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Multiparameter Quantum Minkowski Space-Time and Quantum Maxwell Hierarchy

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Introduction

Invariant differential equations play a very important role in the description of physical symmetries - recall, e.g., the examples of Dirac, Maxwell equations, (for more examples cf., e.g., [BR]).

It is important to construct systematically such equations for the setting of quantum groups, where they are expected as (multiparameter) q-difference equations.

In the present talk we consider the construction of deformed multiparameter analogs of some conformally invariant equations, in particular, the Maxwell equations, following the approach of [D].

We start with the classical situation and we first write the Maxwell equations in an indexless formulation, trading the indices for two conjugate variables $z, \bar{z}$.

This formulation has two advantages. First, it is very simple, and in fact, just with the introduction of an additional parameter, we can describe a whole infinite hierarchy of equations, which we call the *Maxwell hierarchy* .

Second, we can easily identify the variables $z, \bar{z}$ and the four Minkowski coordinates with the six local coordinates of a flag manifold of $SL(4)$ and $SU(2,2)$.

Next we need the deformed analogs of the above constructions. The specifics of the approach of [D] is that one needs also the complexification of the algebra in consideration.

Thus we have used the multiparameter deformations $U_{q,\mathbf{q}}(gl(m))$ and $U_{q,\mathbf{q}}(sl(m))$ in the case $m = 4$. We know that these multiparameter deformations depend maximally on $(m^2 - m + 2)/2$ parameters.

Thus, we obtain initially a 7-parameter deformation of *Minkowski space-time*. Under various conditions we consider several variants with less parameters.

Using the corresponding representations and intertwiners of deformed $U(sl(4))$ we also derive infinite hierarchies of deformed Maxwell and related equations.

Classical setting

It is well known that Maxwell equations may be written in several equivalent forms:

$$\partial^\mu F_{\mu\nu} \;=\; J_\nu \;, \quad \partial^{\mu *} F_{\mu\nu} \;=\; 0 \qquad (1)$$

or,

$$\begin{aligned}
&\partial_k E_k \;=\; J_0 \;(= 4\pi\rho), \\
&\partial_0 E_k - \varepsilon_{k\ell m}\partial_\ell H_m \;=\; J_k \;(= -4\pi j_k), \\
&\partial_k H_k \;=\; 0 \;, \\
&\partial_0 H_k + \varepsilon_{k\ell m}\partial_\ell E_m \;=\; 0 \;,
\end{aligned} \qquad (2)$$

where $E_k \;\equiv\; F_{k0}$, $H_k \;\equiv\; (1/2)\varepsilon_{k\ell m}F_{\ell m}$,
or

$$\partial_k F_k^{\pm} \;=\; J_0 \;, \quad \partial_0 F_k^{\pm} \pm i\varepsilon_{k\ell m}\partial_\ell F_m^{\pm} \;=\; J_k \;, \; (3)$$

where

$$F_k^{\pm} \;\equiv\; E_k \pm iH_k \;. \qquad (4)$$

Not so well known is the fact that the eight equations in (3) can be rewritten as two conjugate scalar equations in the following way:

$$I^{+}\ F^{+}(z)\ =\ J(z,\bar{z})\ , \tag{5a}$$

$$I^{-}\ F^{-}(\bar{z})\ =\ J(z,\bar{z})\ , \tag{5b}$$

$$\begin{aligned} I^{+} \quad &= \quad \bar{z}\partial_{+} + \partial_{v}\ - \qquad\qquad (6a)\\ &\quad - \frac{1}{2}\Big(\bar{z}z\partial_{+} + z\partial_{v} + \bar{z}\partial_{\bar{v}} + \partial_{-}\Big)\partial_{z}\ ,\\ I^{-} \quad &= \quad z\partial_{+} + \partial_{\bar{v}}\ - \qquad\qquad (6b)\\ &\quad - \frac{1}{2}\Big(\bar{z}z\partial_{+} + z\partial_{v} + \bar{z}\partial_{\bar{v}} + \partial_{-}\Big)\partial_{\bar{z}}\ , \end{aligned}$$

$$\begin{aligned} &x_{\pm} \equiv x_0 \pm x_3, \quad v \equiv x_1 - ix_2, \quad \bar{v} \equiv x_1 + ix_2,\\ &\partial_{\pm} \equiv \partial/\partial x_{\pm}, \quad \partial_{v} \equiv \partial/\partial v, \quad \partial_{\bar{v}} \equiv \partial/\partial\bar{v}, \end{aligned} \tag{7}$$

$$\begin{aligned} F^{+}(z) \quad &\equiv \quad z^2(F_1^{+} + iF_2^{+}) - 2zF_3^{+}\ -\\ &\quad -\ (F_1^{+} - iF_2^{+})\ , \qquad\qquad (8)\\ F^{-}(\bar{z}) \quad &\equiv \quad \bar{z}^2(F_1^{-} - iF_2^{-}) - 2\bar{z}F_3^{-}\ -\\ &\quad -\ (F_1^{-} + iF_2^{-})\ ,\\ J(z,\bar{z}) \quad &\equiv \quad \bar{z}z(J_0 + J_3) + \bar{z}(J_1 - iJ_2)\ +\\ &\quad +\ z(J_1 + iJ_2) + (J_0 - J_3)\ , \end{aligned}$$

where we continue to suppress the x_μ, resp., $x_\pm, v, \bar{v}$, dependence in F and J. (The conjugation mentioned above is standard and in our terms it is : $I^+ \longleftrightarrow I^-$, $F^+(z) \longleftrightarrow F^-(\bar{z})$.)

It is easy to recover (3) from (5) - just note that both sides of each equation are first order polynomials in each of the two variables z and $\bar{z}$, then comparing the independent terms in (5) one gets at once (3).

Writing the Maxwell equations in the simple form (5) has also important conceptual meaning. The point is that each of the two scalar operators I^+, I^- is indeed a single object, namely, I^+, I^- are **intertwiners** of the conformal group, while the individual components in (1) - (3) do not have this interpretation.
This also is a restatement of the well known fact that the Maxwell equations are conformally invariant.

Let us be more explicit. The physically relevant representations T^χ of the 4-dimensional conformal algebra $su(2,2)$ may be labelled by $\chi = [n_1, n_2; d]$, where n_1, n_2 are non-negative integers fixing finite-dimensional irreducible representations of the Lorentz subalgebra, and d is the conformal weight (or dimension, or energy). (In the literature these Lorentz representations are labelled also by $(j_1, j_2) = (n_1/2, n_2/2)$.)

Then the intertwining properties of the operators in (6) are given by:

$$\begin{aligned} I^+ &: C^+ \longrightarrow C^0 \ , \\ I^+ \circ T^+ &= T^0 \circ I^+ \ , \end{aligned} \tag{9a}$$

$$\begin{aligned} I^- &: C^- \longrightarrow C^0 \ , \\ I^- \circ T^- &= T^0 \circ I^- \ , \end{aligned} \tag{9b}$$

where $T^a = T^{\chi^a}$, $a = 0, +, -$, $C^a = C^{\chi^a}$ are the representation spaces, and the signatures are given explicitly by:

$$\chi^+ = [2, 0; 2] \ , \quad \chi^- = [0, 2; 2] \ , \quad \chi^0 = [1, 1; 3] \ , \tag{10}$$

as anticipated. Indeed, $(n_1, n_2) = (1, 1)$ is the four-dimensional Lorentz representation, (carried by J_μ above), and $(n_1, n_2) = (2, 0), (0, 2)$ are the two conjugate three-dimensional Lorentz representations, (carried by $F^\pm_k$ above), while the conformal dimensions are the canonical dimensions of a current ($d = 3$), and of the Maxwell field ($d = 2$).

We see that the variables $z, \bar{z}$ are related to the spin properties and we shall call them 'spin variables'. More explicitly, a Lorentz spin-tensor $G(z, \bar{z})$ with signature (n_1, n_2) is a polynomial in $z, \bar{z}$ of order n_1, n_2, resp.

The intertwining properties of the operators in (6) and some more, namely,

$$\chi(\phi) = [0, 0; 0], \quad \chi(\Phi) = [0, 0; 4], \quad \chi(A_\mu) = [1, 1; 1]$$

are presented in the following diagram:

$$
\begin{tikzcd}[row sep=huge, column sep=huge]
\phi \arrow[r, leftrightarrow, "/" marking, "/" {marking, pos=0.8}] \arrow[d, "\partial_\mu"'] & \Phi \\
A_\mu \arrow[r, leftrightarrow, "/" marking, "/" {marking, pos=0.8}] \arrow[d, "d_{12}"'] \arrow[dr, "d_{23}"' near start] & J_\mu \cong \chi^0 \arrow[u, "\partial^\mu"'] \\
F^-_{\lambda\mu} \cong \chi^- \arrow[r, leftrightarrow, "/" marking, "/" {marking, pos=0.8}] \arrow[ur, "d_{23} = I^-" near end] & F^+_{\lambda\mu} \cong \chi^+ \arrow[u, "d_{12} = I^+"']
\end{tikzcd}
$$

Simplest example of diagram with conformal invariant operators
(arrows are differential operators, dashed arrows are integral operators)

$\partial_\mu = \frac{\partial}{\partial x_\mu}$, A_μ electromagnetic potential, $\partial_\mu \phi = A_\mu$

$F = F^+ \oplus F^-$ electromagnetic field, $\partial_{[\lambda} A_{\mu]} = \partial_\lambda A_\mu - \partial_\mu A_\lambda = F_{\lambda\mu}$

J_μ electromagnetic current, $\partial^\lambda F_{\lambda\mu} = J_\mu$, $\partial^\mu J_\mu = \Phi$

d_{12}, d_{23} linear invariant operators corresponding to the roots α_{12}, α_{23}

Formulae (9), (10) are part of an infinite hierarchy of couples of first order intertwiners. Explicitly, instead of (9), (10) we have [D]:

$$\begin{aligned} I^+_n \ &: \ C^+_n \longrightarrow C^0_n \ , \\ I^+_n \circ T^+_n \ &= \ T^0_n \circ I^+_n \ , \end{aligned} \tag{11a}$$

$$\begin{aligned} I^-_n \ &: \ C^-_n \longrightarrow C^0_n \ , \\ I^-_n \circ T^-_n \ &= \ T^0_n \circ I^-_n \ , \end{aligned} \tag{11b}$$

where $T^a_n = T^{\chi^a_n}$, $C^a_n = C^{\chi^a_n}$, and the signatures are:

$$\begin{aligned} &\chi^+_n = [n+2, n; 2] \ , \quad \chi^-_n = [n, n+2; 2] \ , \\ &\chi^0_n = [n+1, n+1; 3] \ , \quad n \in \mathbb{Z}_+ \ , \end{aligned} \tag{12}$$

while instead of (5) we have:

$$I^+_n \ F^+_n(z,\bar{z}) \ = \ J_n(z,\bar{z}) \ , \tag{13a}$$

$$I^-_n \ F^-_n(z,\bar{z}) \ = \ J_n(z,\bar{z}) \ , \tag{13b}$$

where ($n \in \mathbb{Z}_+$)

$$I_n^+ \;=\; \frac{n+2}{2}\Big(\bar{z}\partial_+ + \partial_v\Big) \;- \qquad \text{(14a)}$$
$$- \;\frac{1}{2}\Big(\bar{z}z\partial_+ + z\partial_v + \bar{z}\partial_{\bar{v}} + \partial_-\Big)\partial_z \;,$$
$$I_n^- \;=\; \frac{n+2}{2}\Big(z\partial_+ + \partial_{\bar{v}}\Big) \;- \qquad \text{(14b)}$$
$$- \;\frac{1}{2}\Big(\bar{z}z\partial_+ + z\partial_v + \bar{z}\partial_{\bar{v}} + \partial_-\Big)\partial_{\bar{z}} \;,$$

while $F_n^+(z,\bar{z})$, $F_n^-(z,\bar{z})$, $J_n(z,\bar{z})$, are polynomials in $z,\bar{z}$ of degrees $(n+2,n)$, $(n,n+2)$, $(n+1,n+1)$, resp., as explained above.

If we want to use the notation with indices as in (1), *then* $F_n^+(z,\bar{z})$ *and* $F_n^-(z,\bar{z})$ *correspond to* $F_{\mu\nu,\alpha_1,\dots,\alpha_n}$ *which is antisymmetric in the indices* μ,ν*, symmetric in* $\alpha_1,\dots,\alpha_n$*, and traceless in every pair of indices, while* $J_n(z,\bar{z})$ *corresponds to* $J_{\mu,\alpha_1,\dots,\alpha_n}$ *which is symmetric and traceless in every pair of indices. Note, however, that the analogs of* (1) *would be much more complicated if one wants to write explicitly all components.*

The crucial advantage of (13) is that the operators $I_n^\pm$ are given just by a slight generalization of $I^\pm = I_0^\pm$.

We call the hierarchy of equations (13) the *Maxwell hierarchy*. The Maxwell equations are the zero member of this hierarchy.

Formulae (13),(11),(12) are part of a much more general classification scheme [D], involving also other intertwining operators, and of arbitrary order:

$$
\begin{tikzcd}[row sep=huge, column sep=huge]
\Lambda^{-}_{p\nu n} \arrow[r, leftrightarrow] \arrow[d, "d_2^{\nu}"'] & \Lambda^{+}_{p\nu n} \\
\Lambda'^{-}_{p\nu n} \arrow[r, leftrightarrow] \arrow[d, "d_{12}^{n}"'] \arrow[dr, "d_{23}^{p}" near start] & \Lambda'^{+}_{p\nu n} \arrow[u, "d_{13}^{\nu}"'] \\
\Lambda''^{-}_{p\nu n} \arrow[r, leftrightarrow] \arrow[ur, "d_{23}^{p}"' near end] & \Lambda''^{+}_{p\nu n} \arrow[u, "d_{12}^{n}"']
\end{tikzcd}
$$

The general sextet of invariant differential operators valid for
$so(4,2)$, $so(5,1)$ and $so(3,3) \cong sl(4,\mathbb{R})$.

p, ν, n are three natural numbers, the shown simplest case is when $p = \nu = n = 1$,

d_2^{ν}, d_{13}^{ν} linear invariant operators of order ν corresponding to the roots α_2, α_{13}

d_{12}^{n}, d_{23}^{p} linear invariant operators of order n, p corresponding to the roots α_{12}, α_{23}

But first we go back and we rewrite (14) in the following form:

$$I_n^+ = \frac{1}{2}\Big((n+2)I_1 I_2 - (n+3)I_2 I_1\Big),$$
$$I_n^- = \frac{1}{2}\Big((n+2)I_3 I_2 - (n+3)I_2 I_3\Big) \quad (15)$$

where

$$I_1 \equiv \partial_z\,,\ I_2 \equiv \bar{z}z\partial_+ + z\partial_v + \bar{z}\partial_{\bar{v}} + \partial_-\,,\ I_3 \equiv \partial_{\bar{z}}\,. \quad (16)$$

It is important to note that group-theoretically the operators I_a correspond to the right action of the three simple roots $\alpha_1, \alpha_2, \alpha_3$ of $sl(4)$, while the operators $I_n^{\pm}$ are obtained from the lowest possible singular vectors corresponding to the two non-simple non-highest roots $\alpha_{12} \equiv \alpha_1 + \alpha_2$, $\alpha_{23} = \alpha_2 + \alpha_3$ [D].

This is the form that we generalize for the deformed case. In fact, we can write at once the general form, which follows from the analysis of [D]:

$$\begin{aligned}
\widehat{I}^+_n &= \frac{1}{2}\Big([n+2]_q \widehat{I}_1 \widehat{I}_2 - [n+3]_q \widehat{I}_2 \widehat{I}_1\Big), \\
\widehat{I}^-_n &= \frac{1}{2}\Big([n+2]_q \widehat{I}_3 \widehat{I}_2 - [n+3]_q \widehat{I}_2 \widehat{I}_3\Big)
\end{aligned} \qquad (17)$$

Here $\widehat{I}^\pm_n$ are obtained from the lowest possible singular vectors of $U_q(sl(4))$, corresponding (as above) to the two non-simple non-highest roots [D].

• In addition to the differential operators on the last sextet diagram there are operators arising from singular vectors in doublets :

$$ {}_1\Lambda^-_{p\nu} \xrightarrow{\qquad\qquad(I_{12})^p\qquad} {}_1\Lambda^+_{p\nu} $$

$$ {}_2\Lambda^-_{pn} \xrightarrow{\qquad\qquad\mathcal{D}_{p,n}\qquad} {}_2\Lambda^+_{pn} $$

$$ {}_3\Lambda^-_{\nu n} \xrightarrow{\qquad\qquad(I_{23})^n\qquad} {}_3\Lambda^+_{\nu n} $$

where p, ν, n are natural numbers, $\mathcal{D}_{p+n}$ is an invariant differential operator of order $p+n$.

The above sextets and doublets exhaust the so(4,2) invariant differential operators arising from singular vectors. Besides those there are invariant differential operators arising from Casimirs and from subsingular vectors (two cases for so(4,2)).

Multiparameter quantum Minkowski space-time

The variables $x_\pm, v, \bar{v}, z, \bar{z}$ have definite group-theoretical meaning, namely, they are six local coordinates on the flag manifold
$\mathcal{Y} = GL(4)/\tilde{B} = SL(4)/B$,
where $\tilde{B}, B$ are the Borel subgroups of $GL(4)$, $SL(4)$, respectively, consisting of all upper diagonal matrices. Under a natural conjugation (cf. also below) this is also a flag manifold of the conformal group $SU(2,2)$.

Explicitly, for this is used the triangular Gauss decomposition:

$$\begin{pmatrix} g_{11} & g_{12} & g_{13} & g_{14} \\ g_{21} & g_{22} & g_{23} & g_{24} \\ g_{31} & g_{32} & g_{33} & g_{34} \\ g_{41} & g_{42} & g_{43} & g_{44} \end{pmatrix} = \tag{18}$$

$$= \begin{pmatrix} 1 & 0 & 0 & 0 \\ z & 1 & 0 & 0 \\ v & x_- & 1 & 0 \\ x_+ & \bar{v} & \bar{z} & 1 \end{pmatrix} \begin{pmatrix} * & 0 & 0 & 0 \\ 0 & * & 0 & 0 \\ 0 & 0 & * & 0 \\ 0 & 0 & 0 & * \end{pmatrix} \begin{pmatrix} 1 & * & * & * \\ 0 & 1 & * & * \\ 0 & 0 & 1 & * \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

In the deformed case we use the same decomposition which gives us the commutation relations of the non-commutative coordinates on the multiparameter $\mathcal{Y}_{q,\mathbf{q}}$ flag manifold. There is a technicality here, namely, that we start from the multiparameter deformation $GL_{q,\mathbf{q}}(m)$ of $GL(m)$ (given by Sudbery) which depends on the maximal possible number of parameters, i.e., on the $(m^2 - m + 2)/2$ parameters q, q_{ij}, $1 \leq i < j \leq m$.
(The parametrisation is such that the standard *one-parameter* deformation is obtained for all $q_{ij} = q$.)
Thus, the flag manifold $\tilde{\mathcal{Y}}_{q,\mathbf{q}} = GL_{q,\mathbf{q}}(m)/\tilde{B}_{q,\mathbf{q}}(m)$ depends on the same number of parameters.

Thus, for $m = 4$ we have a seven parameter quantum Minkowski space-time the explicit relations being ($\lambda \equiv q - q^{-1}$):

$$
\begin{aligned}
x_+ v &= \frac{q_{23} q_{34}}{q_{24}} v x_+ \,, \qquad & \bar{v} x_+ &= \frac{q_{14}}{q_{12} q_{24}} x_+ \bar{v} \,, \\
x_- v &= \frac{q_{13}}{q_{12} q_{23}} v x_- \,, \qquad & \bar{v} x_- &= \frac{q_{13} q_{34}}{q_{14}} x_- \bar{v} \,, \\
\bar{v} v &= \frac{q_{13} q_{34}}{q_{12} q_{24}} v \bar{v} \,, \\
\frac{q\, q_{24}}{q_{23} q_{34}} x_+ x_- &= \frac{q_{12} q_{24}}{q\, q_{14}} x_- x_+ + \lambda v \bar{v} \,,
\end{aligned}
\tag{19}
$$

The commutation relations involving the spin variables $z, \bar{z}$ are:

$$
\begin{aligned}
\bar{z}z &= \frac{q_{13}q_{24}}{q_{14}q_{23}} z\bar{z} \ , \\
\bar{z}x_{+} &= \frac{q_{13}q_{34}}{q_{14}} x_{+}\bar{z} \ , \\
\bar{z}x_{-} &= \frac{q_{23}q_{34}}{q^2 q_{24}} x_{-}\bar{z} \ + \ \lambda\bar{v} \ , \\
\bar{z}\bar{v} &= \frac{q_{23}q_{34}}{q_{24}} \bar{v}\bar{z} \ , \\
\bar{z}v &= \frac{q_{13}q_{34}}{q^2 q_{14}} v\bar{z} \ + \ \lambda x_{+} \ , \\
x_{+}z &= \frac{q_{14}}{q_{12}q_{24}} z x_{+} \ , \\
x_{-}z &= \frac{q^2 q_{13}}{q_{12}q_{23}} z x_{-} \ - \ \lambda v \ , \\
vz &= \frac{q_{13}}{q_{12}q_{23}} z v \ , \\
\bar{v}z &= \frac{q^2 q_{14}}{q_{12}q_{24}} z\bar{v} \ - \ \lambda x_{+} \ .
\end{aligned}
\tag{20}
$$

Now we point out several special cases:

• When all deformation parameter are phases, i.e., $|q| = 1$, $|q_{ij}| = 1$, and in addition holds the following relations:

$$q_{13} = \frac{q_{12}q_{24}}{q_{34}} , \quad q_{14} = \frac{q_{12}q_{24}^2}{q_{23}q_{34}} , \qquad (21)$$

then the commutation relations (19) and (20) are preserved by an anti-linear anti-involution ω acting as :

$$\omega(x_\pm) = x_\pm , \quad \omega(v) = \bar{v} , \quad \omega(z) = \bar{z} . \qquad (22)$$

• Further, we recall from [DoPa] that the dual quantum algebra $U_{q,\mathbf{q}}(gl(m))$ has the quantum algebra $U_{q,\mathbf{q}}(sl(m))$ as a commutation subalgebra, but not as a co-subalgebra. In order to achieve the latter we have to impose some relations between the parameters, thus the genuine multiparameter deformation $U_{q,\mathbf{q}}(sl(m))$ as co-subalgebra of $GL_{q,\mathbf{q}}(m)$ depends on $(m^2 - 3m + 4)/2$ parameters.

Thus, in the case of $m = 4$ for the genuine $U_{q,\mathbf{q}}(sl(4))$ we have four parameters instead of seven. Explicitly, we achieve this by imposing that the parameters $q_{i,i+1}$ are expressed through the rest:

$$q_{12} = \frac{q^3}{q_{13}q_{14}} \ , \quad q_{23} = \frac{q^4}{q_{13}q_{14}q_{24}} \ , \quad q_{34} = \frac{q^3}{q_{14}q_{24}} \tag{23}$$

Thus, the four-parameter quantum Minkowski space-time and the embedding quantum flag manifold $\mathcal{Y}_{q,\mathbf{q}}$ are given by (19) and (20) with (23) enforced.

• If we would like to enforce also the conjugation (22) then there are more relations between the deformation parameters, namely, we get:

$$q_{12} = q_{23} = q_{34} = \frac{q^2}{q_{14}} \ , \quad q_{13} = q_{24} = q \ . \tag{24}$$

Thus, we have a two-parameter deformation the analogs of (19) and (20) becoming:

$$
\begin{aligned}
x_+ v &= p\, v x_+ \ , \qquad & \bar{v} x_+ &= p^{-1} x_+ \bar{v} \ , \\
x_- v &= p^{-1} v x_- \ , \qquad & \bar{v} x_- &= p\, x_- \bar{v} \ , \\
\bar{v} v &= v \bar{v} \ , & & \\
\frac{q}{p} x_+ x_- &= \frac{p}{q} x_- x_+ + \lambda v \bar{v} \ , & &
\end{aligned}
\tag{25}
$$

$$
\begin{aligned}
\bar{z} z &= z \bar{z} \ , \\
\bar{z} x_+ &= p\, x_+ \bar{z} \ , \\
\bar{z} x_- &= \frac{p}{q^2} x_- \bar{z} + \lambda \bar{v} \ , \\
\bar{z} \bar{v} &= p\, \bar{v} \bar{z} \ , \\
\bar{z} v &= \frac{p}{q^2} v \bar{z} + \lambda x_+ \ , \\
x_+ z &= p^{-1} z x_+ \ , \\
x_- z &= \frac{q^2}{p} z x_- - \lambda v \ , \\
v z &= p^{-1} z v \ , \\
\bar{v} z &= \frac{q^2}{p} z \bar{v} - \lambda x_+ \ ,
\end{aligned}
\tag{26}
$$

where $p \equiv q^3 / q_{14}^2$.

Quantum Maxwell equations hierarchy

The order of variables hinted in (19),(20) is related to the normal ordered basis of the quantum flag manifold $\mathcal{Y}_{q,\mathbf{q}}$ considered as an associative algebra:

$$\widehat{\varphi}_{ijk\ell mn} \ = \ z^i \, v^j \, x_-^k \, x_+^\ell \, \bar{v}^m \, \bar{z}^n \, , \quad i,j,k,\ell,m,n \in \mathbb{Z}_+ \, . \tag{27}$$

We introduce now the representation spaces C^χ corresponding to the signatures $\chi = [n_1, n_2; d]$. The elements of C^χ, which we shall call (abusing the notion) functions, are polynomials in $z, \bar{z}$ of degrees n_1, n_2, resp., and formal power series in the quantum Minkowski variables. Namely, these functions are given by:

$$\widehat{\varphi}_{n_1,n_2}(\bar{Y}) \quad = \quad \sum_{\substack{i,j,k,\ell,m,n\in\mathbb{Z}_+ \\ i\leq n_1, \ n\leq n_2}} \mu^{n_1,n_2}_{ijk\ell mn} \ \widehat{\varphi}_{ijk\ell mn} \ , \tag{28}$$

where $\bar{Y}$ denotes the set of the six coordinates on $\mathcal{Y}_{q,\mathbf{q}}$. Thus the quantum analogs of $F_n^{\pm}$, J_n, cf. (13), are :

$$\widehat{F}_n^{+} = \widehat{\varphi}_{n+2,n}(\bar{Y}) , \quad \widehat{F}_n^{-} = \widehat{\varphi}_{n,n+2}(\bar{Y}) ,$$
$$\hat{J}_n = \widehat{\varphi}_{n+1,n+1}(\bar{Y}) . \tag{29}$$

Using the above machinery we can present a deformed version of the Maxwell hierarchy of equations. First, we mention that the explicit form of the operators I_a in (16) is obtained by the infinitesimal right action of the three simple root generators of $sl(4)$ on the flag manifold $\mathcal{Y}$ (following the procedure of [D]). Thus, in the deformed case for the right action of $U_{q,\mathbf{q}}(sl(4))$ on $\mathcal{Y}_{q,\mathbf{q}}$ we have:

$$\hat{I}_a = \pi_R(X_a^-) \tag{30}$$

From this we obtain the multi-parameter quantum Maxwell hierarchy of equations by substituting the operators of (30) in (17), i.e., the final result is:

$$\hat{I}^+_n \ \hat{F}^+_n \ = \ \hat{J}_n \ , \tag{31a}$$

$$\hat{I}^-_n \ \hat{F}^-_n \ = \ \hat{J}_n \ . \tag{31b}$$

The reason that we can use (17) is that the multiparameter $U_{q,\mathbf{q}}(sl(4))$ depends only on q as a commutation subalgebra, while the dependence on the other parameters is exhibited only in its co-algebra structure and in the explicit expressions of $\pi_R(X^-_a)$.

Formulae (31) are part of a much more general classification scheme (mentioned above, cf. [D]) involving also other intertwining operators, and of arbitrary order. A subset of this scheme are two infinite two-parameter families of representations which are intertwined by the same operators (14), cf. [D]. Explicitly:

$$I^{+}_{n_1^+,n_2^+} \;:\; C^{+}_{n_1^+,n_2^+} \longrightarrow C^{0+}_{n_1^+,n_2^+} \ , \tag{32a}$$
$$I^{+}_{n_1^+,n_2^+} \circ T^{+}_{n_1^+,n_2^+} \;=\; T^{0+}_{n_1^+,n_2^+} \circ I^{+}_{n_1^+,n_2^+} \ ,$$
$$I^{-}_{n_1^-,n_2^-} \;:\; C^{-}_{n_1^-,n_2^-} \longrightarrow C^{0-}_{n_1^-,n_2^-} \ , \tag{32b}$$
$$I^{-}_{n_1^-,n_2^-} \circ T^{-}_{n_1^-,n_2^-} \;=\; T^{0-}_{n_1^-,n_2^-} \circ I^{-}_{n_1^-,n_2^-} \ ,$$

where $T^{a}_{n_1^\pm,n_2^\pm} = T^{\chi^a_{n_1^\pm,n_2^\pm}}$, $C^{a}_{n_1^\pm,n_2^\pm} = C^{\chi^a_{n_1^\pm,n_2^\pm}}$, $a = \pm$, or $a = 0\pm$, and

$$\chi^{+}_{n_1^+,n_2^+} \;=\; [n_1^+, n_2^+; \frac{n_1^+ - n_2^+}{2} + 1] \tag{33a}$$
$$\chi^{0+}_{n_1^+,n_2^+} \;=\; [n_1^+ - 1, n_2^+ + 1; \frac{n_1^+ - n_2^+}{2} + 2],$$
$$n_1^+ \in I\!N,\ n_2^+ \in Z\!\!Z_+ \ ,$$
$$\chi^{-}_{n_1^-,n_2^-} \;=\; [n_1^-, n_2^-; \frac{n_2^- - n_1^-}{2} + 1] \tag{33b}$$
$$\chi^{0-}_{n_1^-,n_2^-} \;=\; [n_1^- + 1, n_2^- - 1; \frac{n_2^- - n_1^-}{2} + 2],$$
$$n_1^- \in Z\!\!Z_+,\ n_2^- \in I\!N \ .$$

Then instead of (13) in the $q = 1$ case and (31) in the q-deformed case, we have:

$$
\begin{aligned}
{}_qI^{+}_{n_1^+} \; F^{+}_{n_1^+,n_2^+}(z,\bar{z}) \quad &= \quad J^{+}_{n_1^+,n_2^+}(z,\bar{z}), \quad \text{(34a)} \\
{}_qI^{-}_{n_2^-} \; F^{-}_{n_1^-,n_2^-}(z,\bar{z}) \quad &= \quad J^{-}_{n_1^-,n_2^-}(z,\bar{z}), \quad \text{(34b)}
\end{aligned}
$$

where ${}_qI^{+}_{n_1^+}$, ${}_qI^{-}_{n_2^-}$, are given by (17), while $F^{\pm}_{n_1^\pm,n_2^\pm}(z,\bar{z})$, $J^{\pm}_{n_1^\pm,n_2^\pm}(z,\bar{z})$, are polynomials in $z,\bar{z}$ of degrees $(n_1^\pm, n_2^\pm)$, $(n_1^\pm \mp 1, n_2^\pm \pm 1)$, resp.

The crucial feature which unifies these representations is the form of the operators ${}_qI^{\pm}_{n}$, which is not generalized anymore in equations (34).

We call the hierarchy of equations (34) the **generalized q - Maxwell hierarchy**. The q - Maxwell hierarchy is obtained in the partial case when $\chi^{0+}_{n_1^+,n_2^+} = \chi^{0-}_{n_1^-,n_2^-} = \chi^0_n$ which fixes three of the four parameters: $n_1^+ - 2 = n_2^+ = n_1^- = n_2^- - 2 = n$.

- Another one parameter subhierarchy of the generalized q-Maxwell hierarchy involves the two signatures of $\chi_n^+ = [n+2, n; 2]$, $\chi_n^- = [n, n+2; 2]$, and in addition

$$\chi_n^{00} = [n+1, n+1; 1] = \{n+2, -1-n, n+2\}, \tag{35}$$

($n \in \mathbb{Z}_+$). The intertwining relations are:

$$\begin{aligned} I_{n-1}^+ &: C_n^{00} \longrightarrow C_n^-, & (36)\\ I_{n-1}^+ \circ T_n^{00} &= T_n^- \circ I_{n-1}^+, \\ I_{n-1}^- &: C_n^{00} \longrightarrow C_n^+, & (37)\\ I_{n-1}^- \circ T_n^{00} &= T_n^+ \circ I_{n-1}^-, & (38) \end{aligned}$$

where $T_n^{00} = T^{\chi_n^{00}}$, $C_n^{00} = C^{\chi_n^{00}}$. Here the equations are:

$$\begin{aligned} {}_qI_{n-1}^+ \; {}_qA_n &= {}_qF_n^-, & (39a)\\ {}_qI_{n-1}^- \; {}_qA_n &= {}_qF_n^+, & (39b) \end{aligned}$$

where ${}_qI_n^{\pm}$ are given by (17), ${}_qA_n$ has the signature χ_n^{00}.

This hierarchy will be called the **potential q-Maxwell hierarchy** . The reason is that the lowest member obtained for $n = 0$ and $q = 1$ is just:

$$\partial_{[\mu}A_{\nu]} \ = \ F_{\mu\nu} \ . \tag{40}$$

q - d'Alembert equations hierarchy

Here we consider another one parameter sub-hierarchy of the generalized q-Maxwell hierarchy which is obtained from (33) for $n_1^+ = n_2^- = r \in I\!N$, $n_1^- = n_2^+ = 0$, i.e.

$$\chi_r^{d+} = [r, 0; \frac{r}{2} + 1], \qquad (41a)$$

$$\chi_r^{d0+} = [r-1, 1; \frac{r}{2} + 2], \quad r \in I\!N,$$

$$\chi_r^{d-} = [0, r; \frac{r}{2} + 1], \qquad (41b)$$

$$\chi_r^{d0-} = [1, r-1; \frac{r}{2} + 2], \quad r \in I\!N,$$

where the two conjugated equations follow from (34):

$$_qI_r^+ \ F_r^{d+} = J_r^{d+} \ , \qquad (42a)$$

$$_qI_r^- \ F_r^{d-} = J_r^{d-} \ , \qquad (42b)$$

where $_qI_r^{\pm}$ are given by (17).

For the minimal possible value of the parameter $r = 1$ we obtain the two conjugate *q - Weyl equations*.

The case $r = 2$ gives the q-Maxwell equations (note that $J_2^{d+} = J_2^{d-}$). This is the only intersection of the present hierarchy with the q-Maxwell hierarchy.

We call this hierarchy *q - d'Alembert hierarchy* following the classical case, (cf. [D]), due to the following. We consider the representations $\chi_a^{d\pm}$ for the excluded above value $r = 0$, when they coincide. Thus, we set: $\chi^d \equiv \chi_0^{d\pm} = [0,0;1] = \chi(A_q)$. Then the relevant equation is the q-d'Alembert equation [D]:

$$\Box_q \ A_q \ = \ J_q \tag{43}$$

where $\chi(J_q) \ = \ [0,0;3]$,

$$\Box_q \ = \ \left(\widehat{\mathcal{D}}_{\bar{v}}\widehat{\mathcal{D}}_v - q\widehat{\mathcal{D}}_-\widehat{\mathcal{D}}_+ T_v T_{\bar{v}}\right) T_v T_{\bar{v}} T_+ T_- \tag{44}$$

where $\widehat{\mathcal{D}}_y$, T_y are standard q-difference, q-shift, operators.

q - Weyl gravity equations hierarchy

Here we study another hierarchy which is given as follows:

$$\begin{array}{ccccc} & & C^+_m & & \\ & \nearrow & & \searrow & \\ C^h_m & & & & C^T_m \\ & \searrow & & \nearrow & \\ & & C^-_m & & \end{array} \qquad (45)$$

where $m \in I\!N$, and the corresponding signatures are:

$$\begin{aligned} \chi^+_m &= [2m, 0; 2], \quad \chi^-_m = [0, 2m; 2], \\ \chi^h_m &= [m, m; 2-m], \quad \chi^T_m = [m, m; 2+m] \end{aligned} \qquad (46)$$

For future reference we give also the *Dynkin labels* $\chi = \{m_1, m_2, m_3\}$ of these representations:

$$\begin{aligned} \chi^+_m &= \{2m+1, -m-1, 1\}, \\ \chi^-_m &= \{1, -m-1, 2m+1\}, \\ \chi^h_m &= \{m+1, -1, m+1\}, \\ \chi^T_m &= \{m+1, -2m-1, m+1\} \end{aligned} \qquad (47)$$

The arrows on (45) represent invariant differential operators of order m. It is a partial case of the general conformal scheme parametrized by three natural numbers p, ν, n, (cf. Fig. 2), setting here: $\nu = 1$, $p = n = m$. This hierarchy intersects with the Maxwell hierarchy for the lowest value $m = 1$. Here we consider the linear conformal (Weyl) gravity which is obtained for $m = 2$.

q - Linear conformal gravity

We start with the $q = 1$ situation and we first write the linear conformal gravity equations, or Weyl gravity equations in our indexless formulation, trading the indices for two conjugate variables $z, \bar{z}$.

Weyl gravity is governed by the *Weyl tensor* $C_{\mu\nu\sigma\tau}$ which is given in terms of the Riemann curvature tensor $R_{\mu\nu\sigma\tau}$, Ricci curvature tensor $R_{\mu\nu}$, scalar curvature R :

$$\begin{aligned} C_{\mu\nu\sigma\tau} \; &= \; R_{\mu\nu\sigma\tau} - \tfrac{1}{2}(g_{\mu\sigma}R_{\nu\tau} + g_{\nu\tau}R_{\mu\sigma} \; - \\ &\quad - \; g_{\mu\tau}R_{\nu\sigma} - g_{\nu\sigma}R_{\mu\tau}) \; + \\ &\quad + \; \tfrac{1}{6}(g_{\mu\sigma}g_{\nu\tau} - g_{\mu\tau}g_{\nu\sigma})R \; , \end{aligned} \qquad (48)$$

where $g_{\mu\nu}$ is the metric tensor. Linear conformal gravity is obtained when the metric tensor is written as: $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$, where $\eta_{\mu\nu}$ is the flat Minkowski metric, $h_{\mu\nu}$ are small so that all quadratic and higher order terms are neglected. In particular:

$$R_{\mu\nu\sigma\tau} = \frac{1}{2}(\partial_\mu\partial_\tau h_{\nu\sigma} + \partial_\nu\partial_\sigma h_{\mu\tau} - \partial_\mu\partial_\sigma h_{\nu\tau} - \partial_\nu\partial_\tau h_{\mu\sigma})$$

The equations of linear conformal gravity are:

$$\partial^\nu \partial^\tau C_{\mu\nu\sigma\tau} = T_{\mu\sigma} \ , \qquad (49)$$

where $T_{\mu\nu}$ is the energy-momentum tensor. From the symmetry properties of the Weyl tensor it follows that it has ten independent components. These may be chosen as follows (introducing notation for future use):

$$\begin{aligned} &C_0 = C_{0123} \ , \quad C_1 = C_{2121} \ , \quad C_2 = C_{0202} \ , \\ &C_3 = C_{3012} \ , \quad C_4 = C_{2021} \ , \quad C_5 = C_{1012} \ , \\ &C_6 = C_{2023} \ , \quad C_7 = C_{3132} \ , \quad C_8 = C_{2123} \ , \\ &C_9 = C_{1213} \ . \end{aligned} \qquad (50)$$

Furthermore, the Weyl tensor transforms as the direct sum of two conjugate Lorentz irreps, which we shall denote as $C^\pm$ (cf. (46) for $m = 2$). The tensors $T_{\mu\nu}$ and $h_{\mu\nu}$ are symmetric and traceless with nine independent components.

Further, we shall use again the fact that a Lorentz irrep (spin-tensor) with signature (n_1, n_2) may be represented by a polynomial $G(z,\bar{z})$ in $z, \bar{z}$ of order n_1, n_2, resp. More explicitly, for the Weyl gravity representations mentioned above we use:

$$\begin{aligned}
C^+(z) &= z^4 C_4^+ + z^3 C_3^+ + z^2 C_2^+ + z C_1^+ + C_0^+ \ , \\
C^-(\bar{z}) &= \bar{z}^4 C_4^- + \bar{z}^3 C_3^- + \bar{z}^2 C_2^- + \bar{z} C_1^- + C_0^- \ , \\
T(z,\bar{z}) &= z^2\bar{z}^2 T'_{22} + z^2\bar{z} T'_{21} + z^2 T'_{20} + \\
&\quad + z\bar{z}^2 T'_{12} + z\bar{z} T'_{11} + z T'_{10} + \\
&\quad + \bar{z}^2 T'_{02} + \bar{z} T'_{01} + T'_{00} \ , \\
h(z,\bar{z}) &= z^2\bar{z}^2 h'_{22} + z^2\bar{z} h'_{21} + z^2 h'_{20} + \\
&\quad + z\bar{z}^2 h'_{12} + z\bar{z} h'_{11} + z h'_{10} + \\
&\quad + \bar{z}^2 h'_{02} + \bar{z} h'_{01} + h'_{00} \ .
\end{aligned} \tag{51}$$

The components $C_k^{\pm}$ are given in terms of the Weyl tensor components as follows:

$$
\begin{aligned}
C_0^+ &= C_2 - \frac{1}{2}C_1 - C_6 + i(C_0 + \frac{1}{2}C_3 + C_7) \\
C_1^+ &= 2(C_4 - C_8 + i(C_9 - C_5)) \\
C_2^+ &= 3(C_1 - iC_3) \\
C_3^+ &= 8(C_4 + C_8 + i(C_9 + C_5)) \\
C_4^+ &= C_2 - \frac{1}{2}C_1 + C_6 + i(C_0 + \frac{1}{2}C_3 - C_7) \\
C_0^- &= C_2 - \frac{1}{2}C_1 - C_6 - i(C_0 + \frac{1}{2}C_3 + C_7) \\
C_1^- &= 2(C_4 - C_8 - i(C_9 - C_5)) \\
C_2^- &= 3(C_1 + iC_3) \\
C_3^- &= 2(C_4 + C_8 - i(C_9 + C_5)) \\
C_4^- &= C_2 - \frac{1}{2}C_1 + C_6 - i(C_0 + \frac{1}{2}C_3 - C_7)
\end{aligned}
\tag{52}
$$

while the components T'_{ij} are given in terms of $T_{\mu\nu}$ as follows:

$$
\begin{aligned}
T'_{22} &= T_{00} + 2T_{03} + T_{33} \\
T'_{11} &= T_{00} - T_{33} \\
T'_{00} &= T_{00} - 2T_{03} + T_{33} \\
T'_{21} &= T_{01} + iT_{02} + T_{13} + iT_{23} \\
T'_{12} &= T_{01} - iT_{02} + T_{13} - iT_{23} \\
T'_{10} &= T_{01} + iT_{02} - T_{13} - iT_{23} \\
T'_{01} &= T_{01} - iT_{02} - T_{13} + iT_{23} \\
T'_{20} &= T_{11} + 2iT_{12} - T_{22} \\
T'_{02} &= T_{11} - 2iT_{12} - T_{22}
\end{aligned}
\qquad (53)
$$

and similarly for h'_{ij} in terms of $h_{\mu\nu}$.

In these terms all linear conformal Weyl gravity equations (49) (cf. also (45)) may be written in compact form as the following pair of equations:

$$I^{+}\, C^{+}(z) \;=\; T(z,\bar{z})\,, \qquad\qquad I^{-}\, C^{-}(\bar{z}) \;=\; T(z,\bar{z})\,, \tag{54}$$

where the operators $I^{\pm}$ are given as follows:

$$
\begin{aligned}
I^{+} \;=\;& \Big(z^2\bar{z}^2\partial_+^2 + z^2\partial_v^2 + \bar{z}^2\partial_{\bar{v}}^2 + \partial_-^2 + \\
&+2z^2\bar{z}\partial_v\partial_+ + 2z\bar{z}^2\partial_+\partial_{\bar{v}} + \\
&+2z\bar{z}(\partial_-\partial_+ + \partial_v\partial_{\bar{v}}) + \\
&+2\bar{z}\partial_-\partial_{\bar{v}} + 2z\partial_v\partial_-\Big)\partial_z^2 - \\
&-6\Big(z\bar{z}^2\partial_+^2 + z\partial_v^2 + 2z\bar{z}\partial_v\partial_+ + \bar{z}^2\partial_+\partial_{\bar{v}} + \\
&+\bar{z}(\partial_-\partial_+ + \partial_v\partial_{\bar{v}}) + \partial_v\partial_-\Big)\partial_z + \\
&+12\Big(\bar{z}^2\partial_+^2 + \partial_v^2 + 2\bar{z}\partial_v\partial_+\Big) \; , \qquad (55) \\
I^{-} \;=\;& \Big(z^2\bar{z}^2\partial_+^2 + z^2\partial_v^2 + \bar{z}^2\partial_{\bar{v}}^2 + \partial_-^2 + \\
&+2z^2\bar{z}\partial_v\partial_+ + 2z\bar{z}^2\partial_+\partial_{\bar{v}} + \\
&+2z\bar{z}(\partial_-\partial_+ + \partial_v\partial_{\bar{v}}) + \\
&+2\bar{z}\partial_-\partial_{\bar{v}} + 2z\partial_v\partial_-\Big)\partial_{\bar{z}}^2 - \\
&-6\Big(z^2\bar{z}\partial_+^2 + \bar{z}\partial_{\bar{v}}^2 + 2z\bar{z}\partial_+\partial_{\bar{v}} + z^2\partial_v\partial_+ + \\
&+z(\partial_-\partial_+ + \partial_v\partial_{\bar{v}}) + \partial_-\partial_{\bar{v}}\Big)\partial_{\bar{z}} + \\
&+12\Big(z^2\partial_+^2 + \partial_{\bar{v}}^2 + 2z\partial_+\partial_{\bar{v}}\Big),
\end{aligned}
$$

To make more transparent the origin of (54) and in the same time to derive the quantum group deformation of (54), (55) we first introduce the following parameter-dependent operators:

$$
\begin{aligned}
I^+_n &= \frac{1}{2}\Big(n(n-1) I_1^2 I_2^2 - 2(n^2-1) I_1 I_2^2 I_1 \; + \\
&\quad + \; n(n+1) I_2^2 I_1^2 \Big), \qquad (56) \\
I^-_n &= \frac{1}{2}\Big(n(n-1) I_3^2 I_2^2 - 2(n^2-1) I_3 I_2^2 I_3 \; + \\
&\quad + \; n(n+1) I_2^2 I_3^2 \Big) \; ,
\end{aligned}
$$

where $I_1 = \partial_z$, $I_2 = \bar{z}z\partial_+ + z\partial_v + \bar{z}\partial_{\bar{v}} + \partial_-$, $I_3 = \partial_{\bar{z}}$, are from (16).

The operators $I^\pm_n$ correspond to the singular vectors for the two non-simple non-highest $sl(4)$ roots. More precisely, the operator I^+_n corresponds to the singular vector of weight $2\alpha_{12}$, while the operator I^-_n corresponds to weight $2\alpha_{23}$. The parameter $n = \max(2j_1, 2j_2)$.

It is easy to check that we have the following relation:

$$I^{\pm} = I_4^{\pm} \ , \tag{57}$$

i.e., (54) are written as:

$$I_4^+ \, C^+(z) \ = \ T(z,\bar{z}) \, , \qquad I_4^- \, C^-(\bar{z}) \ = \ T(z,\bar{z}) \, . \tag{58}$$

Using the same operators we can write down the pair of equations which give the Weyl tensor components in terms of the metric tensor:

$$I_2^+ \, h(z,\bar{z}) \ = \ C^-(\bar{z}) \, , \qquad I_2^- \, h(z,\bar{z}) \ = \ C^+(z) \, . \tag{59}$$

The above equations are immediately generalizable to the deformed case.

Using the $U_q(sl(4))$ formula for the singular vector given in [D] we obtain for the q-analogue of (56):

$$
\begin{aligned}
{}_qI_n^+ \quad &= \quad \frac{1}{2}\Big([n]_q\,[n-1]_q \;\; {}_qI_1^2\, {}_qI_2^2 \;\; - \\
&\quad - \;[2]_q\,[n-1]_q\,[n+1]_q\, {}_qI_1\, {}_qI_2^2\, {}_qI_1 + \\
&\quad + [n]_q\,[n+1]_q\, {}_qI_2^2\, {}_qI_1^2 \Big)\;, \qquad (60) \\
{}_qI_n^- \quad &= \quad \frac{1}{2}\Big([n]_q\,[n-1]_q\, {}_qI_3^2\, {}_qI_2^2 \;\; - \\
&\quad - \;[2]_q\,[n-1]_q\,[n+1]_q\, {}_qI_3\, {}_qI_2^2\, {}_qI_3 + \\
&\quad + [n]_q\,[n+1]_q\, {}_qI_2^2\, {}_qI_3^2 \Big)\;,
\end{aligned}
$$

where the q-deformed ${}_qI_a$ were given above.

Then the **q-Weyl gravity equations** are (cf. (58)):

$$
{}_qI_4^+\, C^+(z) \;=\; T(z,\bar{z})\,, \qquad {}_qI_4^-\, C^-(\bar{z}) \;=\; T(z,\bar{z})\,, \tag{61}
$$

while q-analogues of (59) are:

$$
{}_qI_2^+\, h(z,\bar{z}) \;=\; C^-(\bar{z})\,, \qquad {}_qI_2^-\, h(z,\bar{z}) \;=\; C^+(z)\,. \tag{62}
$$

THANK YOU !