ell_2(v_3, v_1), v_2\,\big) = 0\,,\end{aligned}$$

the Jacobi identity for ℓ_2 is required to hold only up to ℓ_1 exact (total derivative) terms.

L_∞ algebras are natural to deal with deformations. The proof of the Formality Theorem by Kontsevich is based on the notion of L_∞.

Example: Lie Algebra

Suppose, $\mathbf{L} = L_0 = V$. For any $v \in V$, $\deg(v) = 0$. Since,

$$\deg\big(\ell_n(v_1, \dots, v_n)\big) = n - 2 + \sum_{i=1}^{n} \deg(v_i),$$

the only non-vanishing bracket is $\ell_2 : V \times V \to V$, which is antisymmetric,

$$\ell_2(v_1, v_2) = -\,\ell_2(v_2, v_1)\,,$$

and satisfies the standard Jacobi identity,

$$\ell_2\big(\ell_2(v_1, v_2), v_3\,\big) + \ell_2\big(\ell_2(v_2, v_3), v_1\,\big) + \ell_2\big(\ell_2(v_3, v_1), v_2\,\big) = 0\,.$$

Thus, L_∞ algebra concentrated in L_0 defines a Lie algebra.

Relation to gauge transformations, $\mathbf{L} = L_{-1} \oplus L_0$

$L_{-1} = \{A_a\}$ - classical fields and $L_0 = \{f\}$ - gauge parameters.
Since, $\deg(\ell_n) = n - 2$, the only non-vanishing

$$\ell_{n+1}(f, A^n) \in L_{-1} \qquad \text{and} \qquad \ell_{n+2}(f, g, A^n) \in L_0 \,.$$

Gauge variations are given by:

$$\delta_f A \;= \sum\nolimits_{n\geq 0} \tfrac{1}{n!}(-1)^{\frac{n(n-1)}{2}}\, \ell_{n+1}(f, \underbrace{A, \dots, A}_{n \text{ times}}) = \ell_1(f) + \ell_2(f, A) + \dots \,.$$

L_∞ relations, $\mathcal{J}_{n+2}(f, g, A^n) = 0$, imply the closure condition,

$$[\delta_f, \delta_g]A = \delta_{-C(f,g,A)} A \,,$$

$$C(f, g, A) = \sum_{n\geq 0} \frac{1}{n!}(-1)^{\frac{n(n-1)}{2}}\, \ell_{n+2}(f, g, \underbrace{A, \dots, A}_{n \text{ times}}) = \underbrace{\ell_2(f, g)}_{[f,g]} + \dots \,,$$

field dependent gauge parameters [Berends, Burgers, van Dam' 85].

$$\mathcal{J}_{n+3}(f, g, h, A^n) = 0, \qquad \Rightarrow \qquad \sum_{\text{cycl}} \left[\delta_f, [\delta_g, \delta_h]\right] \equiv 0 \,.$$

Field theory and L_∞^{full} algebra [Hohm, Zwiebach' 17]

Consider nonempty, $L_{-2} = \{\mathcal{F}\}$, containing lhs of eom, $\mathcal{F}_a(A) = 0$,

$$\begin{array}{ccc} L_{-2} & L_{-1} & L_0 \\ \mathcal{F}_a & A_a & f \end{array}$$

Additional non-vanishing brackets

$$\ell_n(A^n) \in L_{-2} \qquad \text{and} \qquad \ell_{n+2}(f, E, A^n) \in L_{-2}\,,$$

The equations of motion are determined as

$$\mathcal{F} := \sum_{n\geq 1} \frac{1}{n!}(-1)^{\frac{n(n-1)}{2}}\,\ell_n(A^n) = \ell_1(A) - \frac{1}{2}\ell_2(A^2) + \cdots = 0\,.$$

The L_∞ relations $\mathcal{J}_{n+1}(f, A^n) = 0$, and $\mathcal{J}_{n+2}(f, E, A^n) = 0$, imply

$$\delta_f \mathcal{F} = \ell_2(f, \mathcal{F}) + \ell_3(f, \mathcal{F}, A) - \frac{1}{2}\ell_4(f, \mathcal{F}, A^2) + \ldots\,,$$

the eom are gauge covariant (invariant on-shell, $\mathcal{F} = 0$).

Field theory and L_∞^{full} algebra

Example: abelian Chern-Simons. Consider the only non-vanishing,

$$\ell_1 : L_0 \to L_1 \,, \qquad \text{with} \qquad \ell_1(f) = \partial_a f \,,$$
$$\ell_1 : L_{-1} \to L_{-2} \,, \qquad \text{with} \qquad \ell_1(A) = \varepsilon^{abc}\partial_b A_c \,.$$

The only L_∞ relation to check: $\ell_1(\ell_1(f)) = \varepsilon^{abc}\partial_b\partial_c f \equiv 0$.

According to the above formulas,

$$\delta_f A_a = \ell_1(f) = \partial_a f \,, \qquad \text{and} \qquad \mathcal{F}^a := \ell_1(A) = \varepsilon^{abc}\partial_b A_c = 0 \,.$$

For the abelian Yang-Mills, $\ell_1(A)^a = \Box A^a - \partial^a(\partial \cdot A)$.
For non-abelian theories one set, $\ell_2(f,g) = [f,g]$, etc.

Massage: L_∞ algebra determines gauge theory and vise versa [Hohm, Zwiebach' 17].

L_∞ bootstrap, arXiv: 1803.00732

Undeformed theory is determined through the given brackets,

$$\ell_1(f) \in L_{-1}\,, \qquad \text{and} \qquad \ell_1(A) \in L_{-2}\,, \qquad \text{with} \qquad \ell_1(\ell_1(f)) \equiv 0\,.$$

The deformation is introduced by setting,

$$\ell_2(f,g) = i[f,g]_\star \in L_0\,.$$

The L_∞ relation, $\mathcal{J}_2(f,g)=0$, becomes an equation on $\ell_2(f,A)$,

$$\ell_1(\ell_2(f,g)) = \ell_2(\overbrace{\ell_1(f)}^{\in L_{-1}},g) + \ell_2(f,\overbrace{\ell_1(g)}^{\in L_{-1}})\,.$$

Then from, $\mathcal{J}_2(f,A)=0$, one finds $\ell_2(A,A)$ and $\ell_2(f,\mathcal{F})$;
After that from, $\mathcal{J}_3(f,g,h)=0$, defines $\ell_3(f,g,A)$, etc.

Gauge variations and field equations are given as before by

$$\begin{aligned} \delta_f A =& \quad \ell_1(f) + \ell_2(f,A) + \ldots\,, \\ \mathcal{F} :=& \quad \ell_1(A) - \tfrac{1}{2}\ell_2(A^2) + \cdots = 0\,. \end{aligned}$$

Solving L_∞ bootstrap equations

Since, $\ell_2(f,g) = -\{f,g\} + \mathcal{O}(\Theta^3)$, and $\ell_1(f) = \partial_a f$, the first relation is:

$$\begin{aligned}\ell_1(\ell_2(f,g)) &= -\{\overbrace{\ell_1(f)}^{\in L_{-1}}, g\} - \{f, \overbrace{\ell_1(g)}^{\in L_{-1}}\} - (\partial_a \Theta^{ij})\, \partial_i f \partial_j g + \mathcal{O}(\Theta^3)\,, \\ &= \ell_2(\ell_1(f), g) + \ell_2(f, \ell_1(g))\,.\end{aligned}$$

which implies that

$$\ell_2(f,A) = i[f, A_a]_\star - \frac{1}{2}(\partial_a \Theta^{ij})\, \partial_i f\, A_j + \mathcal{O}(\Theta^3)\,.$$

- Note that the solution is not unique, one may also set, e.g.,

$$\ell_2'(f,A) = \ell_2(f,A) + s_a^{ij}(x)\, \partial_i f\, A_j\,, \qquad s_a^{ij}(x) = s_a^{ji}(x)\,.$$

However, the symmetric part $s_a^{ij}(x)\, \partial_i f\, A_j$ can be always "gauged away" by L_∞-QISO, physically equivalent to SW map, for more details see arXiv:1806.10314.

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which implies that

$$\ell_2(f,A) = i[f, A_a]_\star - \frac{1}{2}(\partial_a \Theta^{ij})\,\partial_i f\, A_j + \mathcal{O}(\Theta^3)\,.$$

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$\mathsf{L}_\infty^{\mathrm{gauge}}$ algebra

Then, we have to analyze $\mathcal{J}_3(f,g,A)=0$, given by

$$\begin{aligned} 0 &= \ell_2(\ell_2(A,f),g)+\ell_2(\ell_2(f,g),A)+\ell_2(\ell_2(g,A),f) \\ &\qquad -\ell_3(A,\ell_1(f),g)-\ell_3(A,f,\ell_1(g)). \end{aligned}$$

We replace it with $\mathcal{J}_3(g,h,\ell_1(f))=0$, written in the form

$$\begin{aligned} &\ell_3(\ell_1(f),\ell_1(g),h)-\ell_3(\ell_1(f),\ell_1(h),g)=G(f,g,h)\,, \\ &G(f,g,h):= \\ &\ell_2(\ell_2(\ell_1(f),g),h)+\ell_2(\ell_2(g,h),\ell_1(f))+\ell_2(\ell_2(h,\ell_1(f)),g)\,. \end{aligned}$$

By construction, $G(f,g,h)=-G(g,f,h)$. The graded symmetry of $\ell_3(\ell_1(f),\ell_1(g),h)$ implies the graded cyclicity (consistency condition) of $G(f,g,h)$:

$$G(f,g,h)+G(h,f,g)+G(g,h,f)=0\,.$$

Below we show that it holds true as a consequence of the previously solved "Jacobi identity", $\mathcal{J}_2(f,g)=0$.

L_∞^{gauge} algebra

$$
\begin{aligned}
&G(f,g,h)+G(h,f,g)+G(g,h,f)= \\
&\ell_2(\ell_2(\ell_1(h),f),g)+\ell_2(\ell_2(f,g),\ell_1(h))+\ell_2(\ell_2(g,\ell_1(h)),f)+ \\
&\ell_2(\ell_2(\ell_1(g),h),f)+\ell_2(\ell_2(h,f),\ell_1(g))+\ell_2(\ell_2(f,\ell_1(g)),h)+ \\
&\ell_2(\ell_2(\ell_1(f),g),h)+\ell_2(\ell_2(g,h),\ell_1(f))+\ell_2(\ell_2(h,\ell_1(f)),g)\,.
\end{aligned}
$$

Using, $\mathcal{J}_2(f,g)=0$, we rewrite it as

$$
\ell_1\Big[\ell_2(\ell_2(f,g),h)+\ell_2(\ell_2(g,h),f)+\ell_2(\ell_2(h,f),g)\Big]\equiv 0\,.
$$

Thus, the combination (symmetrization in f and g):

$$
\ell_3(\ell_1(f),\ell_1(g),h)=-\frac{1}{6}\Big(G(f,g,h)+G(g,f,h)\Big)\,,
$$

has required graded symmetry and solves, $\mathcal{J}_3(g,h,\ell_1(f))=0$.

L_∞^{gauge} algebra

Setting

$$\ell_3(A,B,h) = \ell_3(\ell_1(f),\ell_1(g),h)\big|_{\ell_1(f)=A;\,\ell_1(g)=B}\ ,$$

one gets in the leading order,

$$\ell_3(A,B,f) = -\frac{1}{6}\Big({G_a}^{ijk} + {G_a}^{jik}\Big)A_iB_j\partial_kf + \mathcal{O}(\Theta^3)\,.$$

with

$${G_a}^{ijk} = \Theta^{im}\partial_m\partial_a\Theta^{jk} - \frac{1}{2}\partial_a\Theta^{jm}\partial_m\Theta^{ki} - \frac{1}{2}\partial_a\Theta^{km}\partial_m\Theta^{ij}\,.$$

- The consistency condition (graded cyclicity) holds true as a consequence of L_∞ construction.
- Even in the associative case one needs higher brackets to compensate the violation of the Leibnitz rule.

Recurrense relations for $L_\infty^{\mathrm{gauge}}$ algebra

For, $\mathcal{J}_{n+2}(g, h, A^n) = 0$, $n > 1$ we proceed in the similar way. First we substitute them by $\mathcal{J}_{n+2}(g, h, \ell_1(f)^n) = 0$,

$$\ell_{n+2}(\ell_1(f)^n, \ell_1(g), h) - \ell_{n+2}(\ell_1(f)^n, \ell_1(h), g) = G(f_1, \dots, f_n, g, h)\,,$$

The graded symmetry of $\ell_{n+2}(\ell_1(f)^n, \ell_1(g), h)$ implies the consistency condition,

$$G(f_1, \dots, f_n, g, h) + G(f_1, \dots, f_{n-1}, g, h, f_n) + G(f_1, \dots, f_{n-1}, h, f_n, g) = 0\,,$$

which follows from the previous L_∞ relations and can be proved by the induction.
The solution is constructed by taking the symmetrization of the r.h.s. in the first $n+1$ arguments, i.e.,

$$\ell_{n+2}\big(\ell_1(f)^n, \ell_1(g), h\big) = -\frac{1}{(n+1)(n+2)}\Big(G(f_1, \dots, f_n, g, h) + G(f_2, \dots, f_n, g, f_1, h) + \dots + G(f_n, \dots, f_{n-1}, h)\Big)\,.$$

Slowly varying field approximation, arXiv:1903.02867

We discard the higher derivatives terms, $\ell_2(f,g) = -\{f,g\}$, then

$$\ell_2(f,A) = -\{f,A_a\} - \frac{1}{2}(\partial_a \Theta^{ij})\,\partial_i f\, A_j\,.$$

Taking, $\Theta^{ij}(x) = 2\theta\varepsilon^{ijk}x^k$, from recurrence relations we see that

$$\delta_f A_a = \partial_a f + \{A_a, f\} + \theta\varepsilon^{abc}A_b\partial_c f + \theta^2\left(\partial_a f\;A^2 - \partial_b f\;A^b A_a\right)\chi(\theta^2 A^2)\,.$$

From the gauge closure condition,

$$[\delta_f,\delta_g]A = \delta_{\{f,g\}}A\,,$$

one finds,

$$\chi(t) = \frac{1}{t}\left(\sqrt{t}\cot\sqrt{t} - 1\right)\,, \qquad \chi(0) = -\frac{1}{3}\,.$$

- NC $su(2)$-like deformation of the abelian gauge transformations in the slowly varying field approximation.

NC Chern-Simons theory, $\mathrm{L}_\infty^{\mathrm{full}}$ algebra; $\mathbf{L} = L_{-2} \oplus L_{-1} \oplus L_0$

The L_∞ bootstrap setup is:

$$\ell_1(f) = \partial_a f\,, \qquad \ell_2(f,g) = -\{f,g\} \qquad \ell_1(A) = \varepsilon_c{}^{ab}\,\partial_a A_b\,.$$

Using the recurrence relations to calculate the lower brackets $\ell_2(A,A)$, $\ell_3(A,A,A)$, etc., one finds the ansatz for the left hand side of the field equations, $\mathcal{F}^a = 0$,

$$\mathcal{F}^a : \;=\; P^{abc}\,(A)\;\partial_b A_c + R^{abc}\,(A)\;\{A_b, A_c\}\;,$$

where

$$\begin{aligned} P^{abc}\,(A) \;&=\; \varepsilon^{abc}\,F\left(\theta^2 A^2\right) + \theta^2 \varepsilon^{abm} A_m\,A^c\,G\left(\theta^2 A^2\right) \\ &\quad +\theta^3 A^a\,A^b\,A^c\,K\left(\theta^2 A^2\right) + \theta\,A^a\,\delta^{bc}\,L\left(\theta^2 A^2\right) \\ &\quad +\theta\,A^b\,\delta^{ac}\,M\left(\theta^2 A^2\right) + \theta\,A^c\,\delta^{ab}\,N\left(\theta^2 A^2\right)\;, \\ R^{abc}\,(A) \;&=\; \varepsilon^{abc}\,S\left(\theta^2 A^2\right) + \theta^2\left(\varepsilon^{abm} A_m\,A^c - \varepsilon^{acm} A_m\,A^b\right)\;T\left(\theta^2 A^2\right) \\ &\quad +\theta\left(\delta^{ab}\,A^c - \delta^{ac}\,A^b\right)\;V\left(\theta^2 A^2\right)\;. \end{aligned}$$

Non-commutative Chern-Simons theory, arXiv:1905.08753

To determine coefficient functions $F(\theta^2 A^2)$, $G(\theta^2 A^2)$, ets., we use the condition that eom should transform covariantly,

$$\delta_f \mathcal{F} = \ell_2(f, \mathcal{F}) = \{f, \mathcal{F}\}\,.$$

Thus,

$$\begin{aligned}
F(t) &= \frac{N(t)}{2} = \frac{\sin\sqrt{t}\cos\sqrt{t}}{\sqrt{t}}\,, \quad G(t) = \frac{2\sqrt{t}\,\cos 2\sqrt{t} - \sin 2\sqrt{t}}{2\,t\sqrt{t}}\,,\\
K(t) &= -4\,T(t) = -\frac{2\sin\sqrt{t}}{t^2}\left(\sqrt{t}\,\cos\sqrt{t} - \sin\sqrt{t}\right)\,,\\
L(t) &= M(t) = -2\,S(t) = -2\,V(t) = -\frac{\sin^2\sqrt{t}}{t}\,.
\end{aligned}$$

Taking into account that, $F(0) = 1$, $G(0) = -4/3$, $K(0) = 2/3$, and $L(0) = -1$ one finds, $\lim_{\theta\to 0}\mathcal{F}^a = \varepsilon^{abc}\partial_b A_c$, since

$$\lim_{\theta\to 0} P^{abc}(A) = \varepsilon^{abc}\,, \qquad \text{and} \qquad \lim_{\theta\to 0} R^{abc}(A) = \frac{1}{2}\,\varepsilon^{abc}\,.$$

Non-commutative field strength

In $3d$ we have constructed a vector $\mathcal{F}_a$, which transforms covariantly, $\delta_f \mathcal{F}_a = \{f, \mathcal{F}_a\}$. Consequently the tensor

$$\mathcal{F}^{ab} := \varepsilon^{abc}\mathcal{F}_c = P^{abcd}(A)\,\partial_c A_d + R^{abcd}(A)\,\{A_c, A_d\} =$$
$$\partial^a\big(A^b F(A)\big) - \partial^b\big(A^a F(A)\big) +$$
$$\theta\varepsilon^{abc}\partial^d\big(A_c A_d L(A)\big) + \theta F(A)\varepsilon^{abc}\partial_c A^2 +$$
$$-\frac{1}{2}\{A^a L(A), A^b\} - \frac{1}{2}\{A^a, A^b L(A)\} - \frac{\theta}{2}L(A)\varepsilon^{abc}\{A_c, A^2\}\,,$$

where

$$F(A) = \frac{\sin\left(2\sqrt{\theta^2 A^2}\right)}{2\sqrt{\theta^2 A^2}}\,, \qquad \text{and} \qquad L(A) = -\frac{\sin^2\sqrt{\theta^2 A^2}}{\theta^2 A^2}\,,$$

is antisymmetric, transforms covariantly, $\delta_f \mathcal{F}^{ab} = \{f, \mathcal{F}^{ab}\}$, and

$$\lim_{\theta\to 0} \mathcal{F}_{ab} = \partial_a A_b - \partial_b A_a\,.$$

We call it the non-commutative field strength. The space L_{-2} also can be treated as the space of the field strength.

NCCS dynamics and action principle

- So, just like in the commutative case the NCCS eom are satisfied if NC field curvature vanishes everywhere.

If $\mathcal{F}_a = \delta\mathcal{L}/\delta A^a$, then

$$\frac{\delta \mathcal{F}_a}{\delta A^b} = \frac{\delta \mathcal{F}_b}{\delta A^a} .$$

One may easily check that, for

$$\mathcal{F}^a : \;=\; P^{abc}(A)\,\partial_b A_c + R^{abc}(A)\,\{A_b, A_c\} \;,$$

this condition does not hold,

$$\frac{\delta \mathcal{F}_a}{\delta A^b} \neq \frac{\delta \mathcal{F}_b}{\delta A^a} ,$$

in particular, because P^{abc} is not an antisymmetric in a and c. Possibly we have a non-trivial deformation of CS theory.

Non-commutative Yang-Mills

Since, $\delta_f \mathcal{F}^{ab} = \{f, \mathcal{F}^{ab}\}$, the following Lagrangian,

$$\mathcal{L} = -\frac{1}{4}\mathcal{F}_{ab}\mathcal{F}^{ab} = -\frac{3}{4}\mathcal{F}_a\mathcal{F}^a \,,$$

also transforms covariantly, $\delta_f \mathcal{L} = \{f, \mathcal{L}\}$, i.e., the action, $S = \int \mathcal{L}$, is gauge invariant.

The NCYM eom are:

$$\mathcal{D}^{abd}\,\mathcal{F}_{ab} = 0\,,$$

where

$$\mathcal{D}^{abd}\,\mathcal{F}_{ab} = \frac{1}{2}P^{abcd}(A)\,\partial_c\mathcal{F}_{ab} - R^{abcd}(A)\,\{A_c, \mathcal{F}_{ab}\}\,.$$

The transformation law is given by

$$\left(\delta_f \mathcal{D}^{abd}\right)\mathcal{F}_{ab} = -\mathcal{D}^{abd}\,\{f, \mathcal{F}_{ab}\}\,.$$

Discussion

- Given undeformed gauge theory and anti-symmetric bi-vector field $\Theta^{ij}(x)$ describing the non-commutativity of the space, we have iterative procedure of the construction of NC gauge theory, which reproduce in the limit $\Theta \to 0$ the undeformed one.
- Our construction is based on the principle that gauge symmetry should be realized by L_∞ and works for any given Θ.
- Open questions:
  - Physical consequences: interaction with the matter fields, quantization, UV/IR? etc.
  - The relation with the previous approaches needs to be better understood.