Diffeomorphisms and approximate invariants on fuzzy sphere

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Keyword:

Matrix regularization

Membrane theory

The Hamiltonian for a (bosonic) closed membrane moving in $\mathbf{R}^{10,1}$ in the light-cone gauge is

$$H = \int_{\Sigma} d^2\sigma \left(\frac{1}{2} p^A p_A + \frac{1}{4}\{x^A, x^B\}\{x_A, x_B\}\right)$$

- Closed surface Σ which represents the membrane.
- Embedding coordinates x^A $(A = 1,2,\ldots,9)$ in $\mathbf{R}^{10,1}$.
- Canonical momenta $p^A = \partial x^A/\partial t$.
- Poisson bracket $\{\,,\}$ induced by the volume form ω on Σ.

Matrix regularization

The Hamiltonian is described in terms of the functions $x^A, p^A \in C^\infty(\Sigma)$ and the Poisson bracket $\{\,,\}$ on Σ.

The matrix regularization is an operation of the following replacement,

[Hoppe, de Wit-Hoppe-Nicolai, Arnlind-Hoppe-Huisken]

Poisson algebra on Σ		Lie algebra of matrices
$(C^\infty(\Sigma), \{\,,\})$	$\longrightarrow$	$(M_N(\mathbf{C}), [\,,])$
Infinite dimension		Finite dimension

which approximate the Poisson algebra by matrices. The accuracy of the approximation improves as $N \to \infty$.

Matrix model for membrane

After the matrix regularization, the Hamiltonian of the membrane theory becomes [Hoppe, de Wit-Hoppe-Nicolai]

$$H = \mathrm{Tr}\left(\frac{1}{2}P^A P_A - \frac{1}{4}[X^A, X^B][X_A, X_B]\right)$$

This coincides with (the bosonic term of) the matrix model which is conjectured to describe M-theory. [BFSS, Susskind, Seiberg]

The matrix regularization is also applied to type IIB string theory and provides a matrix model for the nonperturbative formulation. [IKKT]

Area-preserving diffeomorphisms

In the matrix regularization, the (residual) gauge symmetry of the membrane theory is replaced as

Area-preserving diffs $\xrightarrow{\text{MR}}$ Unitary transformations

$$\delta x^A = \{x^A, v\} \qquad \longrightarrow \qquad \delta X^A = -i[X^A, V]$$

↑

(the Lie algebra is)

isomorphic

↓

Poisson algebra on Σ

↙ (the Lie algebra is) equivalent in $N \to \infty$ at least for S^2

[Hoppe, Pope-Stelle]

Topic of my talk

We study how general diffeomorphisms on Σ act on the matrices in the matrix regularization.

General diffs on Σ ???

$$\delta x^A = u^\mu \partial_\mu x^A \longrightarrow \delta X^A = ???$$

- For constructing a covariant formulation of M-theory.
- For formulating theories of gravity on fuzzy spaces.

[Chamseddine-Connes, Aschieri et al, Hanada-Kawai-Kimura, Steinacker, Nair, Yang, etc.]

Plan of my talk:

1. Berezin-Toeplitz quantization

2. Matrix diffeomorphisms

3. Matrix diffeomorphisms on fuzzy sphere

4. Approximate diffeomorphism invariants

1. Berezin-Toeplitz quantization

Matrix regularization and quantization

The matrix regularization is very similar to quantization on the classical phase space (for a particle moving on the real line),

Noncompact

Poisson algebra on $\mathbf{R}^2$ $\longrightarrow$ Algebra of operators

$$(C^\infty(\mathbf{R}^2), \{\ ,\ \}) \longrightarrow (Q, [\ ,\ ])$$

Infinite dimension

We can say that the matrix regularization is the quantization on a compact curved phase space. $\Longrightarrow$ Berezin-Toeplitz quantization

Berezin-Toeplitz quantization

The quantization is given by a linear map for the canonical variables,

$$(q,p) \in \mathbf{R}^2 \longmapsto (\hat{q},\hat{p}) \;\text{ with }\; [\hat{q},\hat{p}] = i\hbar$$

and fixing the ordering of $(\hat{q},\hat{p})$ in composite operators. The Berezin-Toeplitz quantization is a scheme of the anti-normal ordering:

$$\hat{f} = \frac{1}{\pi\hbar}\int_{\mathbf{R}^2} d^2z\, |z\rangle\langle z|\, f(z)$$

- Complex coordinate $z = (q+ip)/\sqrt{\hbar}$.
- Canonical coherent state $|z\rangle$: $\hat{a}|z\rangle = \frac{1}{\sqrt{\hbar}} z|z\rangle$ for $[\hat{a},\hat{a}^\dagger] = 1$.

Coherent state and Dirac zero modes

We can rewrite the Berezin-Toeplitz map as

$$\langle i|\hat{f}|j\rangle = \int_{\mathbf{R}^2} d^2z\, \psi_j^\dagger(z)\, \psi_i(z)\, f(z)$$

$$\psi_i(z) = \frac{1}{\sqrt{\pi\hbar}} \begin{pmatrix} \langle i|z\rangle \\ 0 \end{pmatrix} \qquad (i = 1,2,\dots,\infty)$$

The spinors $\psi_i(z)$ are characterized as the zero modes of a Dirac operator with a $U(1)$ gauge potential for a constant curvature,

$$D = i\sigma^a\left(\partial_a - \frac{i}{\hbar}A_a\right), \qquad F = dA = dq \wedge dp$$

This formulation can be generalized to general phase spaces.

Berezin-Toeplitz map

The Berezin-Toeplitz map $T_N : C^\infty(\Sigma) \to M_N(\mathbf{C})$ for a closed surface Σ is

[Klimek-Lesniewski, Bordemann-Meinrenken-Schlichenmaier, Ma-Marinescu] c.f.[Terashima]

$$\langle i | T_N(f) | j \rangle = \int_\Sigma d^2\sigma \sqrt{g}\, \psi_j^\dagger \psi_i\, f$$

- Riemannian metric $g_{\mu\nu}$ (which is compatible with ω).
- $U(1)$ gauge potential A_μ with the Chern number $\frac{1}{2\pi}\int_M F = N$.
- Dirac operator $D = i\sigma^\mu(\partial_\mu + \Omega_\mu - iA_\mu)$.
- Orthonormal basis of $\mathrm{Ker} D$: ψ_i $(i = 1, 2, \ldots, N)$.

by the Index theorem

2. Matrix diffeomorphisms

Mapping diffeomorphisms

We map automorphisms of $C^\infty(\Sigma)$ instead of diffeomorphisms of Σ to transformations of matrices:

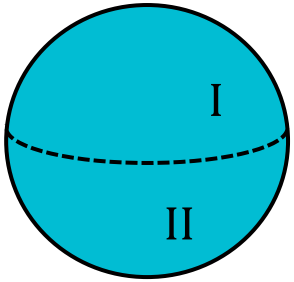
$$
\begin{tikzcd}
f \in C^\infty(\Sigma) \arrow[rr, "\text{B-T map}"] \arrow[d, mapsto, "\substack{\text{automorphism}\\ \text{induced by } \varphi}"] & & F = T_N(f) \arrow[d, mapsto, "\substack{\text{transformation of}\\ \text{matrices}}"] \\
\substack{f \circ \varphi \\ \parallel \\ \varphi^* f \in C^\infty(\Sigma)} \arrow[rr, "\text{B-T map}"] & & F' = T_N(\varphi^* f)
\end{tikzcd}
$$

3. Matrix diffeomorphisms on the fuzzy sphere

Dirac zero modes for S^2

We choose the standard round metric $g = d\theta^2 + sin^2\theta\, d\phi^2$ and the Wu-Yang gauge potential

$$A = \begin{cases} \frac{N}{2}(-\cos\theta + 1)d\phi & \text{(region I)} \\ \frac{N}{2}(-\cos\theta - 1)d\phi & \text{(region II)} \end{cases}$$



With these data, the orthonormal Dirac zero modes are

$$\psi_i(\theta,\phi) = \sqrt{\frac{N}{2\pi}}\begin{pmatrix} \langle i|\Omega\rangle \\ 0 \end{pmatrix} \quad (i = 1, 2, \dots, N)$$

$$|\Omega\rangle = e^{-\mathrm{i}\phi L^3}\, e^{-\mathrm{i}\theta L^2}\, e^{\mathrm{i}\phi L^3}\, |JJ\rangle \quad (N = 2J + 1)$$

N-dim irrep of $SU(2)$ generators

Berezin-Toeplitz map for S^2

The Berezin-Toeplitz map for the embedding functions x^A $(A = 1,2,3)$ of S^2 in $\mathbf{R}^3$ are

$$X^A := T_N(x^A) = \frac{N}{2\pi}\int_{S^2} d\Omega\, |\Omega\rangle\langle\Omega|\, x^A = \frac{L^A}{J+1}$$

This is the well-known configuration of the fuzzy sphere (up to $O(N^{-1})$), which satisfies [Madore]

$$X^A X_A = 1 + O(N^{-1})$$

$$N[X^A, X^B] = 2i\epsilon^{ABC} X_C + O(N^{-1})$$

Holomorphic diffeomorphisms of S^2

We identify the sphere with $\mathbf{C} \cup \{\infty\}$ by the stereographic coordinate $z = e^{i\phi} \tan\frac{\vartheta}{2}$ and focus on the holomorphic diffeomorphisms,

$$\varphi(z) = \frac{az+b}{cz+d}, \qquad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbf{C})$$

Special four types: Isometry & area-preserving This talk

$$\boxed{\begin{array}{c}\text{Rotation}\\[1em] \varphi(z) = e^{i\alpha} z \ (\alpha \in \mathbf{R})\end{array}}$$

$$\boxed{\begin{array}{c}\text{Dilatation}\\[1em] \varphi(z) = e^{t} z \ (t \in \mathbf{R})\end{array}}$$

Translation

$$\varphi(z) = z + b$$

Special conformal

$$\varphi(z) = \frac{z}{cz+1}$$

Mapping dilatation

The transformation of x^A induced by the dilatation is

$$\varphi^* x^A(z) = x^A(e^t z) \quad (t \geq 0)$$

Gauss's hypergeometric function $F(\alpha, \beta, \gamma, s)$

The corresponding transformation of X^A is

$$\langle Jr|X'^+|Js\rangle = \frac{\delta_{r-1s}\, e^{-t}}{J+1}\sqrt{(J-r+1)(J+1)}F(J+r+1,1,2J+3;1-e^{-2t})$$

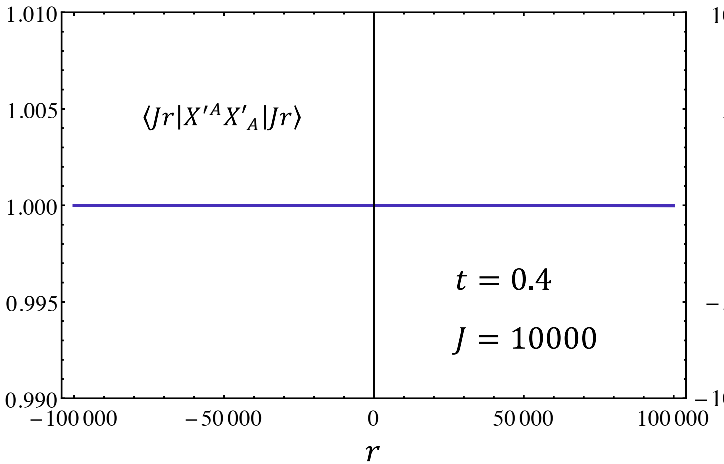
$$\langle Jr|X'^-|Js\rangle = \frac{\delta_{r+1s}\, e^{-t}}{J+1}\sqrt{(J+r+1)(J-1)}F(J+r+2,1,2J+3;1-e^{-2t})$$

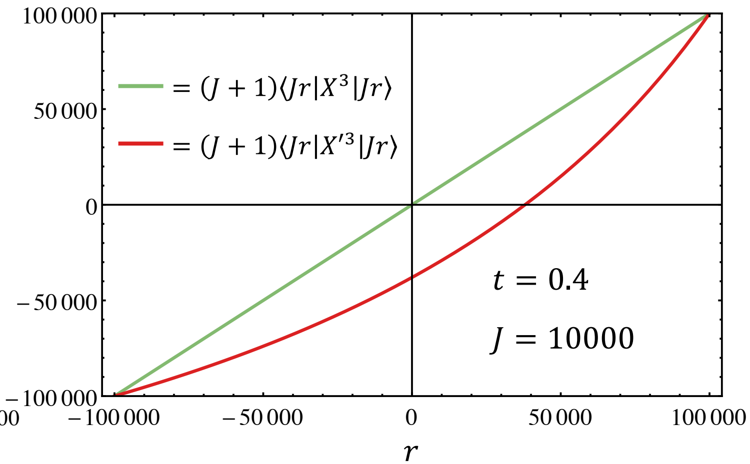
$$\langle Jr|X'^3|Js\rangle = \frac{\delta_{rs}}{2(J+1)}\{(1+e^{-2t})(J+r+1)F(J+r+2,1,2J+3;1-e^{-2t})$$

$$-2(J+1)F(J+r+1,1,2J+2;1-e^{-2t})\}$$

Unitary non-equivalence

The transformation off course does not break the constraint, $X'^A X'_A \simeq 1$, but changes the eigenvalues.





Thus, general diffeomorphisms do not correspond to unitary similarity transformations in the matrix regularization.

4. Approximate diffeomorphism invariants

Approximate diffeomorphism invariants

We propose three kinds of approximate diffeomorphism invariants on the fuzzy sphere in the sense that they are

(i) Invariant exactly under unitary similarity transformations

$$\delta X^A = -i[X^A, V]$$

(ii) Invariant in the large-N limit under general matrix diffs

$$\delta X^A = \frac{N}{2\pi}\int_{S^2} d\Omega\, |\Omega\rangle\langle\Omega| u^\mu \partial_\mu x^A$$

Matrix Dirac operator

From the embedding functions x^A and the matrices X^A of the fuzzy S^2, we define a Dirac type operator,

$$\hat{D} = \sigma^A \otimes (X_A - x_A)$$

We denote the eigenvalues and eigenstates by E_n and $|n\rangle$ such that $|E_0| \leq |E_1| \leq \cdots$. Then we have [de Badyn-Karczmarek-Sabella-Garnier-Yeh]

$$E_0 = \frac{J}{J+1} - 1 = O(N^{-1})$$

$$|0\rangle = U_2 \begin{pmatrix} 1 \\ 0 \end{pmatrix} \otimes |\Omega\rangle$$

Invariance of E_0

Under an infinitesimal matrix diff $X^A \to X^A + \delta X^A$, E_0 transforms as

$$\delta E_0 = \langle 0|\sigma^A \otimes \delta X_A|0\rangle$$

The "locality" of $|\Omega\rangle$

$|\langle\Omega|\Omega'\rangle|^2 \sim J\delta^{(2)}(\Omega-\Omega')$

$$= x^A u^\mu \partial_\mu x^A + O(N^{-1})$$

this term is zero since $x^A x_A = 1$

Thus E_0 is invariant up to $1/N$ corrections.

E_0 has the information of the induced metric for the embedding functions x^A. [Berenstein-Dzienkowski, Ishiki, Schneiderbauer-Steinacker] cf.[Terashima]

Information metric

By using the eigenstate $|0\rangle$, we introduce a density matrix,

$$\rho = |0\rangle\langle 0|$$

This defines an embedding of S^2 into the space of density matrices and gives a metric h on S^2 as the pullback of the information metric,

$$h_{\mu\upsilon} d\sigma^{\mu} d\sigma^{\nu} = \mathrm{Tr}\, d\rho d\rho$$

For general Kähler manifolds, this gives a Kähler metric. [Ishiki-TM-Muraki]

Covariance of $h_{\mu\upsilon}$

Under an infinitesimal matrix diff $X^A \to X^A + \delta X^A$, $|0\rangle$ transforms as

$$\delta|0\rangle = \sum_{n\neq 0} \frac{|n\rangle\langle n|\sigma^A \otimes \delta X_A|0\rangle}{E_0 - E_n} + \text{(puer imaginary)}$$

$$= -u^\mu \partial_\mu |0\rangle + \text{(puer imaginary)} + O(N^{-1})$$

This means $\delta\rho = -u^\mu \partial_\mu \rho + O(N^{-1})$, and so the induced metric h is covariant up to $1/N$ corrections:

$$\delta h_{\mu\upsilon} = -\nabla_\mu u_\upsilon + \nabla_\upsilon u_\mu + O(N^{-1})$$

Heat kernel expansion

For a $2n$-dimensional Riemannian manifold (M, g), the heat kernel

$$K(t) = \mathrm{Tr}\, e^{-t\Delta}$$

$$\Delta = -\frac{1}{\sqrt{g}} \partial_\mu (\sqrt{g} g^{\mu\nu} \partial_\nu)$$

generates diffeomorphism invariants on M as the coefficients of the asymptotic expansion in $t \to +0$:

Einstein-Hilbert action

$$K(t) \sim \frac{1}{(4\pi)^n} \int_M \sqrt{g}\ t^{-n} + \frac{1}{(4\pi)^n} \frac{1}{6} \int_M \sqrt{g}\, R\, t^{-n+1} + \cdots$$

Cosmological term

Heat kernel on fuzzy sphere

We define the heat kernel on the fuzzy sphere by [Sasakura]

$$\hat{K}(t_N, N) = \mathrm{Tr}\, e^{-t_N \hat{\Delta}}$$

$$\hat{\Delta} = (J+1)^2 [X^A, [X_A, \quad]]$$

The matrix Laplacian $\hat{\Delta}$ corresponds to the operator $-\{x^A, \{x_A, \cdot\}\}$ and has the same spectrum with Δ on S^2 up to a UV cutoff:

$$\mathrm{Tr}\, \hat{\Delta} = \sum_{l=0}^{N-1} \sum_{m=-l}^{l} l(l+1) = \sum_{l=0}^{N-1} l(l+1)(2l+1)$$

Double scaling limit

The matrix heat kernel $\widehat{K}$ is regular in $t_N \to +0$ for finite N, but by putting $t_N = N^{-\alpha}$ $(0 < \alpha < 1)$ and taking the limit $N \to \infty$, we have

$\mathrm{Vol}(S^2) = 4\pi$

$$\widehat{K}(t_N, N) \sim 1 \cdot t_N^{-1} + \frac{1}{3} t_N^0 + O(t_N)$$

$R = 2$

The geometric information which $\widehat{K}$ has is based on the metric of $-\{x^A, \{x_A, \cdot\}\} = -g^{\mu\nu}\partial_\mu\partial_\nu + \cdots$ where

$$g^{\mu\nu} = W^{\rho\mu}W^{\sigma\nu}\partial_\rho x^A \partial_\sigma x_A$$

Open string metric in the strong magnetic flux [Seiberg-Witten]

Invariance of $\widehat{K}$

Under a general perturbation $X^A \to X^A + \delta X^A$, we find

$$\delta X^A = \sum_{lm\rho} \delta X_{lm\rho}\, \hat{Y}^A_{lm\rho}$$

Vector fuzzy spherical harmonics
[Ishiki-Shimasaki-Takayama-Tsuchiya]

$$\delta \widehat{K} = 2it_N \delta X_{00-1} \sqrt{\frac{J+1}{J}} \sum_{l=0}^{N-1} e^{-t_N l(l+1)} l(l+1)(2l+1)$$

The mode δX_{00-1} is for $\hat{Y}^A_{00-1} \propto L^A$, which changes the radius of S^2 in $\mathbf{R}^3$ and so violates the constraint $X^A X_A \simeq 1$.

This means that if δX^A is a matrix diffeomorphism, then $\delta \widehat{K} = 0$.

Summary

In the formulation of the matrix regularization, we …

- defined the action of diffeomorphisms on matrices using the Berezin-Toeplitz quantization map.

- proposed three kinds of method of constructing approximate invariants on the fuzzy sphere.

The future work is …

- charactering the matrix diffeomorphisms in terms of purely the matrix geometry.

- applying to formulate gravity on fuzzy spaces.