Fuzzy field theories and related matrix models

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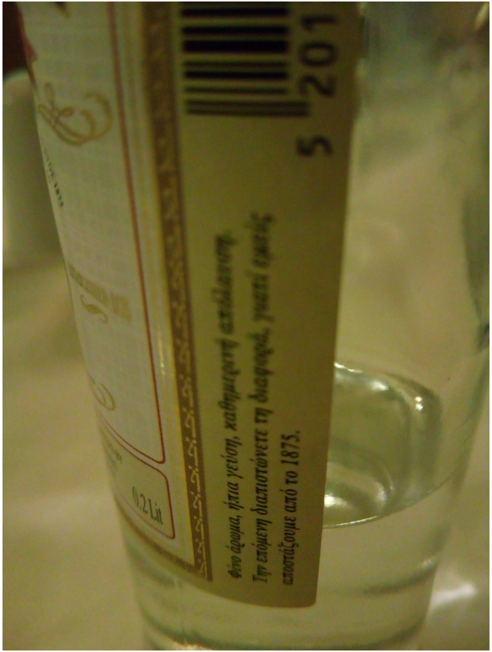

0,2 Lit
φίνο άρωμα, ήπια γεύση, καθημερινή απόλαυση.
Την επόμενη διαπιστώνετε τη διαφορά, γιατί εμείς
αποστάζουμε από το 1875.

Fuzzy spaces

Fuzzy spaces

Fuzzy sphere Hoppe '82; Madore '92; Grosse, Klimčík, Prešnajder '90s

- Functions on the usual sphere are given by

$$f(\theta,\phi)=\sum_{l=0}^{\infty}\sum_{m=-l}^{l}c_{lm}Y_{lm}(\theta,\phi)\ ,$$

  where Y_{lm} are the spherical harmonics

$$\Delta Y_{lm}(\theta,\phi)=l(l+1)Y_{lm}(\theta,\phi)\ .$$

- To describe features at a small length scale we need Y_{lm}'s with a large l.

Fuzzy spaces

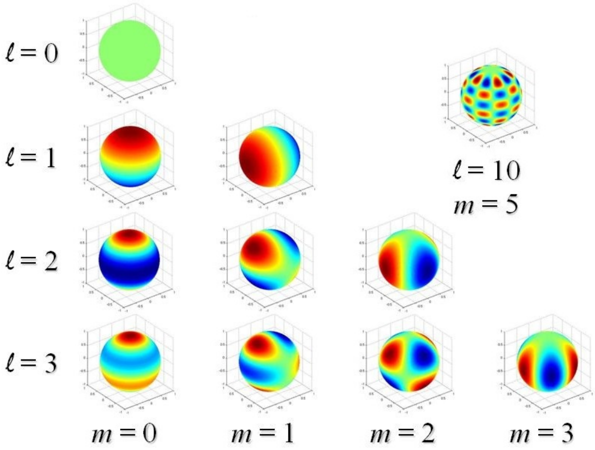


Image taken from http://principles.ou.edu/mag/earth.html

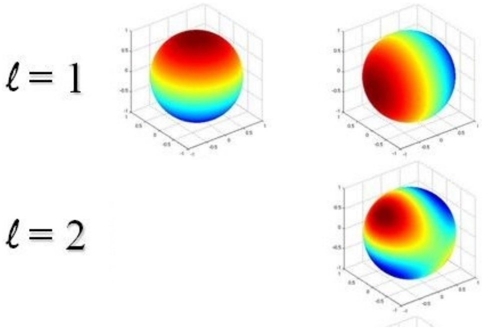
Fuzzy spaces

- If we truncate the possible values of l in the expansion

$$f = \sum_{l=0}^{L} \sum_{m=-l}^{l} c_{lm} Y_{lm}(\theta, \phi) \ ,$$

  we will not be able to see any features of functions under certain length scales.
- Points on the sphere (as δ-functions) cease to exist.
- Expressions defined in this way are not closed under multiplication.

Fuzzy spaces



Fuzzy spaces

- Number of independent functions with $l \leq L$ is N^2, the same as the number of $N \times N$ hermitian matrices.
  The idea is to map the former on the latter and borrow a closed product from there.
- In order to do so, we consider a $N \times N$ matrix as a product of two N-dimensional representations $\underline{N}$ of the group $SU(2)$. It reduces to

$$\begin{array}{ccccccccc} \underline{N} \otimes \underline{N} & = & \underline{1} & \oplus & \underline{3} & \oplus & \underline{5} & \oplus & \ldots \\ & & \downarrow & & \downarrow & & \downarrow & & \\ & = & \{Y_{0m}\} & \oplus & \{Y_{1m}\} & \oplus & \{Y_{2m}\} & \oplus & \ldots \end{array}$$

- We thus have a map $\varphi : Y_{lm} \to M$ and we define the product

$$Y_{lm} * Y_{l'm'} := \varphi^{-1}\left(\varphi\left(Y_{lm}\right)\varphi\left(Y_{l'm'}\right)\right) \ .$$

Fuzzy spaces

- We have a short distance structure, but the prize we had to pay was a noncommutative product $*$ of functions. The space, for which this is the algebra of functions, is called the fuzzy sphere.
- Opposing to some lattice discretization this space still possess a full rotational symmetry

$$Y_{lm} * Y_{l'm'} := \varphi^{-1}\left(\varphi\left(Y_{lm}\right)\varphi\left(Y_{l'm'}\right)\right) \ .$$

- In the limit N or $L \to \infty$ we recover the original sphere.

Fuzzy spaces

- The regular sphere S^2 is given by the coordinates

$$x_i x_i = R^2 \quad , \quad x_i x_j - x_j x_i = 0 \ .$$

- For the fuzzy sphere S^2_F we define

$$x_i x_i = \rho^2 \quad , \quad x_i x_j - x_j x_i = i\theta \varepsilon_{ijk} x_k \ .$$

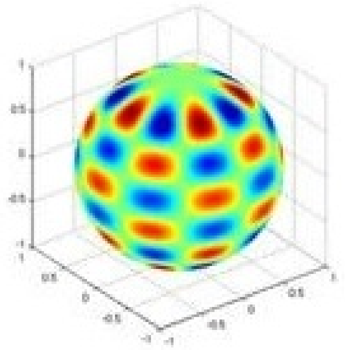
- Can be realized as an $N = 2j+1$ dimensional representation of $SU(2)$

$$x_i = \frac{2r}{\sqrt{N^2-1}} L_i \quad , \quad \theta = \frac{2r}{\sqrt{N^2-1}} \sim \frac{1}{N} \quad , \quad \rho^2 = \frac{4r^2}{N^2-1} j(j+1) = r^2 \ .$$

- In the limit $N \to \infty$ we recover the original sphere.

Fuzzy spaces

- The sphere divided into N cells. Function is given by a matrix M, values on cells represented by eigenvalues.



- No sharp boundaries between the cells, everything is fuzzy.

Take home message

Take home message

- Fuzzy spaces are matrix geometries, which are important as solutions and backgrounds in nonperturbative formulations of string theory. several talks this week
- Fuzzy spaces are toy models of spaces with quantum structure, moreover fuzzy scalar field theories are described by matrix models.
- As such, they are a great laboratory to investigate consequences of quantum structure.

Fuzzy scalar field theory

Fuzzy scalar field theory

- Commutative euclidean theory of a real scalar field is given by an action

$$S(\Phi) = \int d^2x \left[\frac{1}{2} \Phi \Delta \Phi + \frac{1}{2} m^2 \Phi^2 + V(\Phi) \right]$$

and path integral correlation functions

$$\langle F \rangle = \frac{\int d\Phi \, F(\Phi) e^{-S(\Phi)}}{\int d\Phi \, e^{-S(\Phi)}} \; .$$

- We construct the noncommutative theory as an analogue with
  - field $\rightarrow$ matrix,
  - functional integral $\rightarrow$ matrix integral,
  - spacetime integral $\rightarrow$ trace,
  - derivative $\rightarrow$ L_i commutator.

Fuzzy scalar field theory

- **Commutative**

$$S(\Phi) = \int d^2x \left[\frac{1}{2}\Phi\Delta\Phi + \frac{1}{2}m^2\Phi^2 + V(\Phi) \right]$$

$$\langle F \rangle = \frac{\int d\Phi\, F(\Phi) e^{-S(\Phi)}}{\int d\Phi\, e^{-S(\Phi)}}\ .$$

- **Noncommutative** (for S_F^2)

$$S(M) = \frac{4\pi R^2}{N} \mathrm{Tr} \left[\frac{1}{2} M \frac{1}{R^2} [L_i, [L_i, M]] + \frac{1}{2} m^2 M^2 + V(M) \right]$$

$$\langle F \rangle = \frac{\int dM\, F(M) e^{-S(M)}}{\int dM\, e^{-S(M)}}\ .$$

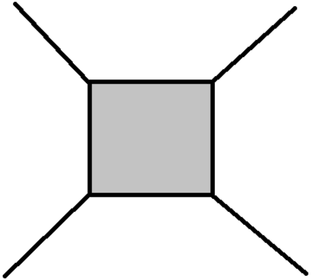
Balachandran, Kürkçüoğlu, Vaidya ’05; Szabo ’03; Ydri ’16

Fuzzy scalar field theory - UV/IR mixing

- The key property of the noncommutative field theories is the UV/IR mixing phenomenon, which arises as a result of the nonlocality of the theory.
  Minwalla, Van Raamsdonk, Seiberg '00; Chu, Madore, Steinacker '01
- In terms of diagrams different properties of planar and non-planar ones.

Fuzzy scalar field theory - UV/IR mixing

$$M = \sum_{l=0}^{N-1} \sum_{m=-l}^{l} c_{lm} T_{lm}$$



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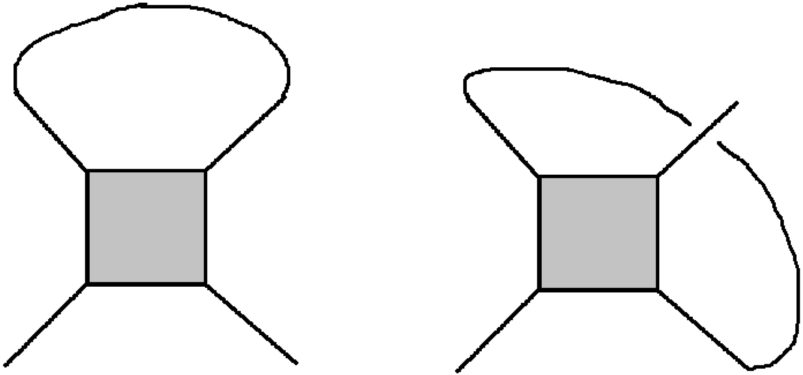
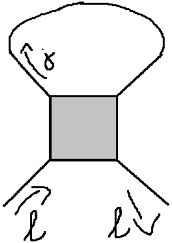
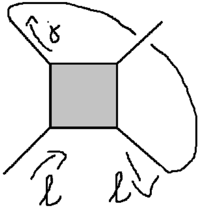


Fuzzy scalar field theory - UV/IR mixing

Chu, Madore, Steinacker '01





$$I^P = \sum_{j=0}^{N-1} \frac{2j+1}{j(j+1)+m^2}$$

$$I^{NP} = \sum_{j=0}^{N-1} \frac{2j+1}{j(j+1)+m^2} (-1)^{l+j+N-1} \left\{ \begin{array}{ccc} l & s & s \\ j & s & s \end{array} \right\} \, , \; s = \frac{N-1}{2}$$

Fuzzy scalar field theory - UV/IR mixing

$$I^{NP} - I^{P} = \sum_{j=0}^{N-1} \frac{2j+1}{j(j+1)+m^2} \left[(-1)^{l+j+N-1} \begin{Bmatrix} l & s & s \\ j & s & s \end{Bmatrix} - 1 \right]$$

- This difference is finite in $N \to \infty$ limit.
- $N \to \infty$ limit of the effective action is different from the standard S^2 effective action.
- In the planar limit $S^2 \to \mathbb{R}^2$ one recovers singularities and the standard UV/IR-mixing.
- The space (geometry) forgets where it came from, but the field theory (physics) remembers its fuzzy origin.

Matrix models of fuzzy field theory

Matrix models of fuzzy field theory

- Random matrix theory = ensemble of hermitian $N \times N$ matrices with a probability measure and expectation values

$$\langle F \rangle = \frac{\int dM\, F(M) e^{-S(M)}}{\int dM\, e^{-S(M)}} \ .$$

Matrix models of fuzzy field theory

- Random matrix theory = ensemble of hermitian $N \times N$ matrices with a probability measure and expectation values

$$\langle F \rangle = \frac{\int dM\, F(M) e^{-S(M)}}{\int dM\, e^{-S(M)}} \; .$$

- This is the very same expression as for the real scalar field.
- Fuzzy field theory = matrix model with

$$S(M) = \frac{1}{2} \mathrm{Tr}\,(M[L_i,[L_i,M]]) + \frac{1}{2} r\, \mathrm{Tr}\,\left(M^2\right) + g\, \mathrm{Tr}\,\left(M^4\right)$$

Matrix models of fuzzy field theories

- The large N limit of the model **without** the kinetic term

$$S(M) = \frac{1}{2}\mathrm{Tr}\left(M[L_i,[L_i,M]]\right) + \frac{1}{2}m^2\,\mathrm{Tr}\left(M^2\right) + g\,\mathrm{Tr}\left(M^4\right)$$

is **well** understood.

Brezin, Itzykson, Parisi, Zuber '78; Shimamune '82

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is **not well** understood

$$\int dM\, F(M) e^{-S(M)} / .$$

Matrix models of fuzzy field theories

"Matrix model begs to be put on a computer."

H. Steinacker @ Humboldt Kolleg

Matrix models of fuzzy field theories

"Matrix model begs to be put on a computer."

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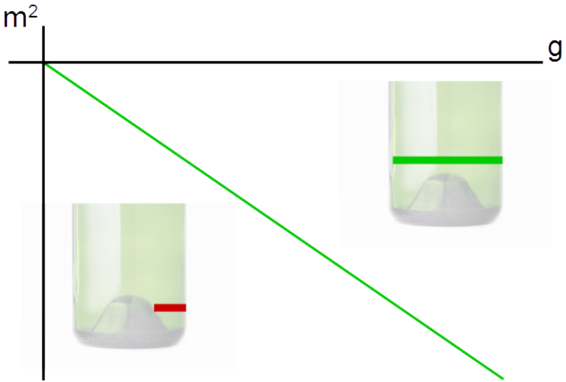
Fuzzy field theory matrix models on a computer:

- spontaneous symmetry breaking patterns,
- behaviour of the correlation functions,
- entanglement entropy.

Spontaneous symmetry breaking

Symmetry breaking in NC field theories

$$S[\phi] = \int d^2x \left(\frac{1}{2}\partial_i\phi\partial_i\phi + \frac{1}{2}m^2\phi^2 + g\phi^4 \right)$$



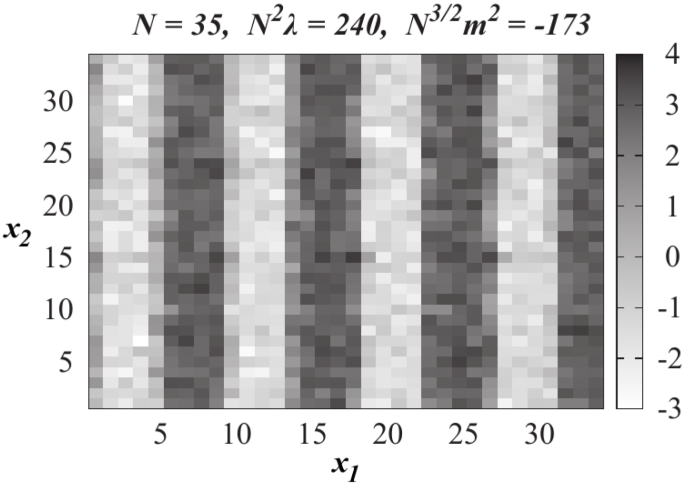
Glimm, Jaffe '74; Glimm, Jaffe, Spencer '75; Chang '76
Loinaz, Willey '98; Schaich, Loinaz '09

Symmetry breaking in NC field theories

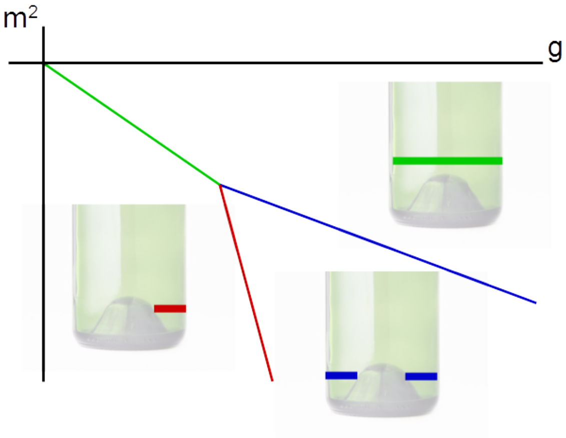
- The phase diagram of noncommutative field theories has one more phase. It is a non-uniform order phase, or a striped phase.
  Gubser, Sondhi '01; Chen, Wu '02
- In this phase, the field does not oscillate around one given value in the whole space. Translational symmetry is broken.
- This has been established in numerous numerical works for variety different spaces.
  Martin '04; García Flores, Martin, O'Connor '06, '09; Panero '06, '07; Ydri '14; Bietenholz, F. Hofheinz, Mejía-Díaz, Panero '14; Mejía-Díaz, Bietenholz, Panero '14; Medina, Bietenholz, D. O'Connor '08; Bietenholz, Hofheinz, Nishimura '04; Lizzi, Spisso '12; Ydri, Ramda, Rouag '16; Kováčik, O'Connor '18
  Panero '15

Symmetry breaking in NC field theories

Mejía-Díaz, Bietenholz, Panero '14 for $\mathbb{R}^2_\theta$



$$S[M] = \mathrm{Tr}\left(\frac{1}{2}M[L_i,[L_i,M]] + \frac{1}{2}m^2M^2 + gM^4\right)$$



Correlation functions

Correlation functions

- Analogues of points in the NC setting are coherent states $|\vec{x}\rangle$.
- "Value" of field M at "point" $\vec{x}$ given by

$$\langle\vec{x}|\, M\, |\vec{x}\rangle = M(x)\ .$$

- Behaviour of

$$\langle M(x)M(y)\rangle = \frac{1}{Z}\int dM\, \langle\vec{x}|\, M\, |\vec{x}\rangle\, \langle\vec{y}|\, M\, |\vec{y}\rangle\, e^{-S(M)}$$

  in the matrix model can be studied numerically.
  Hatakeyama, Tsuchiya '17; Hatakeyama, Tsuchiya, Yamashiro '18 '18
- At the "standard" phase transition, the behaviour of the correlation functions at short distances differs from the commutative theory and seems to agree with the tricritical Ising model.
  A different behaviour at long distances.

Entanglement entropy

Entanglement entropy

- In local theories $S(A) \sim A$.
  Ryu, Takayanagi '06
- In non-local theories this can change.
  Barbon, Fuertes '08; Karczmarek, Rabideau '13; Shiba, Takayanagi '14
- Problem on the fuzzy sphere has been studied numerically.
  Karczmarek, Sabella-Garnier '13; Sabella-Garnier '14; Okuno, Suzuki, Tsuchiya '15; Suzuki, Tsuchiya '16; Sabella-Garnier '17; Chen, Karczmarek '17
- For free fields, the EE follows volume law rather than area law.
  In the interacting case much smaller EE than in the free case.

Matrix models of fuzzy field theories - analytical treatment

Matrix models of fuzzy field theories

- The large N limit of the model **with** the kinetic term

$$S(M) = \frac{1}{2}\mathrm{Tr}\left(M[L_i,[L_i,M]]\right) + \frac{1}{2}m^2\,\mathrm{Tr}\left(M^2\right) + g\,\mathrm{Tr}\left(M^4\right)$$

is **not well** understood.

$$\int dM\, F(M) e^{-S(M)}$$

- The key issue being that diagonalization $M = U\ \mathrm{diag}(\lambda_1,\ldots,\lambda_N)U^\dagger$ no longer straightforward.
- Integrals like

$$\langle F\rangle \sim \int \left(\prod_{i=1}^{N} d\lambda_i\right) dU\ F(\lambda_i,U)\ e^{-N^2\left[\frac{1}{2}m^2\frac{1}{N}\sum\lambda_i^2 + g\frac{1}{N}\sum\lambda_i^4 - \frac{2}{N^2}\sum_{i<j}\log|\lambda_i-\lambda_j|\right]}$$

$$\times\ e^{-\frac{1}{2}\mathrm{Tr}\left(U\Lambda U^\dagger[L_i,[L_i,U\Lambda U^\dagger]]\right)}$$

$$\langle F\rangle \sim \int \left(\prod_{i=1}^{N} d\lambda_i\right) dU\ F(\lambda_i, U)\ e^{-N^2\left[S_{eff}(\lambda_i)+\frac{1}{2}m^2\frac{1}{N}\sum\lambda_i^2+g\frac{1}{N}\sum\lambda_i^4-\frac{2}{N^2}\sum_{i<j}\log|\lambda_i-\lambda_j|\right]}$$

$$e^{-N^2 S_{eff}(\lambda_i)} = \int dU\, e^{-\frac{1}{2}\mathrm{Tr}\left(U\Lambda U^\dagger[L_i,[L_i,U\Lambda U^\dagger]]\right)}$$

Steinacker '05

How to compute S_{eff}?

- Expansion in powers of kinetic term.
- Expansion around free field solution (not expansion in g).

Matrix models of fuzzy field theories

$$e^{-N^2 S_{eff}(\lambda_i)} = \int dU \, e^{-\varepsilon \frac{1}{2} \mathrm{Tr}\left(U\Lambda U^\dagger [L_i,[L_i,U\Lambda U^\dagger]]\right)}$$

- Perturbative calculation of the integral show that the S_{eff} contains products of traces of M.
  O'Connor, Sämann '07; Sämann '10
- The most recent result is
  Sämann '15

$$\begin{aligned} S_{eff}(\lambda_i) =& \frac{1}{2}\left[\varepsilon \frac{1}{2}\left(c_2 - c_1^2\right) - \varepsilon^2 \frac{1}{24}\left(c_2 - c_1^2\right)^2 + \varepsilon^4 \frac{1}{2880}\left(c_2 - c_1^2\right)^4\right] - \\ & - \varepsilon^4 \frac{1}{3456}\left[\left(c_4 - 4c_3c_1 + 6c_2c_1^2 - 3c_1^4\right) - 2\left(c_2 - c_1^2\right)^2\right]^2 - \\ & - \varepsilon^3 \frac{1}{432}\left[c_3 - 3c_1c_2 + 2c_1^3\right]^2 \quad , \quad \text{where } c_n = \frac{1}{N}\sum_i \lambda_i^n \end{aligned}$$

- Standard treatment of such multitrace matrix model yields a very unpleasant behaviour close to the origin of the parameter space.

Matrix models of fuzzy field theories

$$e^{-N^2 S_{eff}(\lambda_i)} = \int dU\, e^{-\varepsilon \frac{1}{2} \mathrm{Tr}\left(U\Lambda U^\dagger [L_i,[L_i,U\Lambda U^\dagger]]\right)}$$

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- More reasonable for large values of m^2, g.
  Rea, Sämann '15

Hermitian matrix model of fuzzy field theories

- For the free theory $g = 0$ the kinetic term just rescales the eigenvalues.
  Steinacker '05
- There is a unique parameter independent effective action that reconstructs this rescaling.
  Polychronakos '13

$$S_{eff}(\lambda_i) = \frac{1}{2}\left[\varepsilon\frac{1}{2}\left(c_2 - c_1^2\right) - \varepsilon^2\frac{1}{24}\left(c_2 - c_1^2\right)^2 + \varepsilon^4\frac{1}{2880}\left(c_2 - c_1^2\right)^4 + \ldots\right] -$$

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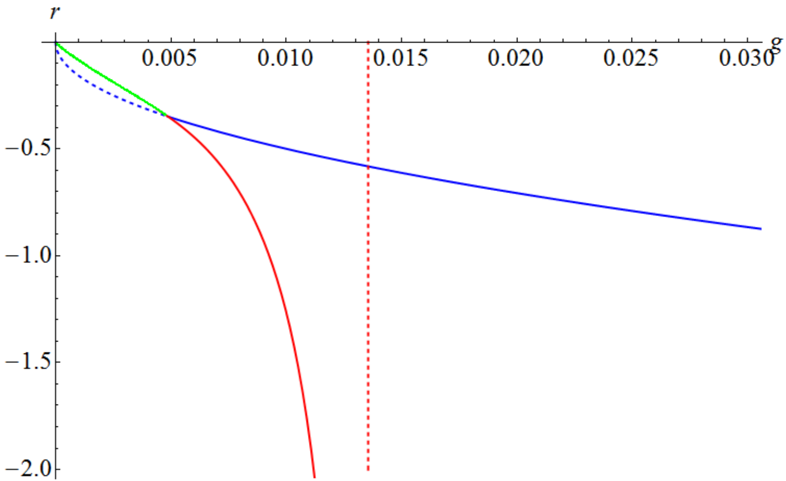
$$S_{eff}(\lambda_i) = \frac{1}{2} F(c_2) + \mathcal{R} = \frac{1}{2} \log \left(\frac{c_2}{1 - e^{-c_2}} \right) + \mathcal{R}$$

- Introducing the asymmetry $c_2 \to c_2 - c_1^2$ we obtain a matrix model

$$S(M) = \frac{1}{2} F(c_2 - c_1^2) + \frac{1}{2} m^2 \operatorname{Tr}\left(M^2\right) + g \operatorname{Tr}\left(M^4\right) \quad , \quad F(t) = \log \left(\frac{t}{1 - e^{-t}} \right)$$

Polychronakos '13; JT

Matrix models of fuzzy field theories



JT '17; Šubjaková, JT '19

Matrix models of fuzzy field theories

- A very good qualitative agreement. A very good quantitative agreement in the critical coupling.
  Kováčik, O'Connor '18
- Different behaviour of the asymmetric transition line for large $-r$.
- We need to include $\mathcal{R}$, or the higher moments of the matrix, in a nonperturbative way.
  work in progress with M. Šubjaková

Challenges

Challenges

Correlation functions

- Quantity $\langle M(x)M(y)\rangle$ is U dependent, so we need to figure out what to do with

$$\int dU\ F(\Lambda, U)\ e^{-\frac{1}{2}\mathrm{Tr}\left(U\Lambda U^\dagger[L_i,[L_i,U\Lambda U^\dagger]]\right)}\ .$$

Entanglement entropy

- We need to extended the model to $\mathbb{R}\times S_F^2$, i.e. $M(t)$

$$S(M)=\int dt\mathrm{Tr}\left(-\frac{1}{2}M\partial_t^2 M+\frac{1}{2}M[L_i,[L_i,M]]+\frac{1}{2}m^2M^2+gM^4\right)$$

Medina, Bietenholz, O'Connor '07; Ihl, Sachse, Sämann '10

Also the U dependence will play a role, but free theory where $\mathcal{R}=0$, is enough.

Take home message

- Fuzzy spaces are matrix geometries important as solutions and background in nonperturbative formulations of string theory.
- Fuzzy spaces are toy models of spaces with quantum structure, scalar field theories are described by matrix models.
- As such, they are a great laboratory to investigate consequences of quantum structure.

Take home message

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Thank you for your attention!

If time permits

If time permits - Towards a matrix model of UV/IR free theory

- The previous method works for any model with a kinetic term $\mathcal{K}$, which is diagonal in T_{lm} basis

$$\mathcal{K}T_{lm} = K(l)T_{lm} \ .$$

- We would like to analyze the more complicated model

$$S = \mathrm{Tr}\left(\frac{1}{2}M[L_i,[L_i,M]] + a12gMQM + \frac{1}{2}m^2M + gM^4\right)$$

where

$$QT_{lm} = \underbrace{-\left(\sum_{j=0}^{N-1}\frac{2j+1}{j(j+1)+m^2}\left[(-1)^{l+j+N-1}\left\{\begin{matrix} l & s & s \\ j & s & s \end{matrix}\right\} - 1\right]\right)}_{Q(l)} T_{lm} \ .$$

This removes the UV/IR mixing in the theory.
Dolan, O'Connor, Prešnajder '01

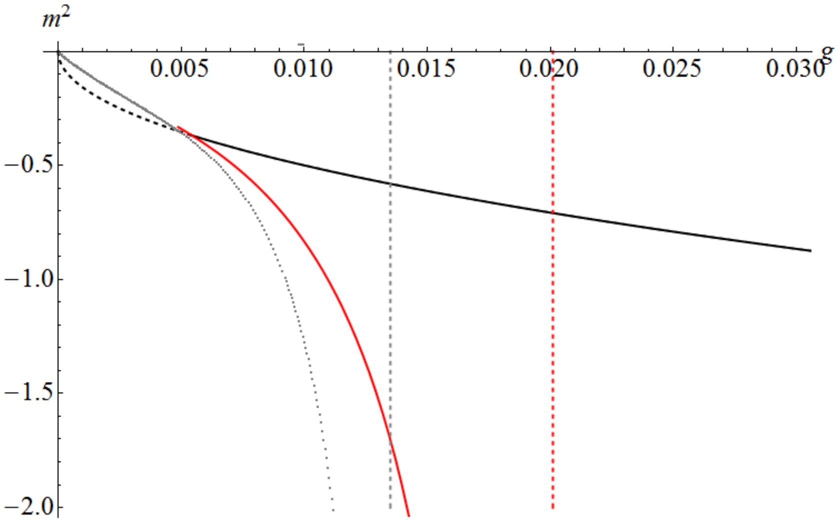
If time permits - Towards a matrix model of UV/IR free theory

- Operator Q can be expressed as a power series in $C_2 = [L_i, [L_i, \cdot]]$

$$Q = q_1 C_2 + q_2 C_2^2 + \dots$$

- As a starting point, it is interesting to see the phase structure of such simplified model.
  O'Connor, Säman '07

Towards a matrix model of UV/IR free theory



If time permits - GW solution

Grosse, Wulkenhaar '09 '14; Grosse, Sako, Wulkenhaar '16; Panzer, Wulkenhaar '18; Grosse, Hock, Wulkenhaar '19 '19

- Model
$$S(M) = \mathrm{Tr}\left(EM^2 + gM^4\right)$$
for a fixed external matrix E has been solved.
- An implicit formula for two point function and formulas for all higher correlation functions in terms of this two point function.
- Challenges: expressions are technically complicated to work with and work only for positive eigenvalues of E.